A mixed eigenvalue problem on domains tending to infinity in several directions

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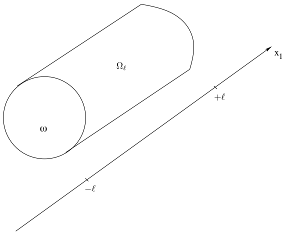

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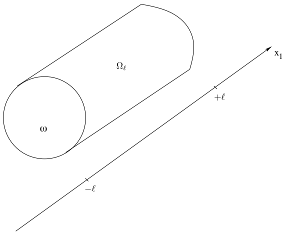

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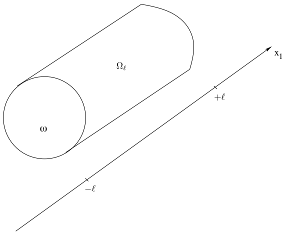

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Theorem [Chipot-Rougirel 08]. $\mu^1 \le \sigma_\ell^k \le \mu^1 + \frac{C_k}{\ell^2}$.

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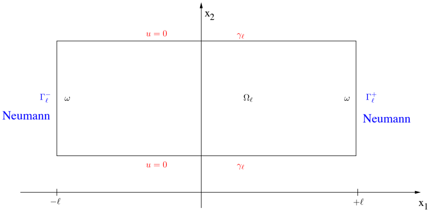

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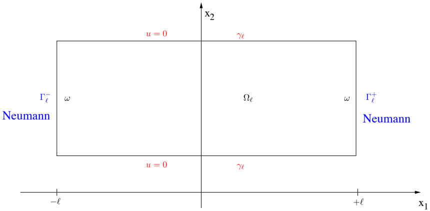

Problem: Find $\lim_{\ell\to\infty} \lambda_\ell^1$ and $\lim_{\ell\to\infty} u_\ell$, with u_ℓ realizing
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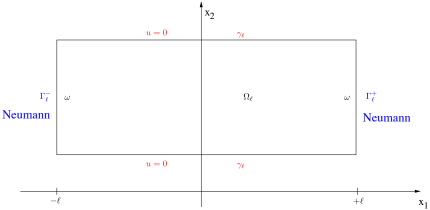

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In particular, can we have $\lim_{\ell\to\infty} \lambda_\ell^1 < \mu^1$?

Dimension reduction: ℓ goes to 0

Theorem.

$$\lim_{\ell \to 0} \lambda_\ell^1 = \Lambda^1 = \inf_{v \in H_0^1(\omega), \int_\omega v^2 = 1} \int_\omega A_{22}(\xi) \nabla v . \nabla v - \frac{|A_{12}(\xi) . \nabla v|^2}{a_{11}(\xi)}.$$

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• "Optimizing over u_{x_1}" allows us to construct a good test function for λ_ℓ^1 (when $\ell\sim 0$).

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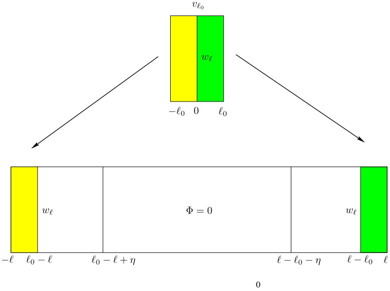

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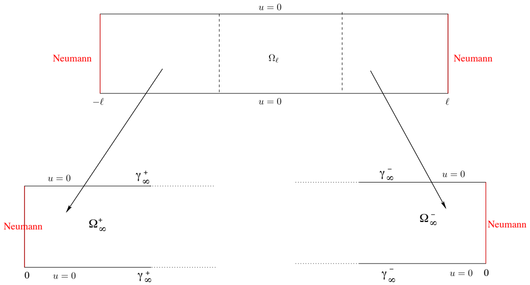

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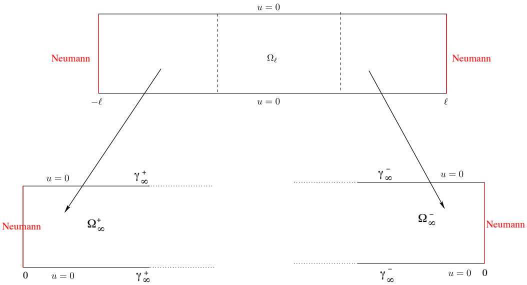

Theorem. $\lim_{\ell\to\infty}\lambda_\ell^1 = \min(\nu_\infty^+, \nu_\infty^-)$.

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Domains tending to infinity in several dimensions (Roy-Sh)

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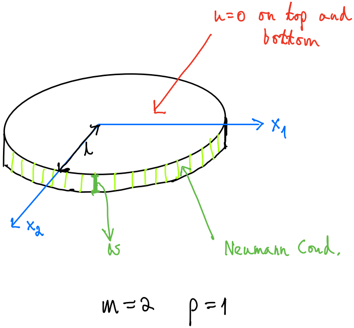

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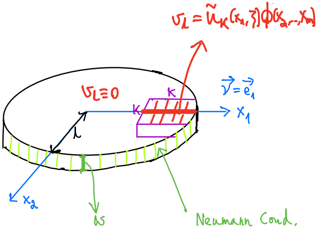
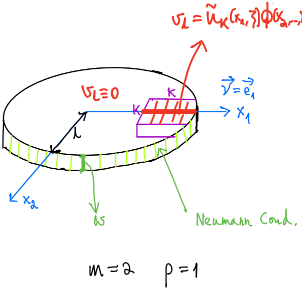


$m = 2 \quad p = 1$



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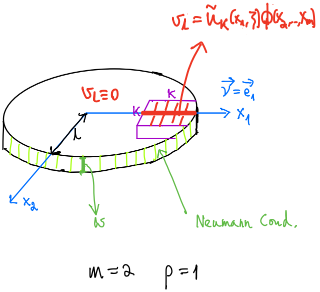

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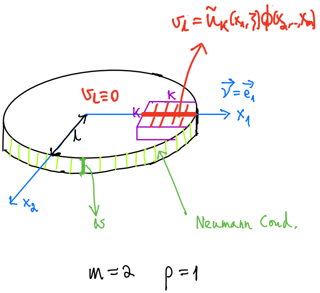

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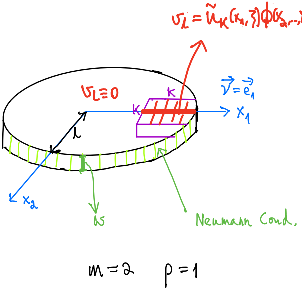

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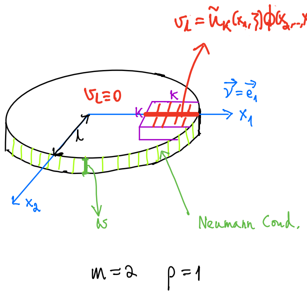
Idea of the proof



Upper bound: Take a test function v_ℓ supported in a "box" of size $K = \ell^\beta$ near the boundary point with normal ν.
Let $v_\ell = \tilde{u}_K(x_1, \xi)\Phi(x_2, \ldots, x_m)$ with $\tilde{u}_K$ an approximated minimizer for the problem on a semi-infinite cylinder for the problem with A_ν.
Lower bound: Decay of u_ℓ in the bulk always holds when $\limsup \lambda_\ell < \mu^1$. Arguing by contradiction leads to decay also near the boundary.But $\int_{\Omega_\ell} u_\ell^2 = 1$. Impossible!

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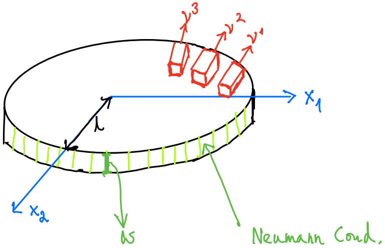
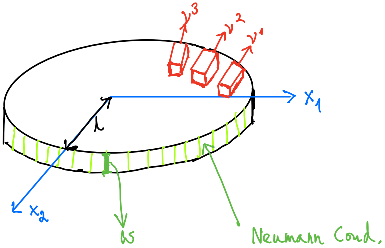
• This is possible since we have $k-1$ linear equations in k unknowns.


ν^3
ν^2
ν^1
x_1
λ
x_2
ω
Neumann Cond.



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Thank you for your attention!