Aleksandrov-Bakelman-Pucci maximum principle for L^n-viscosity solutions of equations with unbounded terms

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(joint work with S. Koike)

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Uniformly elliptic PDE

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- $\Omega \subset \mathbb{R}^n$ is a bounded domain.
- $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n) \to \mathbb{R}$ is measurable, where $\mathbb{S}(n)$ is the set of all $n \times n$ symmetric matrices.
- $r \to F(x, r, p, X)$ is nondecreasing, $F(x, 0, 0, 0) = 0, f \in L^n(\Omega)$.
- F is uniformly elliptic, with fixed ellipticity constants $0 < \lambda \leq \Lambda$, and has gradient dependence coefficient function (drift coefficient) in $L^n(\Omega)$. More precisely:

Uniformly elliptic PDE

$$\begin{cases} \mathcal{P}^-_{\lambda,\Lambda}(X-Y) - \gamma(x)|p-q| - \omega(|r-s|) \\ \qquad \le F(x,r,p,X) - F(x,s,q,Y) \\ \qquad\qquad \le \mathcal{P}^+_{\lambda,\Lambda}(X-Y) + \gamma(x)|p-q| + \omega(|r-s|) \\ \text{for all } r,s \in \mathbb{R}, p,q \in \mathbb{R}^n, X,Y \in \mathbb{S}(n), \text{a.e. } x \in \Omega \\ \text{for some } \gamma \in L^n_+(\Omega) \text{ and a modulus } \omega. \end{cases}$$

$\mathcal{P}^-_{\lambda,\Lambda}, \mathcal{P}^+_{\lambda,\Lambda}$ are the Pucci extremal operators are defined by

$$\mathcal{P}^-_{\lambda,\Lambda}(X) := -\Lambda \mathrm{Tr}(X^+) + \lambda \mathrm{Tr}(X^-), \quad \mathcal{P}^+_{\lambda,\Lambda}(X) := \Lambda \mathrm{Tr}(X^-) - \lambda \mathrm{Tr}(X^+)$$

for $X \in \mathbb{S}(n)$.

ABP maximum principle

• Classical Aleksandrov-Bakelman-Pucci maximum principle: There exist $C = C(n, \lambda, \|\gamma\|_{L^n(\Omega)}) > 0$ such that if $u \in W^{2,n}_{\rm loc}(\Omega) \cap C(\overline{\Omega})$ is a strong subsolution (resp., supersolution) of

$$\mathcal{P}^-(D^2u) - \gamma(x)|Du| = f(x) \text{ in } \Omega$$

$$\Big(\text{resp., } \mathcal{P}^+(D^2u) + \gamma(x)|Du| = -f(x) \text{ in } \Omega\Big)$$

then

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + C \mathrm{diam}(\Omega) \|f\|_{L^n(\Gamma^+(u))}$$

$$\Big(\text{resp., } \max_{\overline{\Omega}} (-u) \le \max_{\partial\Omega} (-u) + C \mathrm{diam}(\Omega) \|f\|_{L^n(\Gamma^+(-u))}\Big).$$

$$\Gamma^+(u) = \Gamma^+(u, \Omega) := \{x \in \Omega \mid \exists p \in \mathbb{R}^n \text{ such that } u(y) \le u(x) + \langle p, y - x\rangle \text{ for } y \in \Omega\}.$$

is the upper contact set of u.

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• Generalized ABP maximum principle with norms over contact sets replaced by norms over Ω when $\gamma \in L^q(\Omega), q > n$, $f \in L^p(\Omega)$ for some $p > p_0 = p_0(n, \Lambda/\lambda)$: strong solutions (Fabes-Stroock, Escauriaza, Cabré,...); L^p-viscosity solutions (Fok, Crandall-Święch, Koike-Święch, ...). Recent generalizations by Krylov for strong solutions when $\gamma \in L^n(\Omega)$ and $p > \bar{p}$ for some constant $\bar{p} < n$ (also Dong-Krylov). Versions with quadratically growing gradient terms (Koike-Swiech), versions for degenerate/singular equations (Imbert, Dávila-Felmer-Quaas), versions in unbounded domains (Amendola, Birindelli, Cabré, Capuzzo-Dolcetta, Leoni, Rossi, Vitolo, ...), ...

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• **Goal:** Prove classical ABP maximum principle with norms over contact sets for L^n-viscosity sub/super-solutions of extremal equations when $\gamma, f \in L^n(\Omega)$.

L^p-viscosity solutions

Definition

A function $u \in C(\Omega)$ is an L^p-viscosity subsolution (resp., supersolution) of (1) if

$$ess \liminf_{y \to x} \left(F(y, u(y), D\varphi(y), D^2\varphi(y)) - f(y) \right) \leq 0$$

$$\left(\text{resp., } ess \limsup_{y \to x} \left(F(y, u(y), D\varphi(y), D^2\varphi(y)) - f(y) \right) \geq 0 \right)$$

whenever for $\varphi \in W^{2,p}_{\text{loc}}(\Omega)$, $u - \varphi$ attains a local maximum (resp., minimum) at $x \in \Omega$. Finally, $u \in C(\Omega)$ is called an L^p-viscosity solution of (1) if it is both an L^p-viscosity subsolution and an L^p-viscosity supersolution of (1).

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• Proof of CCKS ($\gamma \in L^\infty(\Omega) \cap C(\Omega), f \in L^\infty(\Omega)$): Approximation by sup/inf-convolutions to produce semi-convex/semi-concave viscosity sub/super-solutions when $\gamma, f \in L^\infty(\Omega) \cap C(\Omega)$. Reduction to $f \in C(\Omega)$ by solving appropriate extremal equation.

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- No theory of L^n-viscosity solutions when $\gamma \in L^n(\Omega)$.

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Problem: if $\gamma \in L^n(\Omega)$ and $\varphi \in W^{2,n}(\Omega)$ then $\gamma|D\varphi|$ does not have to be in $L^n(\Omega)$. Thus it may not be possible to produce strong solutions to even simple extremal equations

$$\mathcal{P}^{\pm}(D^2u) \pm \gamma(x)|Du| = f(x) \text{ in } \Omega.$$

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• Help: Recent Krylov's extension of the ABP maximum principle: There exists a constant $\bar{p} = \bar{p}(n, \lambda, \Lambda, \|\gamma\|_{L^n(\Omega)}) < n$ such that if $p > \bar{p}, f \in L^p_+(\Omega)$ and $v \in W^{2,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ is a strong subsolution of

$$\mathcal{P}^-(D^2v) - \gamma(x)|Dv| = f(x) \text{ in } \Omega$$

then

$$\max_\Omega v \le \max_{\partial\Omega} v + C(\text{diam}(\Omega))^{2-\frac{p}{n}}\|f\|_{L^p(\Omega)},$$

where $C = C(p, n, \lambda, \Lambda, \|\gamma\|_{L^n(\Omega)})$.

Consequence of Krylov's result

Theorem

Let Ω be a bounded and open domain satisfying the exterior sphere condition. Let $\gamma \in L^n(\Omega)$, $\bar{p} < p < n$ and $f \in L^p_+(\Omega)$. Then extremal equations

$$\begin{cases} \mathcal{P}^{\pm}(D^2 v) \pm \gamma(x)|Dv| = f(x) \text{ in } \Omega \\ v = h \text{ on } \partial\Omega \end{cases}$$

have unique strong solutions $v \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ such that for every $\Omega' \Subset \Omega$

$$\|v\|_{W^{2,p}(\Omega')} \leq C \left(\|f\|_{L^p(\Omega)} + \|h\|_{L^\infty(\partial\Omega)} \right),$$

where $C = C(p, n, \lambda, \Lambda, \|\gamma\|_{L^n(\Omega)}, \gamma, \mathrm{diam}(\Omega), \mathrm{dist}(\Omega', \partial\Omega))$. The dependence of C on γ is through a condition for a number R such that the L^n norms of γ over balls having this radius must be smaller than some prescribed number.

ABP for L^n-viscosity solutions

Theorem

Let $\gamma \in L^n_+(\Omega), f \in L^n_+(\Omega)$. There exist $C = C(n, \lambda, \|\gamma\|_{L^n(\Omega)}) > 0$ such that if $u \in C(\overline{\Omega})$ is an L^n-viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \gamma(x)|Du| = f(x) \text{ in } \Omega$$

then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C\text{diam}(\Omega)\|f\|_{L^n(\Gamma^+(u))}.$$

Preliminary results

• Solvability of modified extremal equations:

B - open ball, $\Omega \subset B$ and the radius of B is equal to $4\,\mathrm{diam}(\Omega)$. We extend γ and g setting $\gamma = g = 0$ on $\mathbb{R}^d \setminus \Omega$. For $0 < \delta < \mathrm{diam}(\Omega)$, we define $\gamma_\delta := \gamma * \eta_\delta$, where η_δ is the standard mollifier.

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Lemma

Let $g \in L^p_+(\Omega)$ *for* $\bar{p} < p < n$. *There exist* $\varepsilon_0 = \varepsilon_0(p, n, \lambda, \Lambda, \gamma, \mathrm{diam}(B)) > 0$ *such that if* $\varepsilon \le \varepsilon_0$ *then equation*

$$\begin{cases} \mathcal{P}^-(D^2 v) - \gamma_\delta(x)|Dv| - \varepsilon |Dv|^{\frac{n}{n-p}} = g(x) \ \text{ in } B \\ v = 0 \ \text{ on } \partial B \end{cases}$$

has a strong solution $v \in W^{2,p}(B)$ *such that*

$$\|v\|_{L^\infty(B)} \le C(\|g\|_{L^p(B)} + \varepsilon),$$

where $C = C(p, n, \lambda, \Lambda, \|\gamma\|_{L^n(\Omega)}, \mathrm{diam}(B))$.

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• $\bar{p} < p' < p$ and set $r = np'/(n-p)$. For $R > 0$, we denote $\mathcal{B}_R := \{v \in W^{1,r}(B) : \|v\|_{W^{1,p*}(B)} \leq R\} \subset W^{1,r}(B)$. For $w \in \mathcal{B}_R$ denote by Tw the unique solution $u \in W^{2,p}(B)$ to

$$\begin{cases} \mathcal{P}^-(D^2u) - \gamma_\delta(x)|Du| - \varepsilon|Dw|^{\frac{n}{n-p}} = g(x) \text{ in } B \\ u = 0 \text{ on } \partial B. \end{cases}$$

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• Use the Schauder fixed point theorem to conclude that $T : \mathcal{B}_R \to \mathcal{B}_R$ has a fixed point which is a strong solution of the equation.

Preliminary results

- L^p-viscosity solutions $\implies$ $L^{p'}$-viscosity solutions for $\bar{p} < p_1 < p \le n$:

Theorem

Let $\bar{p} < p_1 < p \le n$, $f \in L^p(\Omega), \gamma \in L^n_+(\Omega)$. If $u \in C(\Omega)$ is an L^p-viscosity subsolution (resp., supersolution) of $F(x, u, Du, D^2u) = f(x)$ in Ω then u is an L^{p_1}-viscosity subsolution (resp., supersolution) of $F(x, u, Du, D^2u) = f(x)$.

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- Take $\bar{p} < p < n$ and $\varepsilon > 0$. Denote $\gamma_m = \gamma * \eta_{\frac{1}{m}}, f_m = f * \eta_{\frac{1}{m}}$, $\tilde{\gamma}_m = \gamma - \gamma_m$, $\tilde{f}_m = f - f_m$. Using Theorem ($L^n \implies L^p$) and

$$\gamma(x)|Du| \leq \gamma_m(x)|Du| + \varepsilon|Du|^{\frac{n}{n-p}} + C_\varepsilon|\tilde{\gamma}_m(x)|^{\frac{n}{p}},$$

u is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \gamma_m(x)|Du| - \varepsilon|Du|^{\frac{n}{n-p}} = f(x) + C_\varepsilon|\tilde{\gamma}_m(x)|^{\frac{n}{p}} \text{ in } \Omega.$$

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- Use Lemma to find a strong solution $w_m \in W^{2,p}(B_R)$ of

$$\begin{cases} \mathcal{P}^-(D^2w_m) - \gamma_m(x)|Dw_m| - \varepsilon C_*|Dw_m|^{\frac{n}{n-p}} = \tilde{f}_m(x) + C_\varepsilon|\tilde{\gamma}_m(x)|^{\frac{n}{p}} \text{ in } B \\ w_m = 0 \text{ on } \partial B, \end{cases}$$

where $C_* = 2^{\frac{p}{n-p}}$. We have

$$\|w_m\|_{L^\infty(B)} \leq C(\|\tilde{f}_m\|_{L^p(\Omega)} + C_\varepsilon\|\tilde{\gamma}_m\|^{\frac{n}{p}}_{L^n(B)} + \varepsilon).$$

Idea of proof

- Then $u_m = u - w_m$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 u_m) - \gamma_m(x)|Du_m| - \varepsilon C_* |Du_m|^{\frac{n}{n-p}} = f_m(x)$$

in Ω and hence also a C-viscosity subsolution.

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- $A^+_{1/j}[w], A^-_{1/j}[w]$ mean sup- and inf-convolutions of w. It is standard that $A^+_{1/j}[u_m](x) := \sup\{u_m(y) - \frac{j}{2}|x-y|^2 \mid y \in \mathbb{R}^n\}$ is a viscosity subsolution of

$$\mathcal{P}^-(D^2u_m) - \gamma^j_m(x)|Du_m| - \varepsilon C_*|Dv_m|^{\frac{n}{n-p}} = f^j_m(x)$$

where γ^j_m, f^j_m converge uniformly to γ_m, f_m.

Idea of proof

• Using a technique of Crandall-Kocan-Soravia-Święch we show that for sufficiently large $l = l(j)$, the function

$$u_m^j := A^-_{1/l(j)}[A^+_{1/j+1/l(j)}[v_m]]$$

is a viscosity subsolution of the same equation in some increasing regions Ω_j. But u_m^j is $C^{1,1}$ so it is a strong subsolution.

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• Set

$$d = \frac{\max_{\overline{\Omega}} u - \max_{\partial\Omega} u}{\mathrm{diam}(\Omega)}, \quad M = d + 1$$

$$d_j = \frac{\max_{\overline{\Omega}_j} v_m^j - \max_{\partial\Omega_j} v_m^j}{\mathrm{diam}(\Omega_j)}$$

For large j, $d_j < M$.

Idea of proof

- If for $r > 0$,

$$\Gamma_r^+(u_m^j, \Omega_j) := \{x \in \Omega_j \mid \exists p \in B_r(0) \text{ such that } u_m^j(y) \le u_m^j(x) + \langle p, y - x \rangle \text{ for } y \in \Omega_j\},$$

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we have

$$B_{d_j}(0) = Du_m^j(\Gamma_{d_j}^+(u_m^j, \Omega_j))$$

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and hence on $\Gamma_r^+(u_m^j, \Omega_j)$,

$$\varepsilon C_* |Dv_m^j(x)|^{\frac{n}{n-p}} \leq \varepsilon C_* M^{\frac{n}{n-p}} = c_\varepsilon.$$

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Following the standard proof of ABP maximum principle we then get

$$\max_{\overline{\Omega}_j} u_m^j \le \max_{\partial\Omega_j} u_m^j + C\mathrm{diam}(\Omega)(\|f_m^j\|_{L^n(\Gamma^+(u_m^j,\Omega_j))} + c_\varepsilon|\Omega|^{\frac{1}{n}})$$

for $C = C(n,\lambda,\|\gamma_m^j\|_{L^n(\Omega_{1/j})}) \le C(n,\lambda,\|\gamma_m\|_{L^n(\Omega)})$.

Idea of proof

- Since $\limsup_{j\to\infty}\Gamma^+(u_m^j,\Omega_{1/j})\subset\Gamma^+(u_m,\Omega)$, we conclude

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- Send $m\to\infty$, use the uniform convergence of v_m to u on Ω, the fact that $\|\gamma_m-\gamma\|_{L^n(\Omega)}+\|g_m-f\|_{L^n(\Omega)}\to 0$ to obtain

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where $C = C(n, \lambda, \|\gamma\|_{L^n(\Omega)})$.

- Send $\varepsilon \to 0$ to conclude.

Consequences of ABP maximum principle

- Pointwise maximum principle for L^p-viscosity solutions:

Corollary

Let $\gamma \in L^n_+(\Omega), f \in L^n_+(\Omega)$. Let $u \in C(\Omega)$ be an L^n-viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \gamma(x)|Du| = f(x) \text{ in } \Omega$$

and let $u - \varphi$ have a strict local maximum at $x \in \Omega$. Then for every small enough $r > 0$ and $\kappa > 0$, $|\Gamma^+_\kappa(u - \varphi, B_r(x)))| > 0$.

Consequences of ABP maximum principle

- Krylov's version of ABP for L^p-viscosity solutions:

Theorem

Let $\gamma \in L^n_+(\Omega), f \in L^p_+(\Omega)$, *for* $\bar{p} < p \leq n$. *There exist* $C = C(n, p, \lambda, \Lambda, \|\gamma\|_{L^n(\Omega)})$ *such that if* $u \in C(\overline{\Omega})$ *is an* L^p*-viscosity subsolution of*

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then

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C(\mathrm{diam}(\Omega))^{2-\frac{p}{n}} \|f\|_{L^p(\Omega)}.$$

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- Good theory of L^p-viscosity solutions for $\bar{p} < p \le n$: solvability, consistency results, relation to strong solutions, ...

Pointwise properties L^p-viscosity solutions

• Equation is satisfied a.e.:

$\bar{p} < p \leq n$, $\gamma \in L^n_+(\Omega), f \in L^p(\Omega)$. If u is an L^p-viscosity subsolution (resp., supersolution) of $F = 0$, then u is twice pointwise super-differentiable (resp., sub-differentiable) a.e. in Ω and

$$F(x, u(x), Du(x), D^2u(x)) \leq f(x) \quad \text{(resp., } \geq f(x)) \quad \text{a.e. in } \Omega$$

Pointwise properties of L^p-viscosity solutions

- Equivalent definition of L^p-viscosity solutions:

$\bar{p} < p$, $\gamma \in L^n_+(\Omega), f \in L^p(\Omega)$ and $u \in C(\Omega)$ is twice pointwise differentiable a.e. in Ω. Then u is an L^p-viscosity subsolution (resp., supersolution) of $F(x, u, Du, D^2u) = f(x)$ in Ω if and only if $F(x, u(x), Du(x), D^2u(x)) \le f(x)$ (resp., $F(x, u(x), Du(x), D^2u(x)) \ge f(x)$) a.e. in Ω and whenever $\varphi \in W^{2,p}_{\text{loc}}(\Omega)$ and $u - \varphi$ has a local maximum (resp., minimum) at $\hat{x} \in \Omega$, then

$$\operatorname{ess}\limsup_{x\to\hat{x}}(\mathcal{P}^-(D^2(u-\varphi)(x)) - \gamma(x)|D(u-\varphi)(x)|) \ge 0$$

(resp.,

$$\operatorname{ess}\liminf_{x\to\hat{x}}(\mathcal{P}^+(D^2(u-\varphi)(x)) + \gamma(x)|D(u-\varphi)(x)|) \le 0).$$

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F, F_1, F_2 satisfy basic assumptions with the same $\gamma \in L^n_+(\Omega)$, $\bar{p} < p$ and $f_1, f_2 \in L^p(\Omega)$.

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(i) If u is an L^p-viscosity subsolution of $F_1(x, u, Du, D^2u) = f_1(x)$ in Ω, an L^p-viscosity supersolution of $F_2(x, u, Du, D^2u) = f_2(x)$ in Ω and $f(x) := F(x, u(x), Du(x), D^2u(x)) \in L^p(\Omega)$, then u is L^p-viscosity solution of $F(x, u, Du, D^2u) = f(x)$ in Ω.

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Pointwise maximum principle

Notice that if u is twice pointwise differentiable a.e. in Ω then the functions

$$x \to Du(x), \quad x \to D^2u(x)$$

are measurable so the function $x \to F(x, u(x)Du(x), D^2u(x))$ is always measurable.

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Also $f_1 \leq f \leq f_2$ so $f \in L^p(\Omega)$.

Consequences

• Being an L^p-viscosity subsolution/supersolution is more a property of a function than of a particular equation. Once u is an L^p-viscosity subsolution/supersolution of one equation it is an L^p-viscosity subsolution/supersolution of every equation with a similar structure. Thus L^p-viscosity solutions behave like strong solutions and the techniques based on pointwise properties allow to operate on them easily and move them from one equation to another.

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- I expect one should be able to prove standard regularity results for L^p-viscosity solutions of equations with $\gamma \in L^n_+(\Omega)$.

THANK YOU!