DIFFUSIVE HAMILTON-JACOBI EQUATIONS AND THEIR SINGULARITIES

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$$\text{(DHJ)} \qquad \boxed{\begin{cases} u_t - \Delta u = |\nabla u|^p, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}}$$

“Mostly Maximum Principle”, Cortona, May 31, 2022

MOTIVATION

1) Stochastic control problem

Controlled stochastic dynamical system

$$\boxed{dX_s = \alpha_s ds + dW_s, \quad s > 0, \qquad \text{with } X_0 = x \in \Omega}$$

$(W_s)_{s>0}$	=	Brownian Motion with values in $\mathbb{R}^n$
$(X_s)_{s\geq 0}$	=	position of the particle (stochastic process)
$(\alpha_s)_{s>0}$	=	control (the controler can choose the *velocity* of X)
$u_0 \in C_0(\overline{\Omega})$	=	spatial distribution of rewards, i.e.:

At a given time horizon $s = t > 0$, the **final reward** is

$$\begin{cases} u_0(X_t), & \textit{provided } X \text{ stays in } \Omega \text{ until time } t \text{ (i.e. } \tau := \text{first exit time} > t) \\ 0, & \text{otherwise} \end{cases}$$

The **cost** of the control at each time s is $|\alpha_s|^{p/(p-1)}$ (as long as X_s stays in Ω)

Goal (of the controler): maximize the net gain $\boxed{G_t = \chi_{\tau>t} u_0(X_t) - \int_0^\tau |\alpha_s|^{p/(p-1)}\, ds}$

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Controlled stochastic dynamical system

$$\boxed{dX_s = \alpha_s ds + dW_s, \quad s > 0, \qquad \text{with } X_0 = x \in \Omega \text{ (smooth)}}$$

$$\begin{aligned}
(W_s)_{s>0} &= \text{Brownian Motion with values in } \mathbb{R}^n \\
(X_s)_{s\geq 0} &= \text{position of the particle (stochastic process)} \\
(\alpha_s)_{s>0} &= \text{control (the controler can choose the } \textit{velocity} \text{ of } X) \\
u_0 &= \text{spatial distribution of rewards, i.e.:}
\end{aligned}$$

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The **cost** of the control at each time s is $|\alpha_s|^{p/(p-1)}$ (as long as X_s stays in Ω)

Goal (of the controler): maximize the **net gain** $\boxed{G_t = \chi_{\tau>t} u_0(X_t) - \int_0^\tau |\alpha_s|^{p/(p-1)}\, ds}$

Theorem: [Barles-Burdeau CPDE 95, Barles-Da Lio JMPA 04] The value function (maximal gain) is given by the unique (continuous) *viscosity solution* of (DHJ), namely:

$$u(x,t) = \sup_{(\alpha_s)_s} \mathbb{E}\left(G_t \,|\, X_0 = x\right)$$

MOTIVATION

2) KPZ model of surface growth

[Kardar-Parisi-Zhang 86] ($p = 2$) and [Krug-Spohn 88] ($p > 1$)

$$u_t = \nu \Delta u + \lambda |\nabla u|^p + \eta(x,t)$$

- u = height of surface, growing by ballistic deposition of dusts (alumine)
- growth term $\lambda |\nabla u|^p$: deposition of new particles on the surface
- diffusion term $\nu \Delta u$: relaxation of the interface by surface tension
- $\eta(x,t)$: noise term

3) Model case in theory of NL parabolic equations

Among simplest parabolic PDE's with 1st order nonlinearity

Cp. classical problem with zero order nonlinearity (NLH or Fujita equation):

$$u_t - \Delta u = u^p, \quad p > 1$$

BASIC PROPERTIES

$p > 1$, $\Omega \subset \mathbb{R}^n$ smooth bounded, $u_0 \in X_+ := \{v \in C^1(\overline{\Omega});\ v \geq 0,\ v_{|\partial\Omega} = 0\}$

- Local existence-uniqueness, maximal classical solution $T = T(u_0) \in (0, \infty]$.
- Maximum principle estimate:

$$\boxed{0 \leq u(\cdot, t) \leq \|u_0\|_\infty, \quad 0 < t < T}$$

- Blow-up alternative: If $T < \infty$, then **Gradient Blow-up (GBU)**, i.e.:

$$\boxed{\lim_{t \to T} \|\nabla u(t)\|_\infty = \infty}$$

- $p \leq 2$: Global existence and C^1-boundedness for all u_0
- $p > 2$: GBU for large data / Global existence and decay for small data

[Ladyzenskaja 56, Filippov 61, Lieberman 86, Alikakos-Bates-Grant 89, Dlotko 91, Alaa 96, S. 02, Benachour-Dabuleanu 03, Hesaaraki-Moameni 04, S.-Zhang 06, ...]

OTHER TOPICS

• Cauchy problem ($\Omega = \mathbb{R}^n$): all solutions global for any $p > 0$.

Detailed studies of asymptotic behavior:

[Amour, Barles, Ben Artzi, Benachour, Biler, Guedda, Gilding, Karch, Kersner, Koch, Laurençot, Porretta, Quaas, Rodriguez, S., Tabet-Tchamba, Weissler, ...]

• More general diffusions: p-Laplace, fractional, fully nonlinear, ...

[Attouchi, Barles, Bidaut-Véron, Laurençot, Leonori, Magliocca, Quaas, Rodriguez, S., Stinner, Véron, ...]

• Rough initial data, maximal regularity

[Ben Artzi, Benachour, Cirant, Dabuleanu, Goffi, Laurençot, S., Weissler, ...]

• Boundary and initial trace problems

[Bidaut-Véron, Dao, Garcia-Huidobro, Véron, ...]

• Extinction problems ($0 < p < 1$)

[Benachour, Iagar, Laurençot, Schmitt, S., Stinner, ...]

• Null-controllability

[Porretta, Zuazua]

BEHAVIOR OF SOLUTIONS – QUESTIONS

($p > 2$, Ω bounded assumed throughout)

1) GBU singularities:

- Singular set
- Space profile
- Time rate

2) Post-GBU behavior:

- Weak continuation
- Loss of boundary conditions
- Recovery of boundary conditions and regularization
- Oscillations

GBU SET

$B(u_0) := \{x_0 \in \overline{\Omega};\ \nabla u \text{ is unbounded near } (x_0, T)\}.$

Theorem. [S.-Zhang JAM 06]

$$\boxed{B(u_0) \subset \partial\Omega}$$

$$\boxed{|\nabla u| \le C\delta^{-\beta}(x) \quad \text{in} \quad \Omega \times [0,T), \qquad \beta = \frac{1}{p-1}, \quad \delta(x) = \mathrm{dist}(x, \partial\Omega)}$$

Local Bernstein type gradient estimate (elliptic case: [PL Lions, JAM 85])

Remarks.

- Similar result for quasilinear case $u_t - \Delta_p u = |\nabla u|^q \ \ (q > p > 2)$ [Attouchi JDE 12]
- Gradient estim. of this type $\Rightarrow$ Liouville-type thms for ancient solutions in $(-\infty, 0) \times \mathbb{R}^n$.

GBU SET (II)

Question: location of GBU points within the boundary ?

1. Symmetric case: $\Omega = B_R$ and u_0 radial $\Longrightarrow$ $\boxed{B(u_0) = \partial\Omega}$

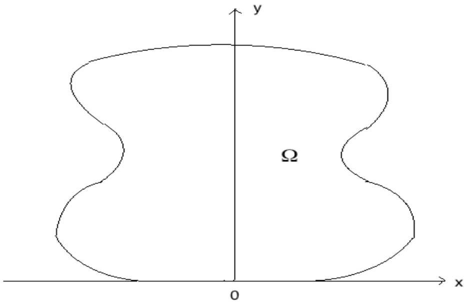
2. [Li-S. CMP 10] Localization in any small open set, assuming $supp(u_0)$ is concentrated

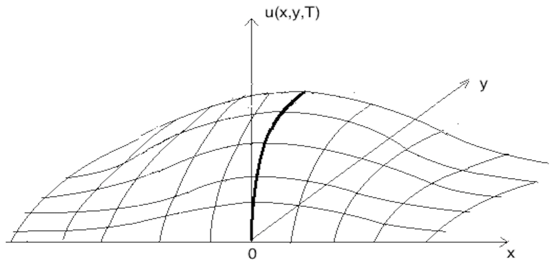
3. [Li-S. CMP 10] Single-point GBU

Theorem. *Assume* $\Omega \subset \mathbb{R}^2$, $0 \in \partial\Omega$, *and*

- *either* Ω *is a disk or* Ω *symm. convex in* x*-direction, locally flat near* 0
- u_0 *symm.* $\searrow$ *in* x, *suitably concentrated near* 0

$\Rightarrow T(u_0) < \infty$ *and* $B(u_0) = \{0\}$





GBU SET

• Remarks

- True for more general (nonflat) symmetric domains [Esteve JMPA 19]

- Nonlinear diffusion $u_t - \Delta_p u = |\nabla u|^q \quad (q > p > 2)$ [Attouchi-S. TAMS 17]

- Possible physical interpretation (KPZ model): the surface tension (diffusion) forces the steep region to become more and more concentrated near a single boundary point

• Ideas of proof

- Auxiliary function $\boxed{J(x, y, t) = u_x + \lambda x y^{-\gamma} u^q}$ $(x, y \text{ small}, q > 1, \ 0 < \gamma < q - 1)$

- Use MP to show $J \leq 0$ ($\rightarrow$ long computations using Bernstein type gradient estimate)

- Integration in $x \implies u(x, y, t) \ll x^{-2/(q-1)} y^{(p-2)/(p-1)}$

- GBU at $x \neq 0$ would contradict nondegeneracy result obtained by barrier arguments (analogue of [Giga-Kohn CPAM 89] for NLH)

• Open problems

- Finiteness of $B(u_0)$ for $n = 2$ and nonradial u_0 ? ([Chen-Matano JDE 89] for NLH)

- Finiteness of $(n - 2)$-Hausdorff measure of $B(u_0)$? ([Velázquez IUMJ 93] for NLH)

SPACE PROFILE (I)

[Filippucci-Pucci-S. CPDE 20]

- **Gradient estimate with sharp constant**

$$|\nabla u| \le (1+\varepsilon) d_p \delta^{-\beta} + C_\varepsilon \quad \text{in } \ \Omega \times [0,T), \qquad \beta = \frac{1}{p-1}, \quad d_p = \beta^{-\beta} \qquad (\forall \varepsilon > 0)$$

- **Sharp GBU profile in normal direction:** For any **GBU point** $a \in \partial\Omega$,

$$\lim_{s\to 0} s^\beta \nabla u(a + s\nu_a, T) = d_p \nu_a$$

(hence $|\nabla u(x,T)| \ \sim \ d_p \delta^{-\beta}$, as $x \to a$, $x - a \perp \partial\Omega$)

- **Main ingredient: elliptic Liouville-type theorem in half-space**

$$(1) \qquad \begin{cases} -\Delta v &= |\nabla v|^p, & x \in \mathbb{R}^n_+ = \{(x_1,\dots,x_n);\ x_n > 0\}, \\ v &= 0, & x \in \partial\mathbb{R}^n_+ \end{cases}$$

Theorem. [Filippucci-Pucci-S. CPDE 20]
Let $p > 2$ and let $v \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ be a solution of (1). Then v depends only on the variable x_n.

Recall whole space case Liouville thm: all solutions are constant [PL Lions, JAM 85]

SPACE PROFILE (II): TANGENTIAL PROFILE

Q (in single-point GBU): along $\partial\Omega$, how fast is u_ν damped away from the GBU point ?

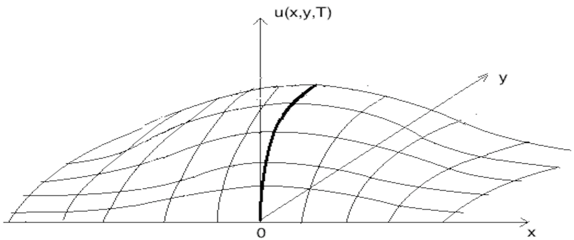
1. General result (csq of above Liouville Thm): For any GBU point a, the final profile is *more singular* in tangential direction (hence anisotropic):

$$\lim_{x\to a,\, x\in\partial\Omega} |x-a|^\beta u_\nu(x,T) = \infty$$

2. Sharp profiles ($n = 2$)

Theorem. *[Porretta-S. IMRN 17] Let $2 < p \le 3$ and consider situation of single point GBU theorem in locally flat case, with u_0 symm. decreasing in x. Then $|u_x| \le C$ and*

$$\boxed{u_y(x,y,T) \approx d_p\Big[y + C|x|^{2(p-1)/(p-2)}\Big]^{-1/(p-1)}} \qquad \textit{for } x, y \textit{ small.}$$



In particular (final profile of normal derivative on $\partial\Omega$):

$$\boxed{u_y(x,0,T) \approx |x|^{-2/(p-2)}}$$

Open problems: $p > 3$? Other profiles ?

TIME RATE OF GBU: Lower estimate

Consider slightly more general KPZ type equation (h smooth)

$$u_t - \Delta u = |\nabla u|^p + h(x).$$

- For *any* GBU solution [Porretta-S. JMPA 19]:

$$\boxed{\|\nabla u(t)\|_\infty \geq C(T-t)^{-1/(p-2)}, \qquad 0 < t < T}$$

Previous partial results [Conner-Grant DIE 96, Guo-Hu DCDS 08]

- **Consequence:** GBU rate is always **type II**, i.e. **never self-similar**

Natural scale invariance would lead to $\dfrac{1}{2(p-1)} \quad \left(< \dfrac{1}{p-2}\right)$

TIME RATE OF GBU: Upper estimate

For time-increasing solutions (sufficient condition: $\Delta u_0 + |\nabla u_0|^p + h \geq 0$), we have

$$C_1(T-t)^{-1/(p-2)} \leq \|\nabla u(t)\|_\infty \leq C_2(T-t)^{-1/(p-2)} \quad (1)$$

provided

- $n = 1$ [Guo-Hu DCDS 08, Porretta-S. JMPA 19]
- $\Omega = B_R$, u_0 radially symmetric [Li-Zhang AMSci 13]
- Ω convex, $2 < p < 3$ [Attouchi-S., CVPDE 20]

Open problem: $p \geq 3$ (also open for some of the elliptic results in [Lasry-Lions, 89])

Ingredients of proofs: MP with tricky auxiliary functions, sharp gradient estimates, zero-number arguments on u_t (1d)

TIME RATE OF GBU:

Faster rates and complete classification in 1d

$$\begin{cases} u_t - u_{xx} = |u_x|^p, & x \in \Omega = (0,R),\ t > 0 \qquad (0 < R \leq \infty) \\ u = 0, & x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Theorem. [Mizoguchi-S., preprint 21]

(a) For *any* $u_0 \in X_+$ with $0 \in \mathcal{B}$, there exist an integer $\ell \geq 1$ and $C > 0$ such that

$$\boxed{\lim_{t \to T} (T-t)^{\frac{\ell}{p-2}}\, u_x(0,t) = C} \tag{1}$$

Moreover, in small boundary layer, u has bubbling space-time behavior, described by

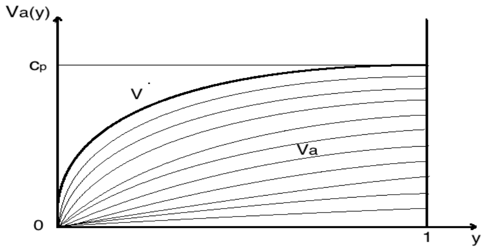
$$\boxed{u = V_{\lambda(t)}(x) + O(x^2), \quad \text{as } t \to T_-, \text{ with } \lambda(t) := c u_x^{1-p}(0,t) \to 0}$$

(b) For *any* integer $\ell \geq 1$, there exists $u_0 \in X_+$ and $C > 0$ such that (1) holds.

TIME RATE OF GBU:

Faster rates and complete classification in 1d (continued)

- **Bubble**



defined by the steady states

$V(x) = c_p x^{(p-2)/(p-1)}$ (singular)

$V_a(x) = V(x+a) - V(a), \quad a > 0$ (regular)

- **Geometric characterization of ℓ:**

ℓ = number of *vanishing intersections* of $u(\cdot, t)$ with U as $t \to T^-$.

- **Stability of GBU time and GBU rate**
  - T continuous w.r.t. initial data iff ℓ odd
  - rate (and profile) stable iff $\ell = 1$

IDEAS OF PROOFS (Part (b))

Based on construction of special solutions with precise space-time behavior, by a modification of Herrero-Velázquez' method for NLH "type-II" solutions (1994)

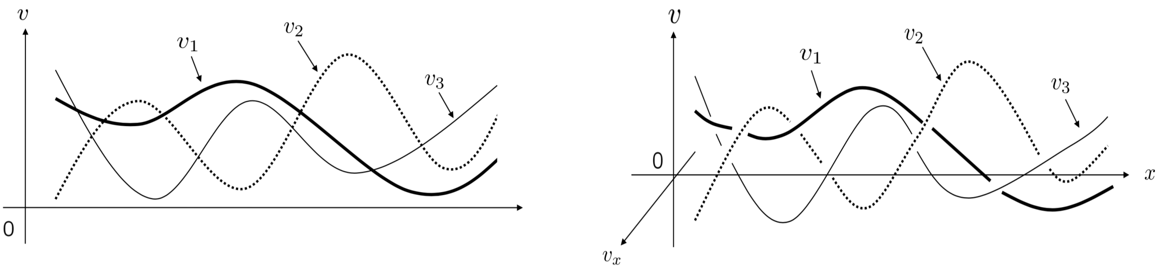
Ingredients:

- similarity variables $y = x/\sqrt{T-t}$, $s = -\log(T-t)$ (cf. Giga-Kohn 1985-89)
- inner/outer expansions: inner region (quasi-stationary behavior) and
  outer region (linearization around singular steady-state → rates given by eigenvalues !)
- heavy a priori estimates
- topological degree

IDEAS OF PROOFS (Part (a))

Based on **zero number** and **braid group techniques** to compare 3 solutions u, U, v, where

- U : singular steady state
- v : special sol. with known rate, s.t. $T^*(v) = T^*(u)$ and $v - U$ has same # of vanishing zeros as $u - U$



Parabolic reduction principle (Matano): denote $G(t) = \text{braid } (v_1, v_2, v_3)$

$$G(t) \text{ loses finitely many } X^2 \text{ or } Y^2 \text{ (up to topological equivalence)}$$

VISCOSITY SOLUTIONS

Viscosity solutions of (DHJ) [Barles-DaLio JMPA 2004]

- $\exists!$ global viscosity solution $\tilde{u} \in BC([0,\infty) \times \overline{\Omega})$, $\tilde{u} \geq 0$
- $u \in C^{1,2}((0,\infty) \times \Omega)$, classical solution inside
- **Boundary conditions** in generalized visc. sense: $\boxed{\min(u, u_t - \Delta u - |\nabla u|^p) \leq 0}$
- $\tilde{u} = u$ on $[0, T(u_0)) \Longrightarrow \tilde{u}$ is a **weak continuation** of u after T.

Equivalent formulation by approximation/truncation:

$$(P_k) \qquad \begin{cases} v_t - \Delta v = F_k(|\nabla v|) := |\nabla v|^2 \min(k^{p-2}, |\nabla v|^{p-2}), & x \in \Omega,\ t > 0, \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ v(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Solution v_k of (P_k) is global in classical sense

$v_k \uparrow \tilde{u}$ in $C^{2,1}_{loc}((0,\infty) \times \Omega)$ But NOT in $C(\overline{\Omega} \times [0,\infty))$!!

Possible loss of BC $\leftrightarrow$ boundary layer phenomenon

VISCOSITY SOLUTIONS: LARGE TIME BEHAVIOR

QUESTIONS:

1) Is there actual loss of boundary conditions after GBU ?

2) Does the solution become eventually classical again ?

3) If yes how does the solution look like in the intermediate time range ?

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Answer to Q2: [Porretta-Zuazua AIHP 12]

There exists $\tilde{T}(u_0) \in [T(u_0), \infty)$ such that $u(\cdot, t) \in C_0^1(\overline{\Omega})$ on $[\tilde{T}(u_0), \infty)$

$\tilde{T}(u_0)$: final regularization time. Moreover,

$$\lim_{t\to\infty} \|\tilde{u}(t)\|_{C^1} = 0$$

LOSS OF BOUNDARY CONDITONS

Answer to Q1 (does LBC occur for GBU solutions ?): YES and NO !

[Porretta-S. AIHP 17] (positive and negative results)
[Quaas-Rodriguez JDE 18] (positive results, also for fully nonlinear problems)

$$\mathcal{L}(u_0) = \big\{x_0 \in \partial\Omega,\ u(x_0,t) > 0 \text{ for some } t > 0\big\} \qquad (p > 2,\ \Omega \text{ bounded})$$

- $\boxed{\exists u_0 \text{ such that } \mathcal{L}(u_0) \neq \emptyset, \text{ and even } \mathcal{L}(u_0) = \partial\Omega}$
- $\boxed{\mathcal{L}(u_0) \text{ can be made arbitrarily close to any given open subset of } \partial\Omega \neq \emptyset}$

- $\boxed{\exists u_0 \text{ such that } T < \infty \text{ and } \mathcal{L}(u_0) = \emptyset}$ i.e., $u = 0$ on $\partial\Omega \times (0,\infty)$
- $\boxed{\text{GBU without loss of BC is exceptional:}}$

$v_0 \geq\not\equiv u_0 \Longrightarrow \mathcal{L}(v_0) \neq \emptyset$

$v_0 \leq\not\equiv u_0 \Longrightarrow T(v_0) = \infty$ (*Threshold* between global classical solutions and GBU)

[Porretta-S. AIHP 17] for $n = 1$, [Filippucci-Pucci-S. CPDE 20] for $n \geq 2$

INTERMEDIATE TIME RANGE BEHAVIOR

Q3: How does the solution look like between $T(u_0)$ and $\tilde{T}(u_0)$?

Precise description in 1d ($\Omega = (0,1)$) [Porretta-S. JMPA 19], [Mizoguchi-S. preprint 20]

$$\mathcal{S}_0 = \Big\{t > 0;\ u(0,t) = 0 \ \text{ and } \ \limsup_{x\to 0} |u_x(x,t)| = \infty\Big\} \qquad (\text{“transition” times})$$

Theorem 1. *[Mizoguchi-S. 20]*

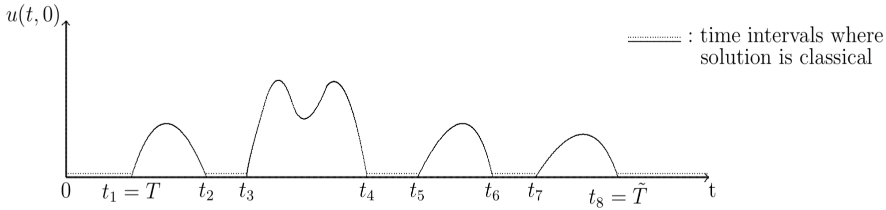
(i) The set $\mathcal{S}_0$ is finite.

(ii) On each interval between two consecutive times $t_1, t_2 \in \mathcal{S}_0$, the solution is either:

(ii1) classical up to $x = 0$, i.e.:

$$u \in C^{1,2}([0,1/2] \times (t_1,t_2)) \quad \text{and} \quad u = 0 \text{ on } \{0\} \times (t_1,t_2)$$

(ii2) or of LBC type at $x = 0$, i.e. $u > 0$ on $\{0\} \times (t_1,t_2)$.



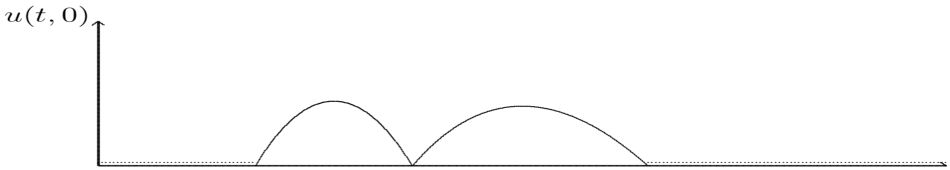
Rem: No “Waiting time” phenomenon is possible: either immediate LBC or immediate regularization after each time $t \in \mathcal{S}_0$

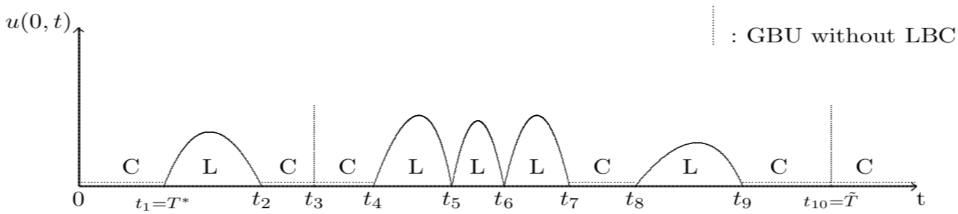
INTERMEDIATE TIME RANGE BEHAVIOR

Theorem 2. [Mizoguchi-S. 20]

(i) For any integer $m \geq 1$, there exist solutions with exactly m times of LBC and m times of regularization (oscillatory behavior)

(ii) More generally, for any finite sequence of interval types "C" or "LBC" in any given order, there exists a solution realizing this sequence. This in particular produces "bouncing" times and times of GBU without LBC.

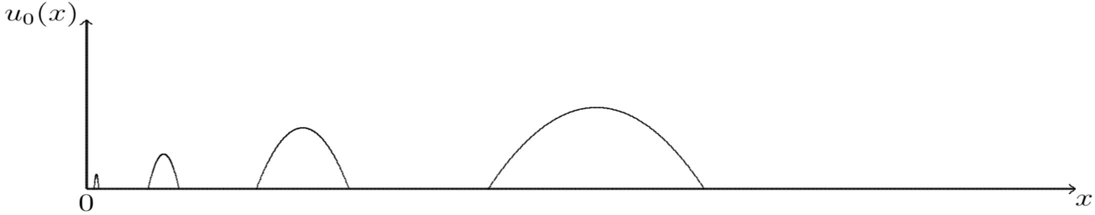




Main tool of proof: **zero number** arguments adapted to viscosity solutions.

INTERMEDIATE TIME RANGE BEHAVIOR

• **Shape of initial data** leading to solution with m LBC (works also in higher dimension):



- Recursive construction of a multiscale, compactly supported initial data
- m suitably rescaled bumps located farther and farther from the boundary
- Bump closest to boundary responsible for first GBU and LBC
- Influence of 2nd bump becomes significant only after some lapse of time, leaving enough time for regularization by diffusion (before producing 2nd GBU and LBC, etc.)
- General case (including bouncing and GBU sol withou LBC) requires delicate argument with arbitrary number of critical parameters

• **Application of multi-bump LBC solutions to stochastic control problem:**

For suitable **multibump spatial distributions of rewards** inside the domain, if a controled Brownian particle starts near the boundary, the net gain can be maximized on different time horizons but **not** on some intermediate times.

INTERMEDIATE TIME RANGE BEHAVIOR: FURTHER RESULTS

• **Rates of recovery of BC:** complete classification in $1d$. [Mizoguchi-S., preprint 21]

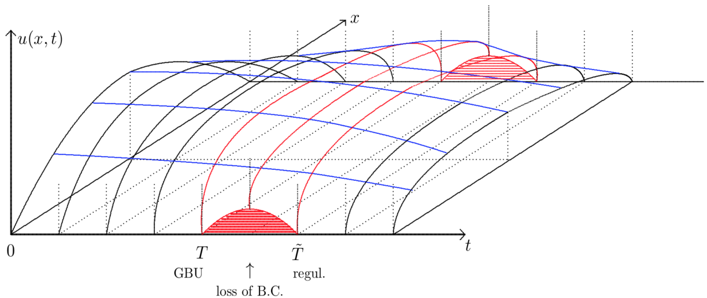
Analogue of above classification of GBU rates but, instead of multiples of $-1/(p-2)$, rates are the **integers**:

$$u(0,t) \sim C(\tilde{T}-t)^{\ell}, \quad \text{as } t \to \tilde{T}^{-}.$$

• For some special classes of solutions in $1d$: [Porretta-S. JMPA 19]

- **Rate of LBC** is $\approx t-T$, as $t \to T^{+}$
- **Rate of final regularization** of $\|u_x(t)\|_\infty \approx (t-\tilde{T})^{-1/(p-2)}$, as $t \to \tilde{T}^{+}$

(no complete classification available so far)



• In higher d: existence of solutions with multiple GBU/LBC and some other partial results, but many open questions...

THANK YOU !!