Nonlinear partial differential equations: theory, numerics and applications
in memory of Maurizio Falcone

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Abstracts

Alessandro Alla
Università Ca' Foscari Venezia

Online identification and control of PDEs via Reinforcement Learning methods

In this talk, we focus on the control of unknown Partial Differential Equations. Our approach is based on the idea to control and identify the model on the fly. The control, in this work, is based on the State Dependent Riccati approach whereas the identification of the model on bayesian linear regression. At each iteration we obtain an estimation of the a-priori unknown coefficients of the PDEs based on the observed data and then we compute the control of the correspondent identified model. We show by numerical evidence the convergence of the method for infinite horizon problems. Joint work with Agnese Pacifico, Michele Palladino and Andrea Pesare.

Martino Bardi
Università di Padova

PDE and control methods for global optimization in deep neural networks

A central problem in deep learning of neural networks is the mini- mization of a highly nonlinear, non-convex, possibly non-smooth function of a large number of variables. A recent method called entropy gradient descent was proposed and studied in [1]. The related paper [2] developed it into an algorithm called Deep Relaxation that is justified by a homoge- nization limit in a system of stochastic differential equations. We reformulate such limit and imbed it into a much larger class of singular perturbation problems for stochastic control systems where some state variables evolve at a faster time scale than the others. One expects that the limit is an averaging of the slow subsystem with respect to a suitable measure. We study these problems by means of the associated 2nd order Hamilton-Jacobi-Bellman equations, using viscosity methods and some ideas from ergodic control. We prove some convergence results that give a rigorous mathematical framework for the algorithm in [2] as well as for various extensions. The results are joint work with Hicham Kouhkouh (Aachen University) [3, 4], and are also part of his PhD thesis.

[1] Chaudhari, P., Choromanska, A., Soatto, S., LeCun, Y., Baldassi, C., Borgs, C., ... & Zecchina, R.: Entropy-SGD: Biasing gradient descent into wide valleys. J. Stat. Mech. 2019.
[2] Chaudhari, P., Oberman, A., Osher, S., Soatto, S., & Carlier, G. Deep relaxation: partial differential equations for optimizing deep neural networks. Res. Math. Sci. 2018.
[3] M. Bardi, H. Kouhkouh, Singular perturbations in stochastic opti- mal control with unbounded data, arXiv:2208.00655 to appear in ESAIM Control Optim. Calc. Var.
[4] M. Bardi, H. Kouhkouh, Deep Relaxation of Controlled Stochastic Gradient Descent via Singular Perturbations, arXiv:2209.05564

Simone Cacace
Sapienza Università di Roma

Numerical solution of optimal control problems on stratified domains

In optimal control problems defined on stratified domains, the dynamics and the running cost may have discontinuities on a finite union of sub-manifolds of the state space. The corresponding value function can be characterized as the unique viscosity solution of a discontinuous Hamilton-Jacobi (HJ) equation satisfying additional viscosity conditions on the sub-manifolds. In this talk, I will present HJSD, an easy-to-use software for the numerical solution of such optimal control problems in the case of flat stratifications on two and three dimensional structured grids. The core algorithm is based on a semi-Lagrangian approximation scheme for the stratified HJ equation, whose convergence is proved relying on a classical stability argument in viscosity solution theory. I will also present some details on ongoing developments to extend HJSD to more general problems, including non-flat stratifications on unstructured grids, more general Hamiltonians and Mean Field Games. Finally, I will present some simulations related to stationary and evolutive problems, showing the particular phenomena that can arise with respect to the classical continuous framework.

Piermarco Cannarsa
Roma Tor Vergata

Aubry-Mather theory for sub-Riemannian control systems

The long-time average behavior of the value function in the classical calculus of variation is known to be connected with the existence of solutions of the so-called critical equation, that is, a stationary Hamilton-Jacobi equation which includes a sort of nonlinear eigenvalue called the critical constant (or effective Hamiltonian). In this talk, we will present results from joint works with C. Mendico addressing similar issues for the dynamic programming equation of an optimal control problem of sub-Riemannian type, for which coercivity of the Hamiltonian is no longer guaranteed. For such control systems, we introduce the Aubry set and we show that any fixed point of the Lax-Oleinik semigroup is horizontally differentiable on such a set. Furthermore, we obtain a variational representation of the critical constant using an adapted notion of closed measures. By further restricting such a class of measures, we give a definition of the Mather set in the sub-Riemannian case and we prove that such a set is included in the Aubry set.

Emiliano Cristiani
IAC-CNR

Detecting congestion and forecasting boundary conditions: How Machine Learning techniques can improve differential traffic models

Vehicular traffic has traditionally been described and forecast by means of mathematical models, either differential and nondifferential. In recent years, machine learning came into play in order to analyze traffic data, trying to discover patterns and substitute, when possible, traditional models. Hybrid approaches like the one based on physics-informed neural networks were also proposed. In this talk we focus on a road segment where traffic flux is continuously counted by means of some fixed sensors. The idea is to couple machine learning and differential models, putting the former at the service of the latter. More precisely, we use a neural network to predict sensor data at the inflow boundary of the road, where the model expects information about the next-to-come traffic flow. Moreover, we use a neural network to detect congestion formations near the sensors. This information is used to recover the car density by inverting the (noninjective) fundamental diagram. Some examples motivated by real scenarios will be discussed. Real data are provided by the Italian company Autovie Venete S.p.A.

Andrea Davini
Sapienza Università di Roma

On the vanishing discount approximation for compactly supported perturbations of periodic Hamiltonians

We consider the discounted approximation of the critical Hamilton-Jacobi equation set on the real line associated with a Hamiltonian $G(x, p) := H(x, p) - V(x)$, where H is a 1-periodic Tonelli Hamiltonian and V is a continuous and compactly supported potential. The critical constant associated with G is characterized as the unique constant for which the associated HJ equation $G(x, u') = a$ can have globally bounded solutions. We prove that the solutions of the discounted equation converge to a specific critical solution, which is identified in terms of projected Mather measures for G and of the asymptotic solution of the unperturbed periodic problem. This is joint work with I. Capuzzo-Dolcetta.

Jean-Denis Durou
University of Toulouse, France.

Photographic 3D-reconstruction: A Tour

In this talk, I will describe several techniques of photogrammetry, which consists in reconstructing a 3D scene from photographs. In particular, I will talk about a work that was in progress between the Toulouse and Rome research groups, when Maurizio suddenly left us.

Adriano Festa
Politecnico Torino

A system of of Hamilton-Jacobi equations characterizing geodesic centroidal tessellations

We introduce a class of systems of Hamilton-Jacobi equations characterizing geodesic centroidal tessellations, i.e. tessellations of domains with respect to geodesic distances where generators and centroids coincide. Typical examples are given by geodesic centroidal Voronoi tessellations and geodesic centroidal power diagrams. An appropriate version of the Fast Marching method on unstructured grids allows computing the solution of the Hamilton-Jacobi system and therefore the associated tessellations. We propose various numerical examples to illustrate the features of the technique.

Diogo Gomes
Kaust

Machine Learning architectures for price formation models with common noise

We propose a machine learning method to solve a mean-field game price formation model with common noise. This involves determining the price of a commodity traded among rational agents subject to a market clearing condition imposed by random supply, which presents additional challenges compared to the deterministic counterpart. Our approach uses a dual recurrent neural network architecture encoding noise dependence and a particle approximation of the mean-field model with a single loss function optimized by adversarial training. We provide a posteriori estimates for convergence and illustrate our method through numerical experiments.

Lars Gruene
University of Bayreuth
Decaying sensitivity and separable optimal value functions

In this talk we look at the possibility to approximate optimal value functions in high space dimensions by sums of lower dimensional functions. This study is motivated by the possibility to approximate optimal value functions with neural networks avoiding the curse of dimensionality, which we will explain at the beginning of the talk. We show that a particular ”decaying sensitivity” property allows to construct such a decomposition and explain this construction. Several examples illustrate optimal control problems in which this is possible.
The talk is based on joint work with Dante Kalise, Luca Saluzzi, and Mario Sperl.

Dante Kalise
Imperial College, London
Learning high-dimensional feedback laws for collective dynamics control

We discuss the control of collective dynamics for an ensemble of high-dimensional particles. The collective behaviour of the system is modelled using a kinetic approach, reducing the problem to efficiently sampling binary interactions between controlled agents. However, as individual agents are high-dimensional themselves, the controlled binary interactions correspond to large-scale dynamic programming problems, for which we propose a supervised learning approach based on discrete-time State-dependent Riccati Equations and recurrent neural networks.

Shigeaki Koike
Waseda University, Tokyo
ABP maximum principle with upper contact sets

In 1989, L. A. Caffarelli established Lp regularity theory for fully nonlinear second order uniformly elliptic equations. Motivated by this result, the notion of Lp-viscosity solutions was introduced by Caffarelli-Crandall-Kocan-Swiech in 1996. As is well-known, the Aleksandrov-Bakelman-Pucci's (ABP for short) maximum principle is the key tool for the regularity theory of fully nonlinear PDEs. In our series of works with A.Swiech, we have studied ABP maximum principle when coefficients to the drift term are Lq functions provided q is bigger than the dimension. However, in our results for ABP maximum principle, the maximum of Ln-viscosity subsolutions is estimated by the Ln norm of the inhomogeneous term, where the Ln norm is taken over the whole domain. However, the ABP maximum principle holds true with Ln norm over a subset (i.e. an upper contact set) in case when strong solutions instead of Ln-viscosity solutions or when the the coefficient to the drift term is bounded (but for Ln-viscosity solutions). In this talk, we point out the importance of the ABP maximum principle ”over” the upper contact sets, and then present recent results when the coefficient to the drift is unbounded. This is a joint work with A.Swiech (Georgia Institute of Technology).

Fabiana Leoni
Sapienza Università di Roma
Principal eigenvalues and related eigenfunctions for fully nonlinear equations in punctured balls

We consider the principal eigenvalues problem for fully nonlinear, uniformly elliptic equations, posed in punctured balls, in presence of radial singular potentials. In particular, for potentials of the form $1/r^\gamma$, we prove the existence of radial bounded eigenfunctions which can be extended in the whole ball if $\gamma < 2$, the existence of radial unbounded eigenfunctions in the case $\gamma = 2$, and the non existence of positive eigenfunctions when $\gamma > 2$. Moreover, for the case $\gamma = 2$ and when the operators are given by Pucci's extremal operators, we explicitly determine the principal eigenvalues, thus obtaining an extension in the fully nonlinear framework of the Hardy-Sobolev constant.

Pierangelo Marcati
GSSI
Quantum fluids and their appplications

I will review the main mathematical ideas that allow a rigorous formulation of quantum hydrodynamics and I will show various applications to problems in physics and engineering.

Roberto Natalini
IAC-CNR
Multiscale models of cell movements and their numerical approximation

In this talk we focus on a quite general class of hybrid mathematical models of collective motions of cells under the influence of chemical stimuli. The models are hybrid in the sense that cells are discrete entities given by ODE, while the chemoattractant is considered as a continuous signal which solves a diffusive equation. For these models it is possible to prove the mean-field limit in the Wasserstein distance to a system given by the coupling of a Vlasov-type equation with the chemoattractant equation. This approach and results are not based on empirical measures, but rather on marginals of an increasing number of individuals densities, and we show the limit with explicit bounds, by proving also existence and uniqueness for the limit system. In the monokinetic case we derive a new pressureless nonlocal Euler-type model with chemotaxis, which will be compared with other macroscopic models of cell movement. Numerical simulation comparing the different scales are presented. These results have been obtained by Roberto Natalini and Thierry Paul in collaboration, for the numerical part, with Marta Menci.

Michele Palladino,
Università dell'Aquila
Optimal Control and Reinforcement Learning

The talk aims at presenting one possible way to model certain tasks in reinforcement learning. The model provides a framework to deal with situations in which the system dynamics are not known and encodes the available information about the state dynamics that an agent has as a measure on the space of functions. In this framework, a natural question is if whether the optimal policies and the value functions converge, respectively, to an optimal policy and to the value function of the real, underlying optimal control problem as soon as more information on the environment is gathered by the agent. We provide a positive answer in the linear-quadratic case as well as in a particular case of a nonlinear, control-affine dynamics with Lagrangian cost. Open problems and questions will be also discussed.

Athena Picarelli
Università di Verona
A semi-Lagrangian scheme for a Hamilton-Jacobi-Bellman equation arising in stochastic exit time control problems.

We study the numerical approximation of parabolic, possibly degenerate, Hamilton-Jacobi-Bellman (HJB) equations in bounded domains. It is well known that convergence of the numerical approximation to the exact solution of the equation (considered here in the viscosity sense) is achieved under the assumptions of monotonicity, consistency and stability of the scheme. While standard finite difference schemes are in general non monotone, the so-called semi-Lagrangian (SL) schemes are monotone by construction. These schemes make use of a wide stencil and, when the equation is set in a bounded domain, this typically causes an overstepping of the boundary. We discuss here a suitable modification of this scheme adapted to the treatment of boundary problems.

Giovanni Russo
Università di Catania
Semilagrangian-spectral methods for the Boltzmann equation of rarefied gas dynamics

Semi-Lagrangian (SL) methods are proposed for the numerical solution of the Boltzmann equation of rarefied gas dynamics. The transport term is treated by a SL method based on a high order conservative non-linear reconstruction, which prevents formation of spurious oscillations, still maintaining conservation. High order accuracy in time is obtained by either Runge-Kutta or multistep schemes. The evaluation of the collisional operator is performed by a Fourier-spectral method which allows at the same time spectral accuracy and efficient computation. An L^2-projection technique is adopted in order restore full conservation at the level of the collision operator. The overall accuracy of the schemes is up to fourth order in time, up to five order in space and spectral in velocity. The SL methodology allows the use of a much larger time step compared to Eulerian based schemes, thus providing improving efficiency. The methods have been implemented on a simplified model 1D in space and 2D in velocity. Several numerical tests illustrates the accuracy and efficiency of the proposed methods

Luca Saluzzi
Imperial College, London
A statistical POD approach for feedback boundary optimal control in fluid dynamics

Hamilton-Jacobi-Bellman (HJB) equation plays a central role in optimal control and differential games, enabling the computation of robust controls in feedback form. The main disadvantage for this approach depends on the so- called curse of dimensionality, since the HJB equation and the dynamical sys- tem live in the same, possibly high dimensional, space. In this talk I will con- sider feedback boundary optimal control problems arising from fluid dynamics and their reduction by the means of a Statistical Proper Orthogonal Decom- position (SPOD) method. The Proper Orthogonal Decomposition (POD) is a well-known technique in the Model Order Reduction community used to reduce the complexity of intensive simulations. The SPOD approach is characterized by the introduction of stochastic terms in the model (e.g. in the initial condi- tion or in the boundary conditions) to enrich the knowledge of the Full Order Model, useful for the definition of a more reliable controlled reduced dynamics. In the offline stage of the method we consider different realizations of the artifi- cial random variables and we compute the corresponding optimal trajectory via the Pontryagin Maximum Principle (PMP), which will form our snapshots set. Afterwards, we construct the reduced basis and we consider the corresponding reduced dynamical system. At this point the HJB can be solved in a reduced domain through the application of a data-driven Tensor Train Gradient Cross [2] based on samples derived by either PMP or the State-Dependent Riccati Equation. Finally, I will show its effectiveness on the optimal control of the incompressible Navier-Stokes equation in a backward step domain.

Joint work with S. Dolgov and D.Kalise

[1] S. Dolgov, D. Kalise, L. Saluzzi, A statistical POD approach for boundary optimal control in fluid dynamics, preprint, 2023.

[2] S. Dolgov, D. Kalise, L. Saluzzi, Data-driven Tensor Train Gradient Cross Approximation for HJB Equations. To appear on SIAM J. Sci. Comp., 2023

Antonio Siconolfi

Sapienza Università di Roma

Homogenization of Hamilton–Jacobi equations on networks

We present an homogenization procedure for time dependent Hamilton–Jacobi equations posed on networks embedded in the Euclidean space $\mathbb{R}^N$, and depending on an oscillation parameter ϵ which becomes infinitesimal.

The peculiarity of the construction is that the limit equation is posed in an Euclidean space whose dimension depends on the topological complexity of the network. Approximating and limit equations are therefore defined on different spaces, which requires an appropriate notion of convergence for the corresponding solutions.

We use closed probability measures defined on an abstract graph underlying the network, and define an equivalent on graph of the so–called Mather α and β functions. The α function plays the role of effective Hamiltonian.

The results have been obtained in collaboration with Marco Pozza and Alfonso Sorrentino.

Silvia Tozza

Università di Bologna

A trip into Image Processing with Maurizio

In this talk I will recall some results obtained in collaboration with Maurizio. The topic is related to analysis and approximation of Hamilton-Jacobi equations with applications in the context of Image Processing. As an example, for the stationary case we focused on the problem of reconstructing the shape of an object starting from one input grey-level image of it (the so-called Shape-from-Shading problem). This is an ill-posed problem since it does not admit a unique solution. For the evolutive case, we worked on the image segmentation problem via a Level-Set approach. This problem is used to detect the boundaries of one or more objects present in a picture, useful in different application fields, e.g. the biomedical or astronomical ones.