On Periodic Homogenization of Nonlocal Hamilton-Jacobi Equations

Erwin Topp Paredes
Departamento de Matemática y Ciencias de la Computación
Universidad de Santiago de Chile

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Sections

The basic problem can be posed as follows: for $\epsilon \in (0,1)$, consider a perturbed problem with the form

$$F(x, \frac{x}{\epsilon}, u, Du, D^2u) = 0,$$

where F is a second-order, degenerate elliptic operator F and periodic in its second variable.

If u^ϵ is a solution for this problem, is there some compactness property for the family $\{u^\epsilon\}_\epsilon$?

If so, can we characterize the limit? Does it solve a particular PDE? Is there a rate of convergence?

Viscosity approach

First-order problems (Lions-Papanicoloau-Varadhan ['86]): Consider the Cauchy problem

$$\begin{cases} \partial_t u^\epsilon + H(x, \frac{x}{\epsilon}, Du^\epsilon) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\cdot, 0) = u_0 \in \mathrm{BUC}(\mathbb{R}^n) \end{cases}$$

for coercive Hamiltonian H. Then, $u^\epsilon \to \bar{u}$ as $\epsilon \to 0$, uniformly in $\mathbb{R}^n \times [0, T]$, for all $T > 0$.

Moreover, $\bar{u}$ is the unique viscosity solution to the problem

$$\begin{cases} \partial_t u + \bar{H}(x, Du) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u(\cdot, 0) = u_0, \end{cases}$$

where, for each x, p, $\bar{H}(x, p)$ is defined as the unique (ergodic) constant $c \in \mathbb{R}$ for which the problem

$$H(x, y, p + D_y \psi) = c \quad \text{for } y \in \mathbb{T}^N,$$

has a viscosity solution. Connection with optimal control.

References (far to be exhaustive):

- Results in divergence form: Bensoussan, Lions, Papanicoloau ['78], Jikov, Kolzov, Olienik ['94]...
- First/second-order problems: Evans [Proc.Edin'91-'92] (perturbed test function method); Alvarez-Bardi [MemAMS'10] (singular perturbation); Camilli-Ley-Loreti [ESAIM'10] (systems)...
- Rate of convergence: Capuzzo-Dolcetta - Ishii [Indiana'01] (first-order); Camilli-Marchi [Nonlty'09] (second-order, convex); Caffarelli-Souganidis [InvMath'10] (fully, nonconvex); Mitake-Tran [ARMA'19], Achdou-Patrizi [M.M.M.Ap.Sci'11] (evolution); Kim-Lee [ARMA'16] (higher order expansions)...
- Lions-Souganidis [AIHP'05] (almost periodic); Barles-Da Lio-Lions-Souganidis [Indiana'08] (Neumann); Caffarelli-Souganidis-Wang [CPAM'05], Armstrong-Cardaliaguet [JEMS'18] (stochastic homogenization)...

- Given $s \in (0,1)$, the fractional Laplacian of order $2s$ is defined as

$$\Delta^s u(x) = \text{P.V.} \int_{\mathbb{R}^N} [u(x+z) - u(x)]|z|^{-(N+2s)} dz,$$

see Di Nezza, Palatucci, Valdinoci [BullSciMath'12], and its non-authonomous version

$$Lu(x) = \text{P.V.} \int_{\mathbb{R}^N} [u(x+z) - u(x)] K(x,z) dz,$$

where $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ is "comparable" to the kernel of Δ^s.

(*) Δ^s is the infinitesimal generator of a jump Lévy process, see Applebaum (2011).

Nonlocal homogenization

- Arisawa [CPDE'09 - Proc.Edin'12]: Periodic homogenization of (linear) nonlocal problems with the form

$$u^\epsilon - a(x/\epsilon)\Delta^s u^\epsilon = \ell(x/\epsilon).$$

  Explicit limit problem: denote $\bar{a} = \int_{\mathbb{T}^N} a(y)^{-1}$, we have

$$\bar{u} - \bar{a}^{-1}\Delta^s \bar{u} = \bar{a}^{-1}\int a^{-1}(y)\ell(y).$$

- Schwab [SIAM'10]: Periodic homogenization of (fully nonlinear) nonlocal problems with the form

$$\inf_\alpha \sup_\beta \left\{ \int_{\mathbb{R}^N} [u(x+z) - u(x)] K_{\alpha,\beta}(x/\epsilon, z)dz + \ell_{\alpha\beta}(x/\epsilon) \right\} = 0$$

- Weak formulation for nonlocal homogenization by Fernández Bonder-Ritorto-Salort [SIAM'17], Piatnitski-Zhizhina [SIAM'17], Kassmann-Piatnitski-Zhizhina [SIAM'19].

Gradient dominance.

Theorem (Bardi, Cesaroni & T (Proc.Edin'20))

Let $s \in (0,1)$, H supercritical and coercive (i.e. $H(p) \sim |p|^m$ with $m > 2s$). Then, we have homogenization for the Cauchy problem

$$\begin{cases} \partial_t u^\epsilon - a(x, \frac{x}{\epsilon})\Delta^s u^\epsilon(x) + H(x, \frac{x}{\epsilon}, Du^\epsilon) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u^\epsilon(\cdot, 0) = u_0 \in \mathrm{BUC}(\mathbb{R}^n). \end{cases}$$

(*) The nonlocal operator can be "non-symmetric".

We observed three regimes:

- $s < 1/2$ (first-order). Gradient dominates. Regularity enough to evaluate Δ^s classically (Barles-Koike-Ley-T. [CVPDE'15]).
- $s > 1/2$ (elliptic). Elliptic cell problem + Fredholm alternative imply representation formula for the effective problem.
- $s = 1/2$ (critical). Solved using "more machinery": smooth correctors (Barles-Ley-T.[Nonlty'17] + Silvestre [Adv.Math'11]), and comparison among Hölder solutions.

Critical case.

Theorem (Ciomaga, Ghilli & T. (CPDE'22))

Consider a nonlocal operator

$$L_{y}u(x) = \text{P.V.} \int_{\mathbb{R}^N} [u(x+z) - u(x)] K(y, z) dz,$$

with a (nonsymmetric) kernel K "of order 1" (i.e. $L \sim -\sqrt{-\Delta}$). Then, we have homogenization for the Cauchy problem

$$\begin{cases} \partial_t u^\epsilon - L_{\frac{x}{\epsilon}} u^\epsilon + H(x, \frac{x}{\epsilon}, Du^\epsilon) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u^\epsilon(\cdot, 0) = u_0 \in \text{BUC}(\mathbb{R}^n), \end{cases}$$

where

$$H(x, y, p) = \sup_{a \in A}\{-f^a(x, y) \cdot p - \ell^a(x, y)\}.$$

(*) No coercivity assumed.

• Non-explicit effective Hamiltonian, but degenerate elliptic: if $\varphi_1(x) = \varphi_2(x)$ and $\varphi_1 \geq \varphi_2$ in $\mathbb{R}^N$, then

$$\bar{H}(x, p, \varphi_1) \leq \bar{H}(x, p, \varphi_2).$$

Difficulty: This ellipticity is not enough to get comparison at the effective level, due to the x-dependence of $\bar{H}$...

• Again, homogenization through regularity: C^α estimates of Chang-Lara and Dávila [JDE'16] allows to show that $w = u - v$ (difference of two solutions of the effective problem) solve the maximal inequality

$$\partial_t w - \mathcal{M}_K^+ w - C|Dw| \leq 0 \quad \text{in } \mathbb{R}^n \times (0, +\infty),$$

and we use MAXIMUM PRINCIPLE to conclude.

Convergence rates.

Theorem (Rodríguez-Paredes & T. (preprint 2020))

We consider stationary nonlocal H-J problems with the form

$$u^\epsilon(x) - H(x, \frac{x}{\epsilon}, Du^\epsilon(x), u^\epsilon) = 0 \quad \textit{in } \mathbb{R}^n,$$

with a nonlocal Hamiltonian given by

$$H(x,y,p,u) = \sup_{a\in A}\{-L_y^a u(x) - f^a(x,y)\cdot p - \ell^a(x,y)\},$$

and L of order $2s > 1$ with symmetric kernels (the diffusion dominates!).
Then, there exists $C > 0$ and $\alpha \in (0,1)$ such that

$$\|u^\epsilon - \bar{u}\|_\infty \leq C\epsilon^\alpha,$$

where $\bar{u}$ is the unique solution to the associated effective problem.

Representation formula for $\bar{H}$.

Following Ishii, Mitake and Tran [JMPA'17], we define the set

$$\mathcal{G}_0 = \{\phi \in C(\mathbb{T}^n \times A) : \exists u \ \sup_{a\in A}\{-L_y^a u(y) - \phi(a,y)\} \le 0, \ y \in \mathbb{T}^N\},$$

and the dual cone

$$\mathcal{G}_0' = \{\mu \in \mathcal{P}(\mathbb{T}^n \times A) \ : \ \int \phi \, d\mu \ge 0 \text{ for all } \phi \in \mathcal{G}_0\}.$$

This set $\mathcal{G}_0'$ is nonempty convex and compact (w.r.t. the $*$-weak convergence).

(*) The definition is inspired by the generalization of Mather measures to second-order PDEs (Gomes [Prog.Nonlin.Diff.Eq.'05]).

Representation formula for $\bar{H}$.

For $x, p \in \mathbb{R}^n, \phi : \mathbb{R}^n \to \mathbb{R}$ smooth, we have

$$\overline{H}(x,p,\phi) = \sup_{\mu \in \mathcal{G}_0'} \left\{ -\overline{L}_\mu \phi(x) - \bar{f}_\mu(x) \cdot p - \bar{\ell}_\mu(x) \right\},$$

where, for each $\mu \in \mathcal{G}_0'$ we have denoted

$$\overline{L}_\mu \phi(x) = \int_{\mathbb{R}^N} [\phi(x+z) - \phi(x)] \overline{K}_\mu(z)\, dz,$$

$$\text{with} \quad \bar{K}_\mu(z) = \int_{\mathbb{T}^N \times A} K_a(y,z) d\mu(y,a);$$

$$\bar{f}_\mu(x) = \int_{\mathbb{T}^N \times A} f(x,y,a) d\mu(y,a);$$

$$\bar{\ell}_\mu(x) = \int_{\mathbb{T}^N \times A} \ell(x,y,a) d\mu(y,a).$$

This representation formula allows us to obtain regularity estimates for the effective problem through a sequence of papers:

- Chang-Lara-Dávila [JDE'16]: C^α estimates.
- Barles-Chasseigne-Ciomaga-Imbert [JDE'12]: Lipschitz estimates.
- Caffarelli-Silvestre [CPAM'09]: $C^{1,\alpha}$ estimates.
- Caffarelli-Silvestre [AnnMath'11] and/or Serra [CVPDE'15]: $C^{2s,\alpha}$ estimates.
- Once $C^{2s,\alpha}$ for the effective problem are at hand, we adapt to the nonlocal framework the arguments of Camilli-Marchi [Nonlty'09] to prove the rate of convergence.
- The idea is to estimate the difference $u^\epsilon - u$ using comparison principle. However, u^ϵ and u solve different equations and this is where the corrector term plays a role, together with the regularity estimates for $\bar{u}$.

Homogenization: Optimal control perspective

Following Bardi-Terrone [CIM-Math.Sci.'15]: Consider the perturbed dynamical system

$$\begin{cases} y' = f(y, \frac{y}{\epsilon}, \alpha), \ t > 0; \\ y(0) = x, \end{cases}$$

and the value function of the optimal control problem

$$v^\epsilon(x) = \inf_{\alpha \in \mathcal{A}} \int_0^{+\infty} e^{-s} \ell(y, \frac{y}{\epsilon}, \alpha) ds,$$

which is the unique viscosity solution of the associated H-J equation.

Define the occupational measures

$$\mu_t = \mu_t(x, y, \alpha) = \frac{1}{t} \int_0^t \delta_{(y^\alpha(s), \alpha(s))} ds,$$

where y^α solves

$$\begin{cases} y' = f(x, y, \alpha), \ t > 0; \\ y(0) = y_0. \end{cases}$$

Denote the set of occupational measures

$$Z(x) = \{\mu : *\text{-weak limit of } \mu_t \text{ for some } y_0 \text{ and some } \alpha\}.$$

Under controlability assumptions on f, $\{v^\epsilon\}_\epsilon$ converges as $\epsilon \to 0$ to the function

$$\bar{v}(x) = \inf\left\{ \int_0^\infty e^{-s}\bar{\ell}(x(s), \mu(s))ds \ : \ x \text{ solves } (*)\right\},$$

where $x = x(s)$ solves the differential inclusion

$$\begin{cases} x' \in \bar{f}(x, Z(x)), \ s > 0, \\ x(0) = x, \end{cases} \qquad (*)$$

with

$$\bar{f}(x,\mu) = \int_{\mathbb{T}^N \times A} f(x,y,a)d\mu(y,a);$$

$$\bar{\ell}(x,\mu) = \int_{\mathbb{T}^N \times A} \ell(x,y,a)d\mu(y,a).$$

Then, we have the characterization of the effective Hamiltonian:

$$\bar{H}(x,p) = \sup_{\mu \in Z(x)} \{-\bar{f}(x,\mu)\cdot p - \bar{\ell}(x,\mu)\}.$$

(similar characterization for second-order problems in Ph.D. thesis of Khoukhou [U. Padova'21])

• Coming back to the nonlocal problem: Our representation formula holds when L depends also on the slow variable:

$$L_{x,y}u(x) = \int_{\mathbb{R}^N} [u(x+z) - u(x)]K(x,y,z)dz,$$

since in that case, $\mathcal{G}_0'$ depends on x! Being the dual cone to

$$\mathcal{G}_0(x) = \{\phi \in C(\mathbb{T}^N \times A) : \exists u \sup_{a\in A}\{-L^a_{x,y}u(y) - \phi(a,y)\} \leq 0,\ y \in \mathbb{T}^N\}.$$

Some (Hölder) continuity in the sense of Haussdorf distance in the set of probability measures in $\mathbb{T}^N \times A$ shall be explored.

Thank you for your attention!