On the vanishing discount approximation for compactly supported perturbations of periodic Hamiltonians

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Nonlinear partial differential equations: theory, numerics and applications.
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The problem

$$H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \qquad \text{continuous}$$

(H0) $H(x+z, p) = H(x, p) \quad \forall (x,p) \in \mathbb{R} \times \mathbb{R}, \ \forall z \in \mathbb{Z}$

(H1) (coercivity) $\inf_{x \in \mathbb{R}} H(x,p) \to +\infty$ $\quad |p| \to +\infty$

(H2) (convexity) $H(x, \cdot)$ convex on $\mathbb{R}$ $\quad \forall x \in \mathbb{R}$

$$V \in C_c(\mathbb{R})$$

set

$$G(x,p) := H(x,p) - V(x) \qquad \forall (x,p) \in \mathbb{R} \times \mathbb{R}$$

Let $\lambda > 0$ and consider the equation

$$\lambda u(x) + G(x, u') = c(G) \quad \text{in } \mathbb{R} \qquad (EG_\lambda)$$

where $c(G)$ is the unique constant for which

$$G(x, u') = c(G) \quad \text{in } \mathbb{R} \qquad (EG_0)$$

may have bounded (viscosity) solutions.

Fact: $u, v \in C_b(\mathbb{R})$, u super-sol., v sub-sol. of (EG_λ)

$\Longrightarrow v \leq u$ in $\mathbb{R}$ $\quad (\lambda > 0!)$

As a consequence, $\exists!\ u_G^\lambda \in C_b(\mathbb{R})$ solution to (EG_λ)

Let $\lambda > 0$ and consider the equation

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where $c(G)$ is the unique constant for which

$$G(x, u') = c(G) \quad \text{in } \mathbb{R} \qquad (EG_0)$$

may have bounded (viscosity) solutions.

<u>Fact 1:</u> (Bounded) solutions to (EG_0) are not unique, not even up to additive constants in general

<u>Fact 2</u>: Solutions to (EG_0) always exist.

<u>Fact 3</u>: Bounded solutions to (EG_0) need not exist.

$$\lambda u_G^\lambda(x) + G(x, d_x u_G^\lambda) = c(G) \quad \text{in } \mathbb{R} \qquad (EG_\lambda)$$

Fact : $\{u_G^\lambda\}_{\lambda>0}$ locally equi-bounded & equi-Lipschitz $\lambda u_G^\lambda \rightrightarrows 0$

By the Ascoli-Arzelá Thm and the stability of the notion of viscosity solution, the functions u_G^λ uniformly converge, along subsequences as $\lambda \to 0^+$, to viscosity solutions of

$$G(x, u') = c(G) \quad \text{in } \mathbb{R} \qquad (EG_0)$$

Question : does $\lim_{\lambda\to 0^+} u_G^\lambda$ exist ?

Theorem (I. Capuzzo-Dolcetta, D., Comm. PDEs, 2023)

The whole family $\{u_G^\lambda\}_{\lambda>0}$ locally uniformly converges on $\mathbb{R}$, as $\lambda \to 0^+$, to a distinguished solution u_G^0 of the critical equation

$$G(x, u') = \underbrace{c(G)}_{\text{critical constant}} \quad \text{in } \mathbb{R}$$

Remark: We also have a characterization of the limit function u_G^0.

Motivation

[LPV] P.L.-Lions, G. Papanicolau, S.R.S. Varhadan,
Homogenization of HJ equations (1987)

$H \in C(T^*M)$ $\qquad M := \mathbb{T}^d \cong \mathbb{R}^d / \mathbb{Z}^d$

(H1) (coercivity) $\displaystyle \inf_{x \in M} H(x,p) \xrightarrow[\|p\| \to +\infty]{} +\infty$

Thm: $\exists!\ c(H) \in \mathbb{R}$ s.t. the equation
$H(x, d_x u) = c(H)$ in M admits (viscosity) solutions

$c(H)$ is known as ergodic constant or Mañe critical constant

Theorem ([DFIZ])

Let $H \in C(T^*M)$ be convex and coercive in the momentum. Denote by u_H^λ the unique solution of

$$\lambda u_H^\lambda(x) + H(x, d_x u_H^\lambda) = c(H) \quad \text{in } M \qquad (EH_\lambda)$$

Then the whole family $\{u_H^\lambda\}_{\lambda>0}$ uniformly converges on M as $\lambda \to 0^+$ to a distinguished solution u_H^0 of the critical equation

$$H(x, d_x u) = c(H) \quad \text{in } M \qquad (EH_0)$$

[DFIZ] A. Davini, A. Fathi, R. Iturriaga, M. Zavidovique, Invent. Math. (2016)

A counterexample for non-convex Hamiltonians

[Z19] B. Ziliotto, J. Math. Pures Appl. (2019)

Example of $H \in C(\mathbb{T} \times \mathbb{R})$ non-convex but otherwise standard Hamiltonian for which

$$\nexists \lim_{\lambda \to 0^+} u_H^\lambda \qquad \text{on} \quad M := \mathbb{T}^1$$

Developments

- H. Mitake, H.V. Tran, Adv. Math. (2017)
- H. Ishii, H. Mitake, H.V. Tran, JMPA (2017)

viscous HJ equations

- A. Davini, M. Zavidovique, Adv. Calc. Var. (2021)
- H. Ishii, Math. Eng. (2021)

weakly coupled HJ systems

- H. Ishii, L. Jin, Calc. Var. PDEs (2020)
- P. Cardaliaguet, A. Porretta, Anal. PDE (2019)

Mean Field Games

- Q. Chen, Adv. Calc. Var. (2021)
- Q. Chen, W. Cheng, H. Ishii, K. Zhao, Comm. PDE (2019)
- M. Zavidovique, Anal. PDE (2022)
- Q. Chen, A. Fathi, M. Zavidovique, J. Zhang (2023)

⋮

Contact type HJ eqts

In all these papers, the vanishing discount problem is studied on a compact Riem. manifold M (typically, $M = \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$)

When M is noncompact

$\boxed{M = \mathbb{R}^d \text{ below}}$

[IS20] H. Ishii, A. Siconolfi, Comm. in PDEs (2020)

Example: $G(x,p) := |p| - f(x)$

The vanishing discount problem is studied with

$c_f(G) = -\inf f$ in place of $c(G)$ and assuming

(A3)/(A3') $\{x \in \mathbb{R}^d : f(x) = -c_f(G)\}$ compact

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[BK22] M. Bardi, H. Kouhkouh, Applied Math. Optim. (2023)

The vanishing discount problem is studied for $G(x,p) := |p| - f(x)$ with $c_f(G) = -\inf f$ in place of $c(G)$ and assuming f bounded, Lipschitz, semiconcave and $\{x \in \mathbb{R}^d : f(x) = -c_f(G)\} \neq \emptyset$.

[CDM88] I. Capuzzo-Dolcetta, J. Menaldi, JDE (1988)

The vanishing discount problem is studied for

$$G(x,p) := -g(x)\cdot p - f(x)$$

with $f \in BUC(\mathbb{R}^d)$ and $-g \in Lip(\mathbb{R}^d;\mathbb{R}^d)$ strongly monotone.

In this case, $\exists!\, x_0 \in \mathbb{R}^d$ such that $g(x_0)=0$.

$$\lambda u_\lambda + G(x, d_x u_\lambda) = c \quad \text{in } \mathbb{R}^d \quad \text{with} \quad c = f(x_0)$$

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They prove that

- $u_\lambda(x_0) = 0 \quad \forall \lambda > 0$;
- $u_\lambda(\cdot) \longrightarrow v$ in $C(\mathbb{R}^d)$, v is the unique solution of

$$-g(x)\cdot Dv - f(x) = c \quad \text{in } \mathbb{R}^d \quad \text{with} \quad v(x_0) = 0.$$

Critical constants: the periodic case

The critical constant $c(H)$ is characterized as the unique real constant for which the H-J equation

$$H(x, Du) = c(H) \quad \text{in } \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$$

admits solutions, equivalently the H-J equation

$$H(x, Du) = c(H) \quad \text{in } \mathbb{R}^d$$

admits $\mathbb{Z}^d$-periodic solutions.

Fact: the following characterization holds:

$$c(H) = \min \left\{ a \in \mathbb{R} : \exists \text{ a } \mathbb{Z}^d\text{-periodic subsolution to } H(x, Du) \leq a \text{ in } \mathbb{R}^d \right\}$$

Fact : Fix $\theta \in \mathbb{R}^d$ and set $H_\theta(x,p) := H(x,\theta+p)$

$$H(x,\theta+Du) = c(H_\theta) \quad \text{in } \mathbb{R}^d \qquad (*)$$

The function $\theta \mapsto c(H_\theta)$ is convex and coercive

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Def.: We define the free critical value as

$$c_f(H) := \inf \{ a \in \mathbb{R} : \exists \text{ a subsolution to } H(x, Du) = a \text{ in } \mathbb{R}^d \}$$

Fact: Inf is a min

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Remark: If u is a $\mathbb{Z}^d$-periodic sol. to $(*)$ then $u_\theta(x) := u(x) + \theta \cdot x$ is a solution of

$$H(x, Dv) = c(H_\theta) \quad \text{in } \mathbb{R}^d$$

Hence $c_f(H) \leq \min_{\theta \in \mathbb{R}^d} c(H_\theta)$.

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Remark: By the same argument one gets that

$$c_f(H) = c_f(H_\theta) \qquad \forall \theta \in \mathbb{R}^d.$$

Fact : Fix $\theta \in \mathbb{R}^d$ and set $H_\theta(x,p) := H(x, \theta + p)$

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Thm (A. Fathi – E. Maderna 2007)

$$c_f(H) = \min_{\theta \in \mathbb{R}^d} c(H_\theta)$$

Guiding example: $H(x, p) = \frac{1}{2}p^2 + \cos(2\pi x)$

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u is a 1-periodic solution of

$$H(x, \theta + u') = c \quad \text{in } \mathbb{R}$$

<u>Question:</u> How do we choose $c = c(H, \theta)$ in function of θ ?

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<u>Fact:</u> $c \geq c_f(H) = 1$

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u is a 1-periodic solution of

$$H(x, \theta + u') = c \qquad \text{in } \mathbb{R}$$

Set $v(x) = \theta x + u(x)$. Then we have to find a solution v of

$$H(x, v') = c \qquad \text{in } \mathbb{R} \qquad (*)$$

such that $\int_0^1 v'(x)\,dx = \int_0^1 (\theta + u'(x))\,dx = \theta + \int_0^1 u'\,dx = \theta$

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Now v is a viscosity solution to $(*)$ if

(a) $H(x, v'(x)) = c$ for a.e. $x \in \mathbb{R}$

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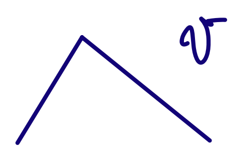
(b) v' can jump downwards



ok

Guiding example: $H(x,p)=\frac{1}{2}p^2+\cos(2\pi x)$

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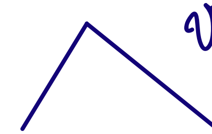
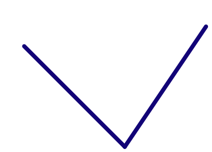
(a) $H(x,v'(x))=c$ for a.e. $x\in\mathbb{R}$

(b) v' can jump downwards

(c) v' cannot jump upwards

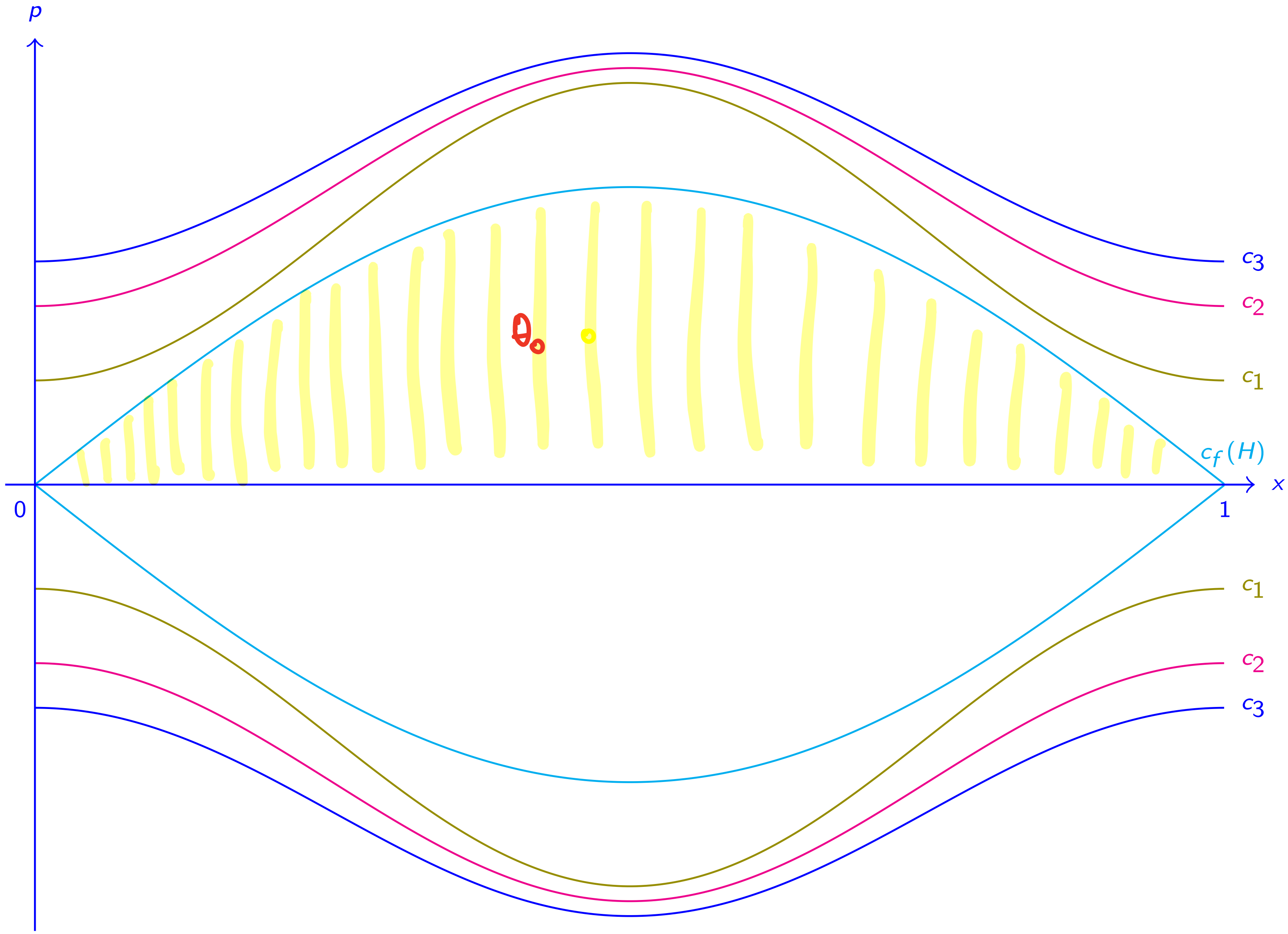


v ok



v No!

Guiding example: $H(x,p) = \frac{1}{2}p^2 + \cos(2\pi x)$



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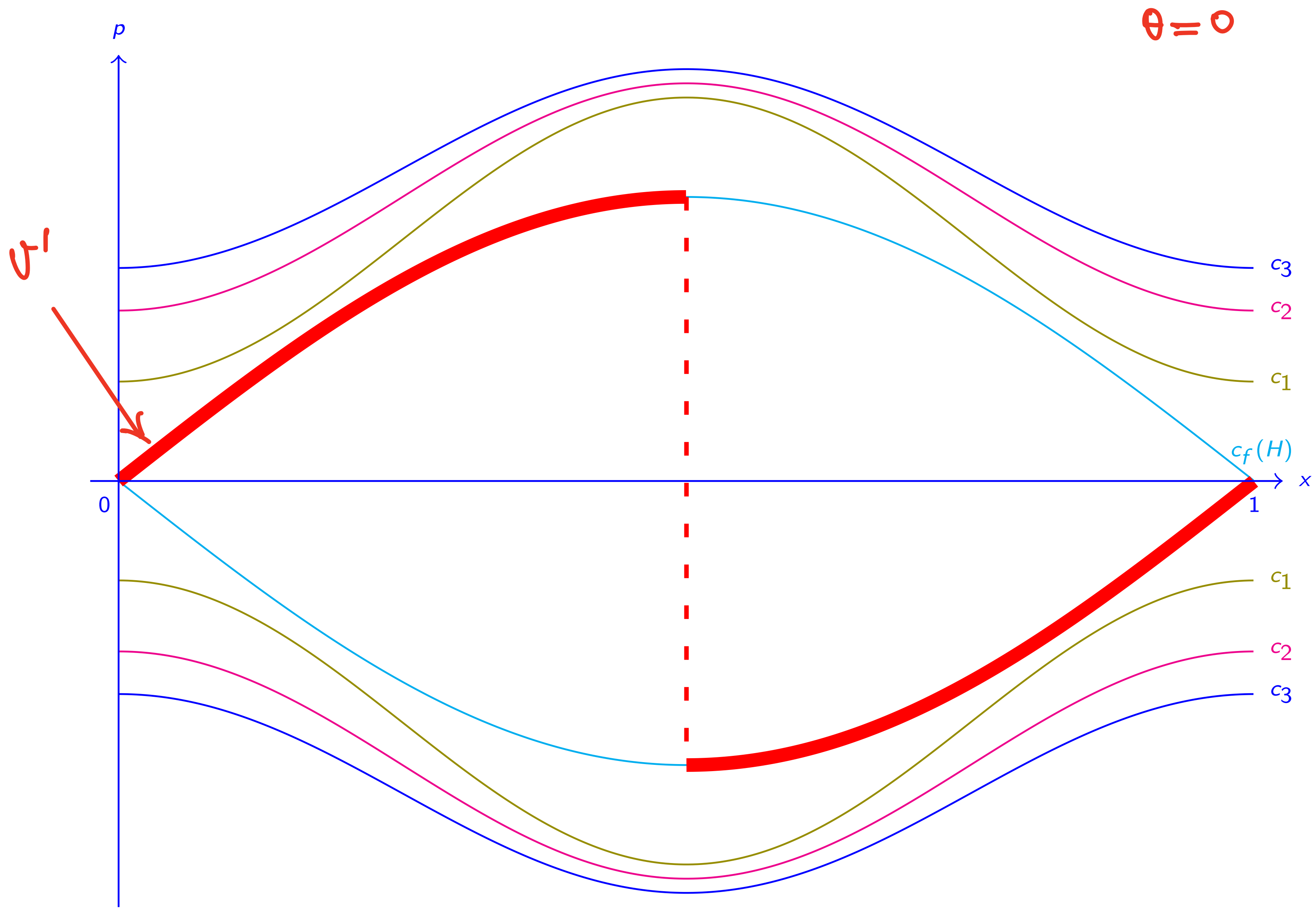
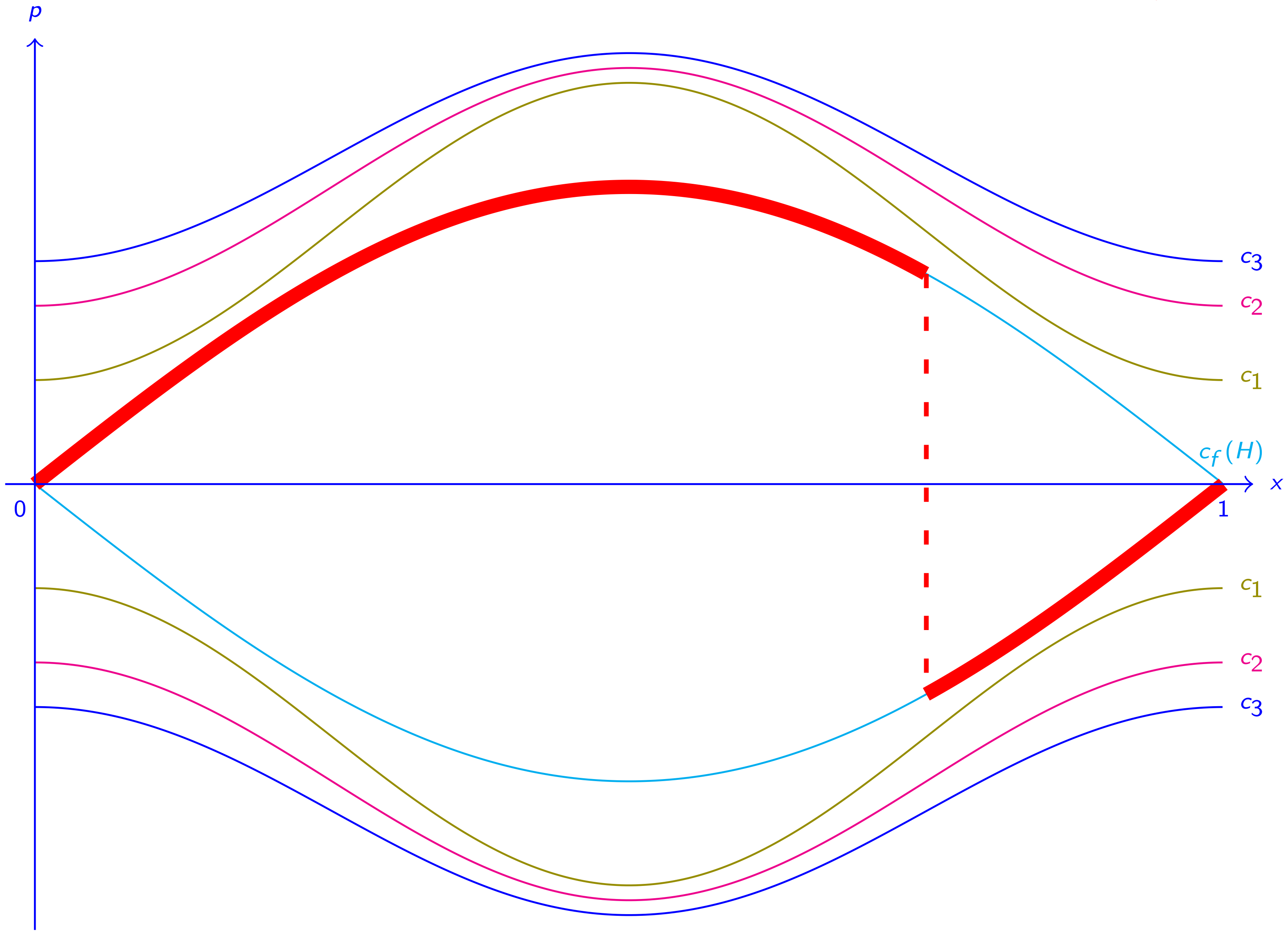


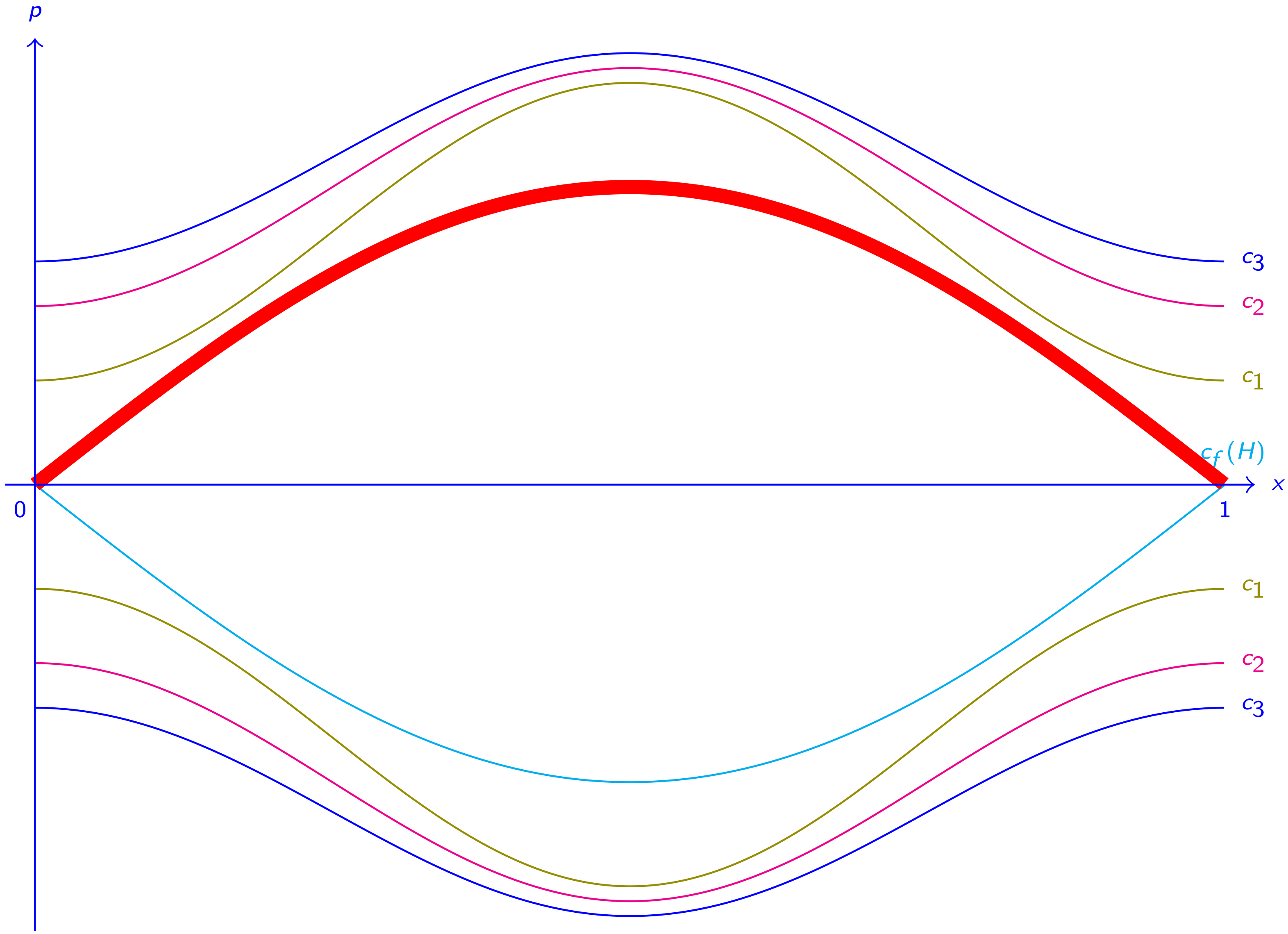
Guiding example: $H(x,p) = \frac{1}{2}p^2 + \cos(2\pi x)$

$0 < \theta < \theta_0$



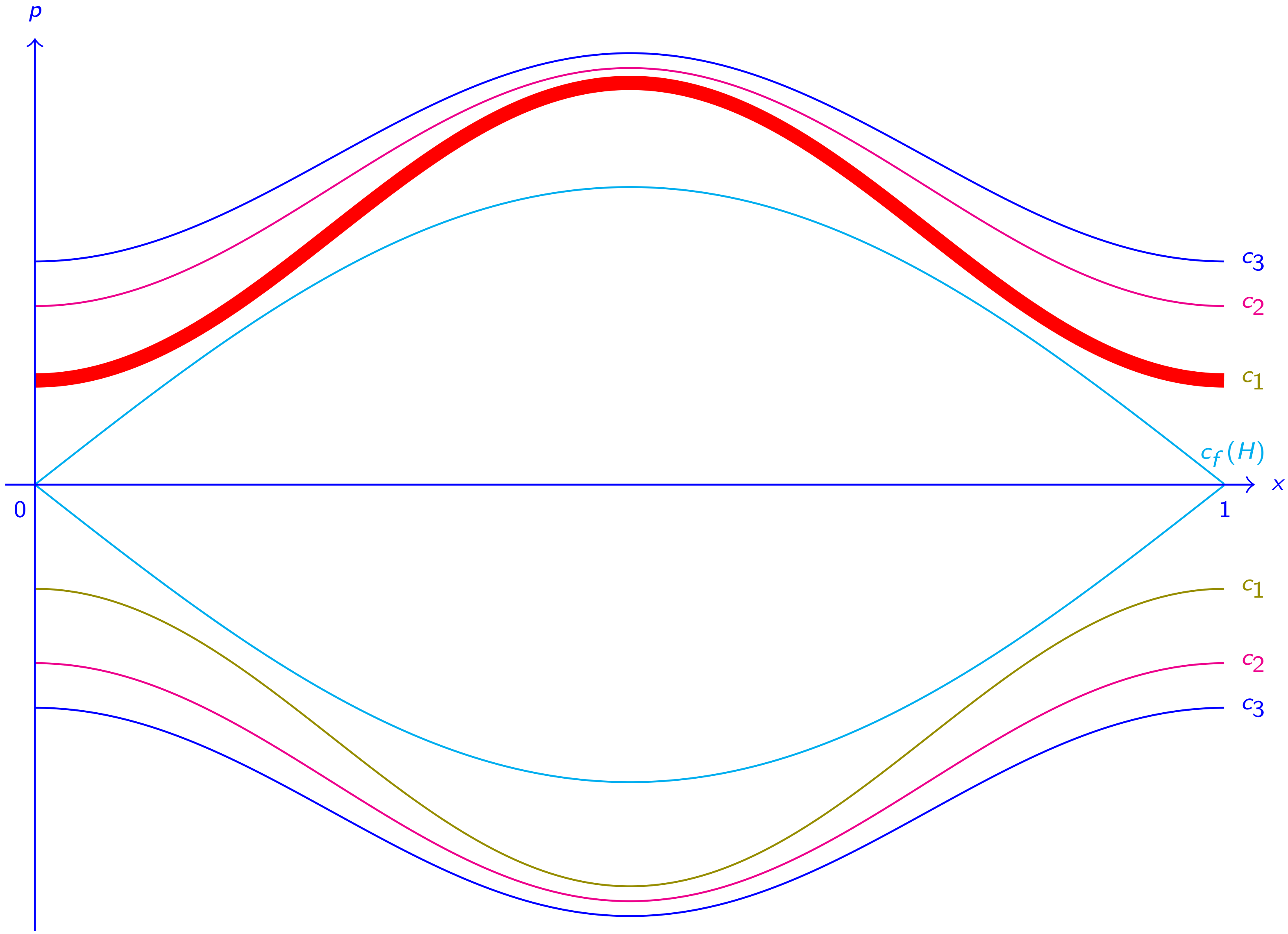
Guiding example: $H(x,p) = \frac{1}{2}p^2 + \cos(2\pi x)$

$\theta = \theta_0$



Guiding example: $H(x,p) = \frac{1}{2}p^2 + \cos(2\pi x)$

$\theta > \theta_0$



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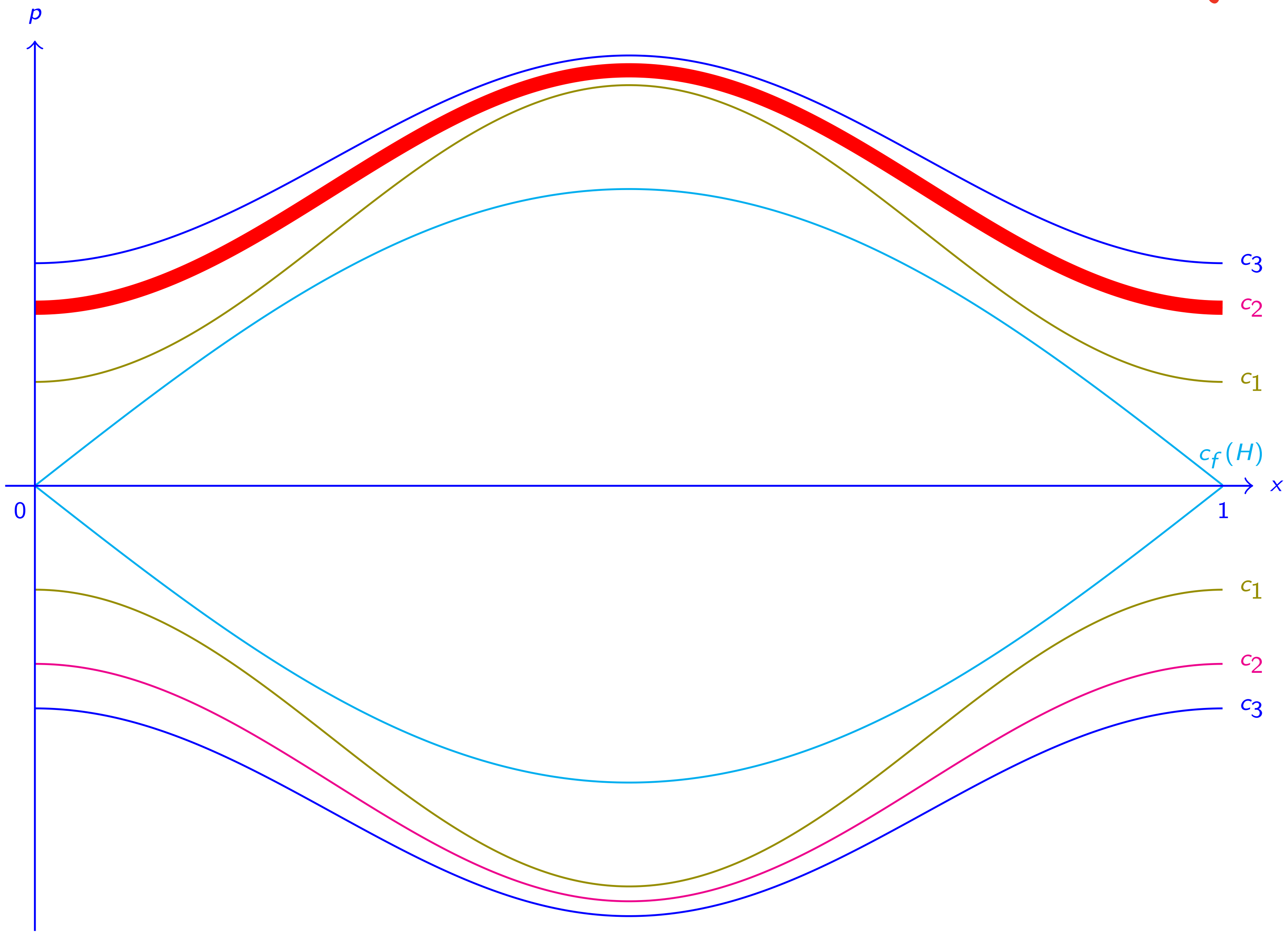

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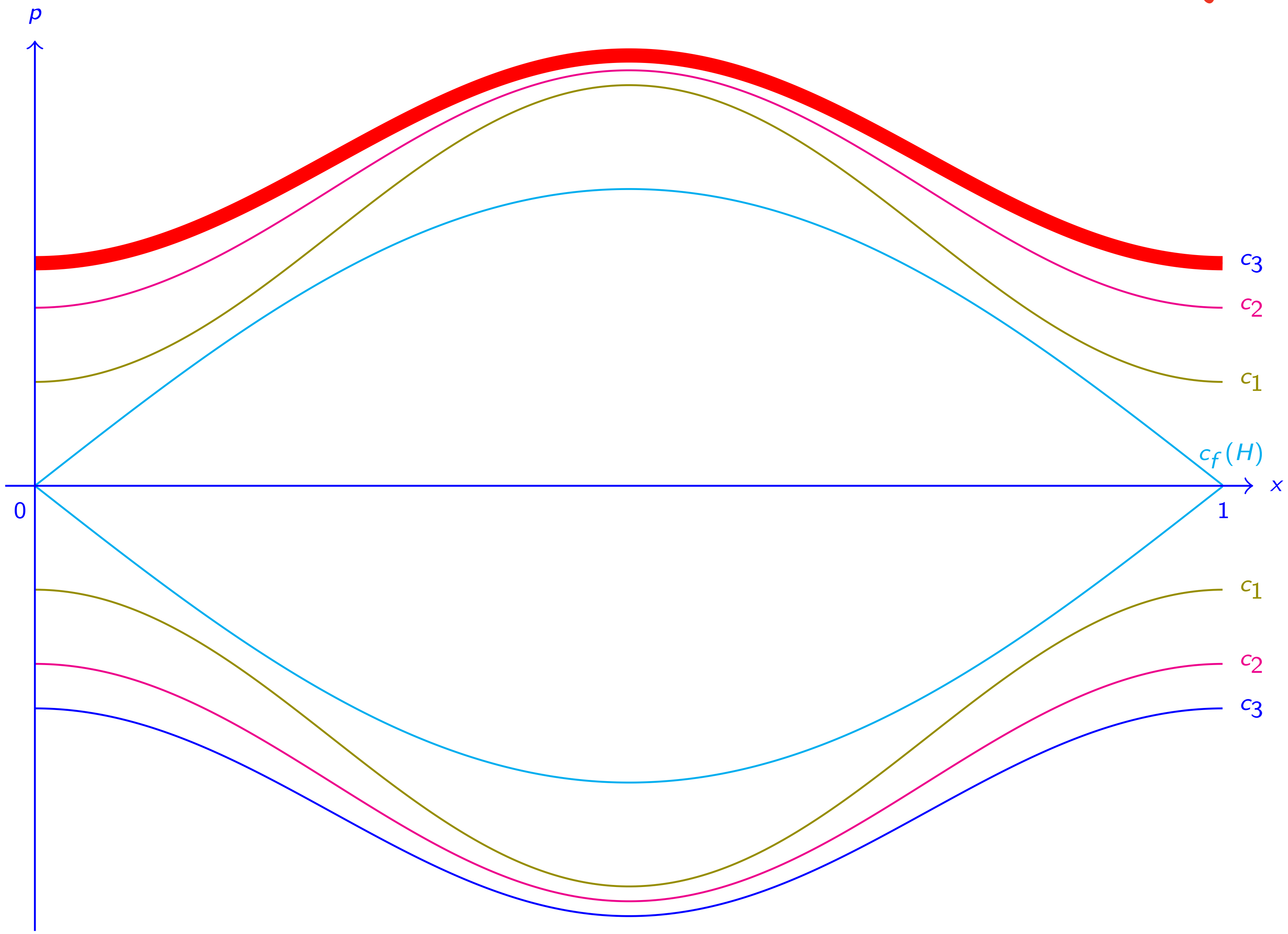

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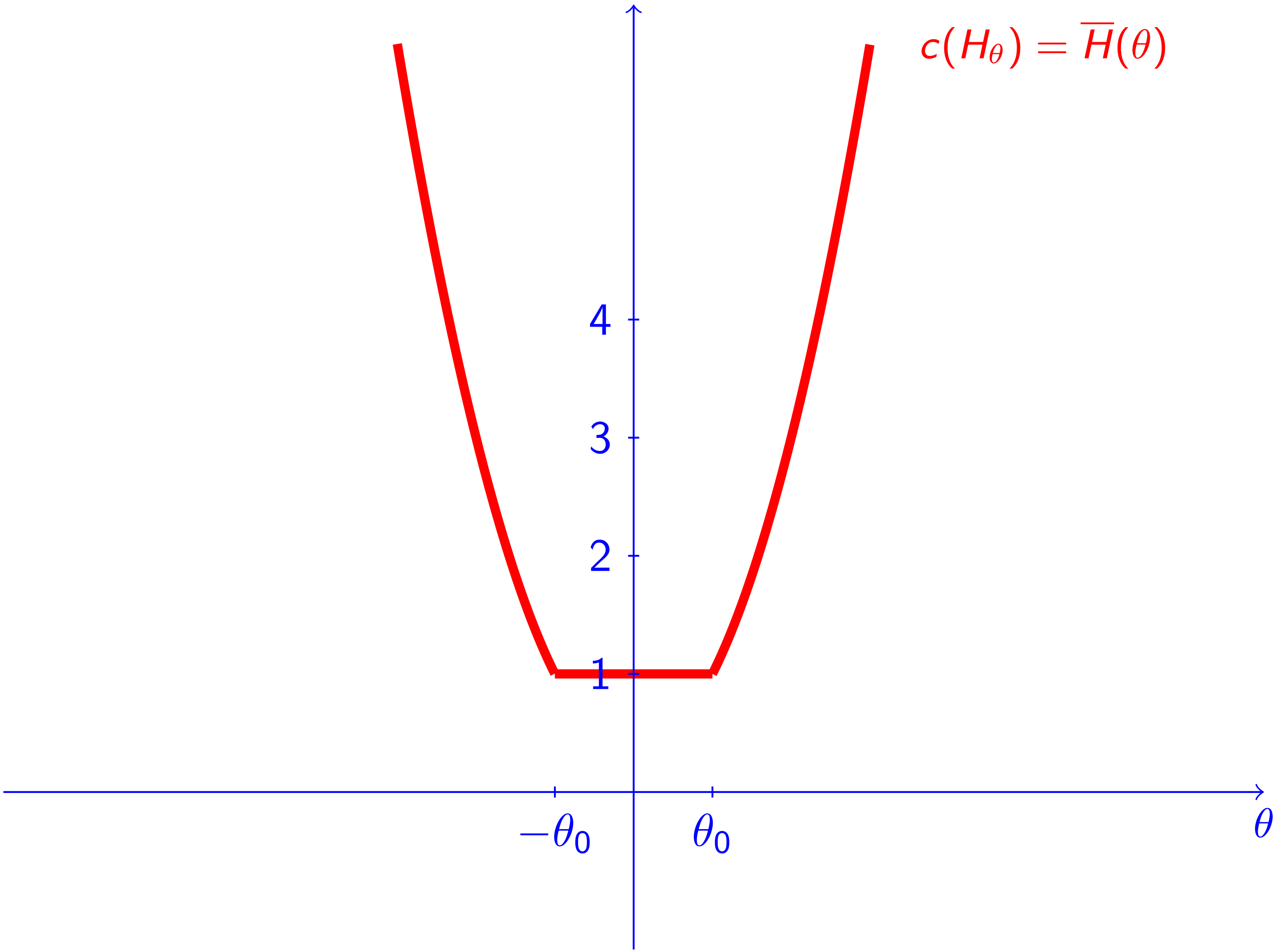

Critical constants: beyond the periodic setting

$G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ continuous

(H1) (coercivity) $\displaystyle \inf_{x \in \mathbb{R}} G(x,p) \longrightarrow +\infty$ $\quad |p| \to +\infty$

(H2) (convexity) $G(x,\cdot)$ convex on $\mathbb{R}$ $\quad \forall x \in \mathbb{R}$

For $a \in \mathbb{R}$, let

$$G(x, u') = a \qquad \text{in } \mathbb{R} \qquad (HJ_a)$$

The free critical value is the real constant defined as

$$c_f(G) := \inf \{ a \in \mathbb{R} : (HJ_a) \text{ admits subsolutions} \}$$

inf is a min

Remark: Fix $\theta \in \mathbb{R}$ and set $G_\theta(x,p) := G(x, \theta + p)$.

Then $c_f(G_\theta) = c_f(G)$.

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$$\boxed{c(G) := \inf\{a \in \mathbb{R} : (HJ_a) \text{ admits bounded subsolutions}\} \quad (*)}$$

$$G(x,u') = a \qquad (HJ_a)$$

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Then $c_f(G_\theta) = c_f(G)$.

Set

$$\boxed{c(G) := \inf \{ a \in \mathbb{R} : (HJ_a) \text{ admits bounded subsolutions} \} \qquad (*)}$$

$$G(x, u') = a \qquad (HJ_a)$$

The definition of $c(H)$ given in $(*)$ is consistent with the usual one adopted in the periodic setting.

Remark: Fix $\theta \in \mathbb{R}$ and set $G_\theta(x,p) := G(x, \theta + p)$.

Then $c_f(G_\theta) = c_f(G)$.

Set

$$\boxed{c(G) := \inf\{a \in \mathbb{R} : (HJ_a) \text{ admits bounded subsolutions}\} \qquad (*)}$$

$$G(x, u') = a \qquad (HJ_a)$$

Proposition: Let H be a 1-periodic Hamiltonian (with respect to x) satisfying (H1)-(H2). Then the value $c(H)$ given by $(*)$ satisfies

$$c(H) = \min\{a \in \mathbb{R} : (HJ_a) \text{ admits 1-periodic subsolutions}\}.$$

Remark: Fix $\theta \in \mathbb{R}$ and set $G_\theta(x,p) := G(x, \theta + p)$.

Then $c_f(G_\theta) = c_f(G)$.

Set

$$\boxed{c(G) := \inf\{a \in \mathbb{R} : (HJ_a) \text{ admits bounded subsolutions}\} \quad (*)}$$

$$G(x,u') = a \qquad (HJ_a)$$

Proposition: Let $w \in \mathrm{Lip}(\mathbb{R})$ be a bounded supersolution to (HJ_a) for some $a \in \mathbb{R}$. Then $a \leqslant c(G)$. In particular, if w is a solution, then $a = c(G)$.

Critical Constants: the perturbed periodic case

$H \in C(\mathbb{R}\times\mathbb{R})$ 1-periodic in x & satisfying (H1)–(H2)

$G(x,p) := H(x,p) - V(x)$ with $V \in C_c(\mathbb{R})$

$$\boxed{G(x,u') = a \quad \text{in } \mathbb{R} \qquad (HJ_a)}$$

$c_f(G) = \min\{a \in \mathbb{R} : (HJ_a) \text{ admits subsolutions}\}$

$c(G) := \inf\{a \in \mathbb{R} : (HJ_a) \text{ admits bounded subsolutions}\}$

<u>Proposition</u>: $c(G) \geq c(H)$ and $c_f(G) \geq c_f(H)$.

Proof of $c(G) \geqslant c(H)$

Let $a > c(G)$ and $v \in \mathrm{Lip}(\mathbb{R})$ be a bounded subsolution of (HJ_a).

Let $(z_n)_n \subset \mathbb{Z}$ with $\lim_n |z_n| = +\infty$ and set

$$v_n(\cdot) = v(\cdot + z_n) - v(z_n), \qquad G_n(\cdot,\cdot) = G(\cdot + z_n, \cdot)$$

Then $$G_n(x, v_n') \leqslant a \quad \text{in } \mathbb{R}.$$

Up to subsequences, $v_n \longrightarrow u$ in $C(\mathbb{R})$.

Since $G_n \to H$ in $C(\mathbb{R} \times \mathbb{R})$, we conclude that u is a bounded subsolution of $H(x, u') \leqslant a$ in $\mathbb{R}$.

Hence $a \geqslant c(H)$, as it was asserted.

The proof of $c_f(G) \geqslant c_f(H)$ is similar. $\square$

<u>Theorem</u>: Let us consider the critical equation

$$G(x, u') = c(G) \quad \text{in } \mathbb{R} \qquad (EG_0).$$

(i) If $c(G) > c(H)$, (EG_0) does not admit bounded solutions.

(ii) If $c(G) = c(H)$, (EG_0) does not admit a bounded subsolution v which is uniformly strict outside some compact set K, i.e., satisfying

$$G(x, v'(x)) < c(G) - \delta \quad \text{for a.e. } \mathbb{R} \setminus K$$

for some $\delta > 0$.

Guiding example : $G(x,p) = H(x,p) - V(x)$

$$H(x,p) = \frac{p^2}{2} + \cos(2\pi x), \quad V \leq 0$$

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$$G(x, \theta + u') = c \quad \text{in } \mathbb{R}.$$

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$$G(x, \theta + u') = c \quad \text{in } \mathbb{R}.$$

Set $v(x) = \theta x + u(x)$. We have to find a function $v \in Lip(\mathbb{R})$ such that

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(b) v' jumps only downwards;

(c) $\exists\, M \geq 0$ such that

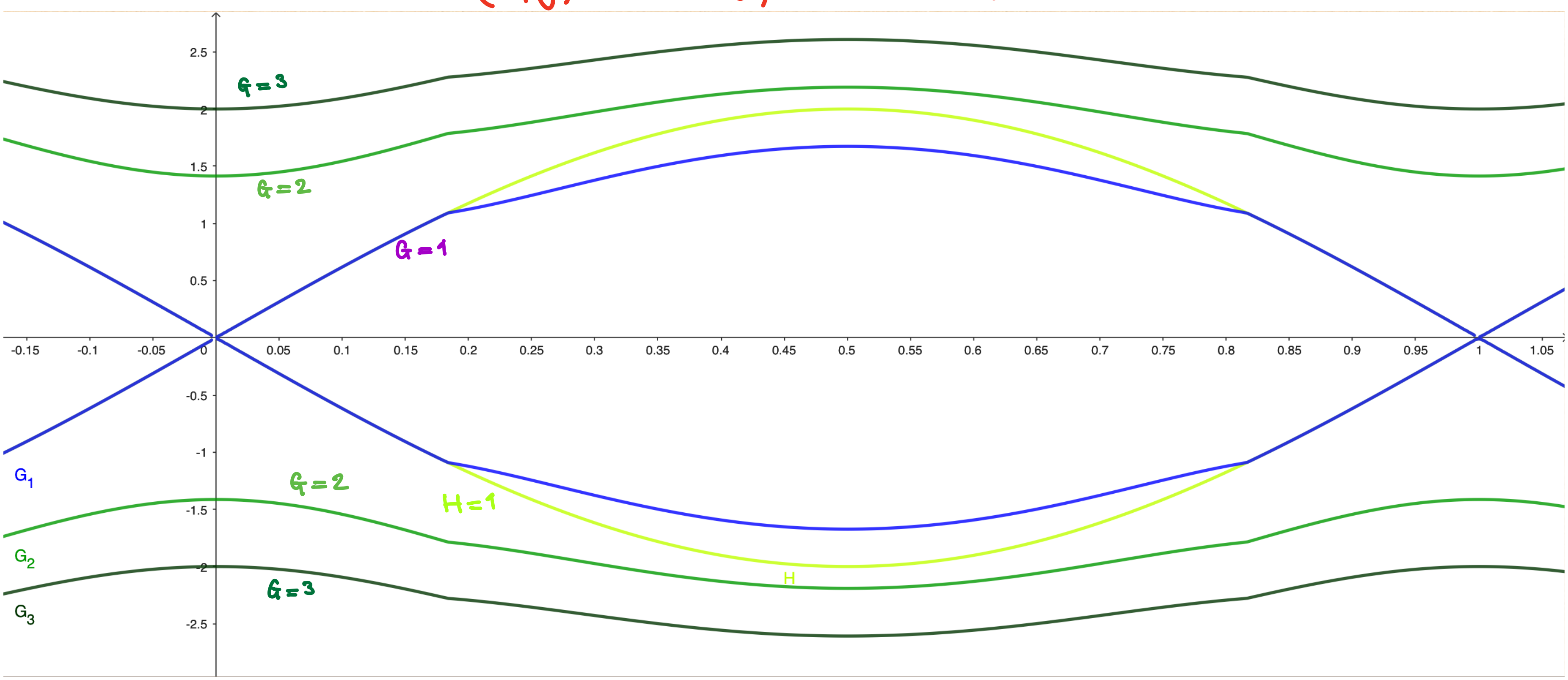
$$\left| \int_0^n (v'(x) - \theta)\, dx \right| \leq M \qquad \forall\, n \in \mathbb{Z}$$

Guiding example : $G(x,p) = H(x,p) - V(x)$

$$H(x,p) = \frac{p^2}{2} + \cos(2\pi x), \quad V \leq 0, \quad \operatorname{supp}(V) \subset \left(\tfrac{1}{10}, \tfrac{9}{10}\right)$$

$|V| << 1$ so that

$$c(G_\theta) = c(H_\theta) \quad \forall\, \theta \in \mathbb{R}$$



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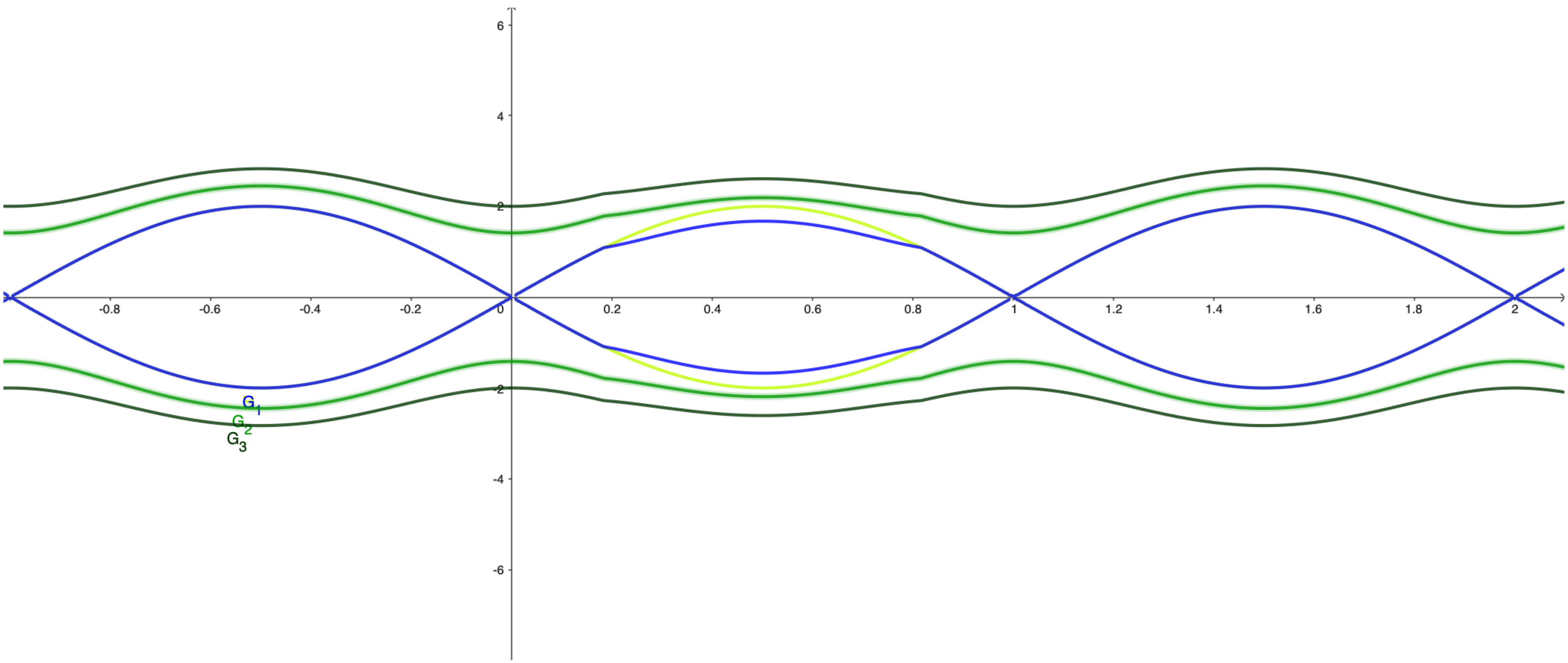

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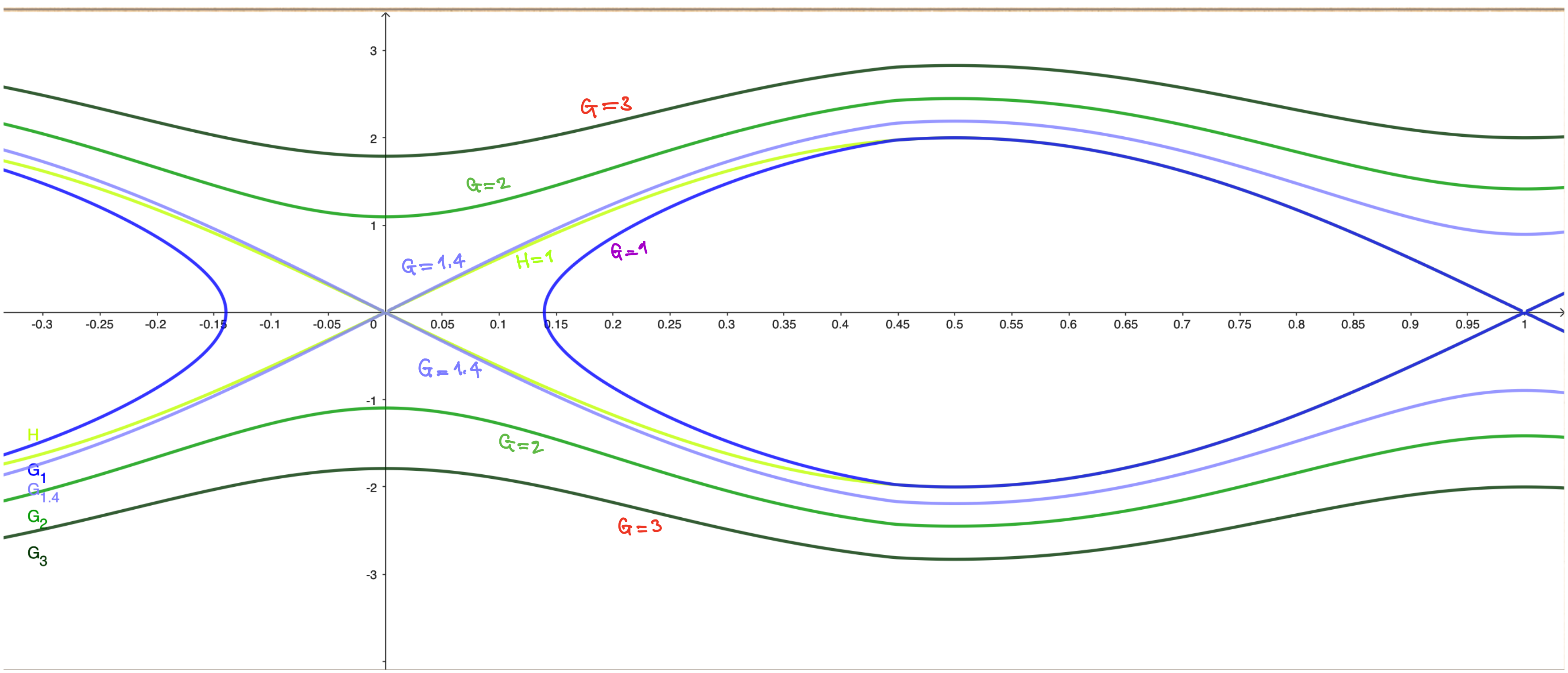

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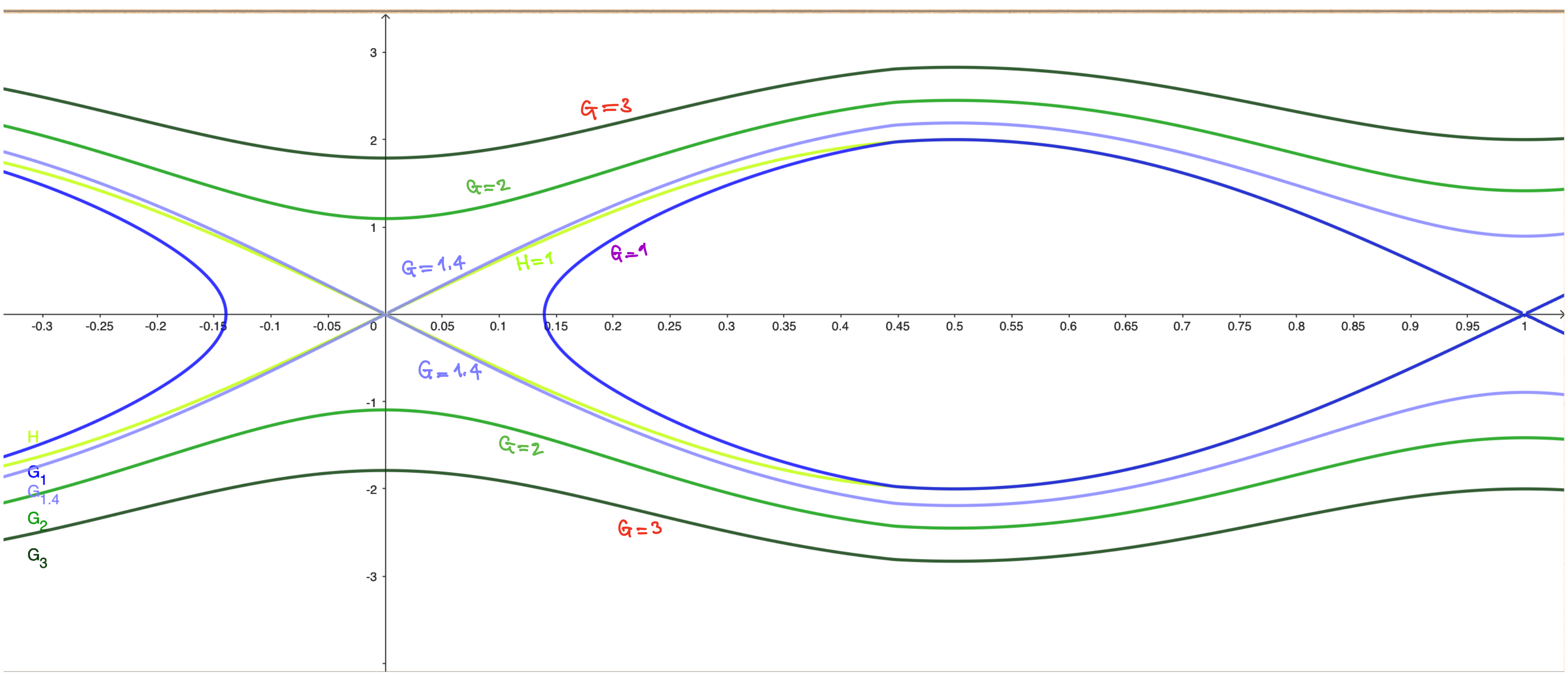

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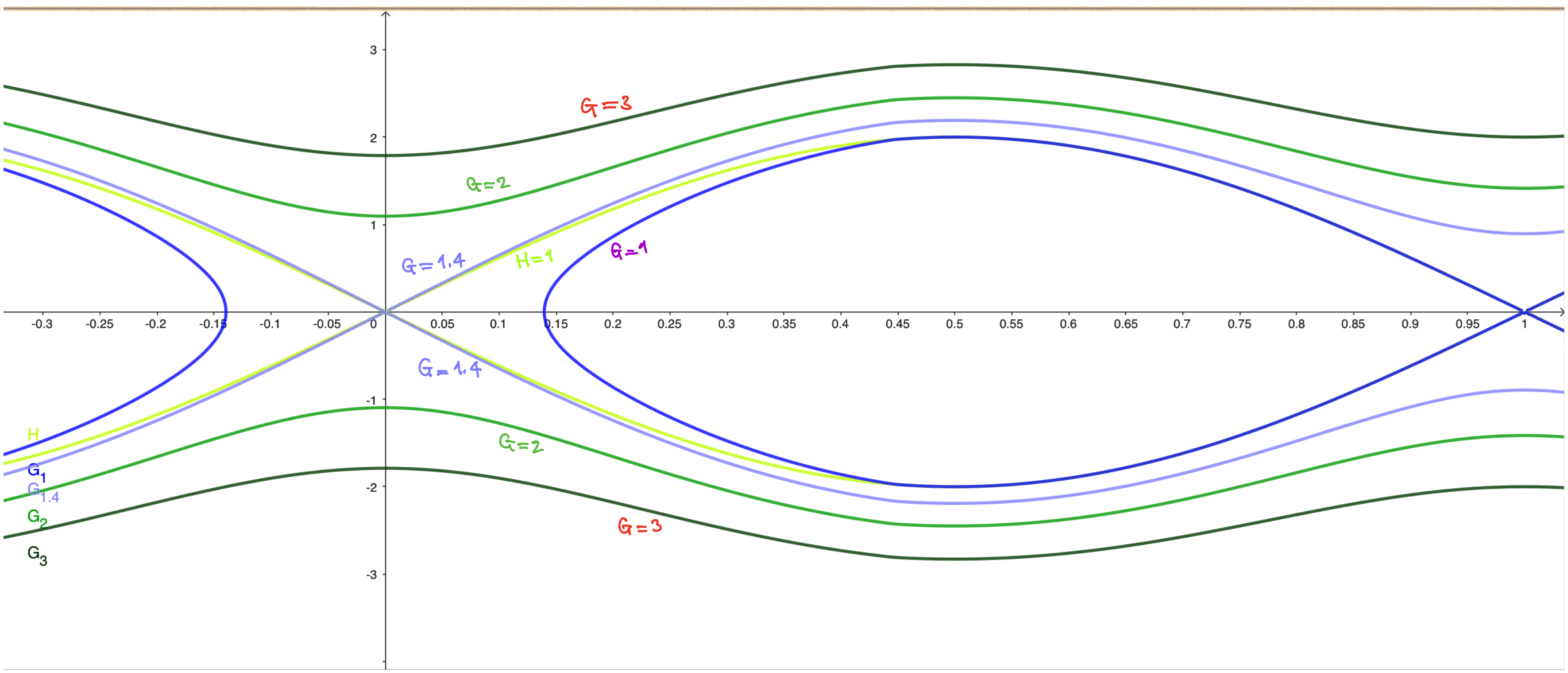
$\nexists$ bounded solutions of $G(x, \theta + u') = c(H_\theta)$ for $\theta \in (-\theta_0, \theta_0)$.



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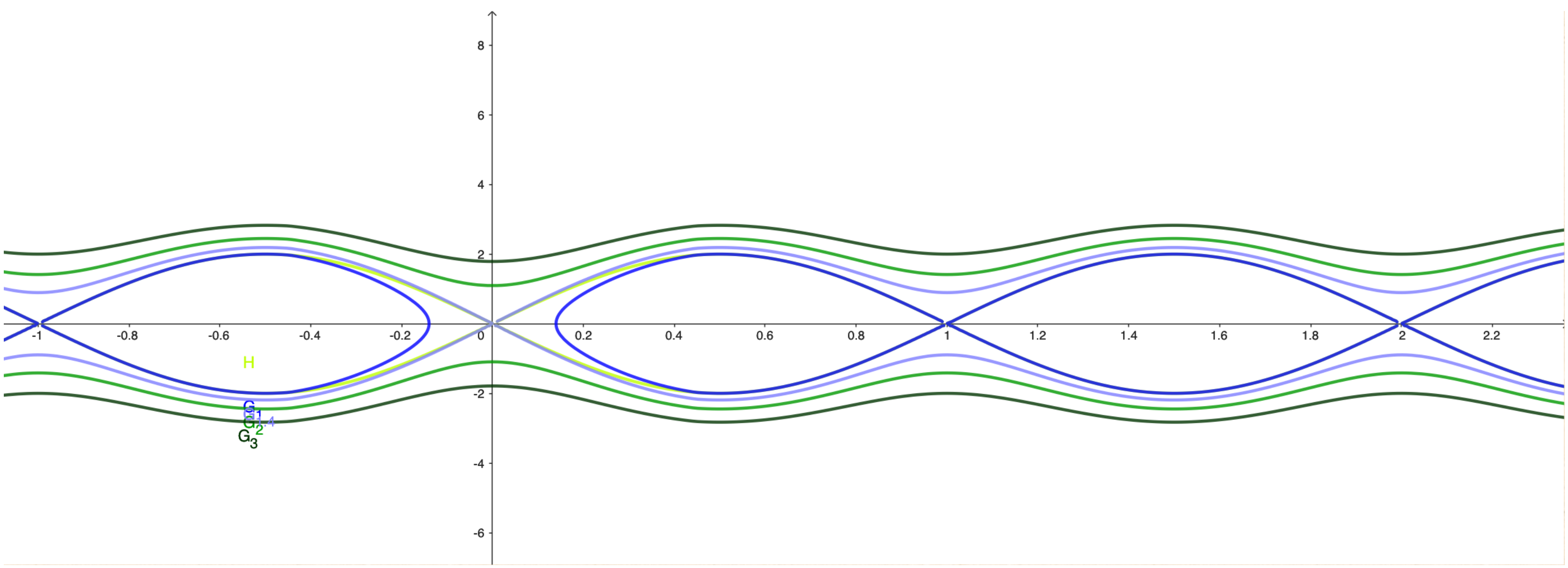
$c(G_\theta) > c(H_\theta)$ $\forall\, \theta \in (\theta^-, \theta^+)$ & $c_f(G) > c_f(H) = 1$



The vanishing discount problem: the periodic case

$H \in C(T^*M)$, $\qquad M := \mathbb{T}^d \cong \mathbb{R}^d / \mathbb{Z}^d$

(H1') (superlinearity) $\quad \inf_{x \in M} \dfrac{H(x,p)}{|p|} \longrightarrow +\infty \quad$ as $\quad |p| \to +\infty$

(H2) (convexity) $\quad H(x,\cdot)$ convex $\quad \forall x \in M$.

Theorem ([DFIZ]) The unique bounded solution u_H^λ of

$$\lambda u_H^\lambda(x) + H(x, d_x u_H^\lambda) = c(H) \quad \text{in } M \qquad (EH_\lambda)$$

uniformly converges on M as $\lambda \to 0^+$ to a solution u_H^0 of

$$H(x, d_x u) = c(H) \quad \text{in } M \qquad (EH_0)$$

Furthermore

Mather measures

$$u_H^0(x) := \sup\left\{ v(x) : v \text{ critical subsolution}, \int_{TM} v \, d\tilde{\mu} \le 0 \quad \forall \tilde{\mu} \in \widetilde{\mathfrak{M}}(H) \right\}, \quad x \in M.$$

Definition: A measure $\tilde{\mu} \in \mathcal{P}(TM)$ is termed closed if

(a) $\displaystyle\int_{TM} |q| \, d\tilde{\mu}(x,q) < +\infty$;

(b) $\displaystyle\int_{TM} \langle d_x\varphi, q \rangle \, d\tilde{\mu}(x,q) = 0 \qquad \forall \varphi \in C^1(M)$.

Example: Let $\gamma : [0,T] \to M$ be an absolutely continuous closed curve. Define a measure $\tilde{\mu}_\gamma \in \mathcal{P}(TM)$ via

$$\langle \tilde{\mu}_\gamma, \psi \rangle := \frac{1}{T} \int_0^T \psi(\gamma(s), \dot{\gamma}(s)) \, ds \qquad \forall \psi \in C_b(TM)$$

(a) $\displaystyle\int_{TM} |q| \, d\tilde{\mu}_\gamma(x,q) = \frac{1}{T}\int_0^T |\dot{\gamma}(s)| \, ds < +\infty$

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(b) $\forall \varphi \in C^1(M)$

$$\int_{TM} \langle d_x\varphi, q \rangle \, d\tilde{\mu}_\gamma(x,q) = \frac{1}{T} \int_0^T \langle d_{\gamma(s)}\varphi, \dot{\gamma}(s) \rangle \, ds = \frac{1}{T} \int_0^T \frac{d}{ds} \varphi(\gamma(s)) \, ds$$

$$= \frac{1}{T} \left(\varphi(\gamma(T)) - \varphi(\gamma(0)) \right) = 0 \qquad \text{since } \gamma(T) = \gamma(0).$$

Thm:

$$\min_{\substack{\tilde{\mu} \in \mathcal{P}(TM) \\ \tilde{\mu} \text{ closed}}} \int_{TM} L_H(x,q)\, d\tilde{\mu}(x,q) = -c(H).$$

$$L_H(x,q) := \sup_p \langle p,q \rangle - H(x,p) \qquad \forall (x,q) \in TM$$

Definition: The set of Mather measures is defined as

$$\tilde{\mathfrak{M}}(H) := \left\{ \tilde{\mu} \in \mathcal{P}(TM) : \tilde{\mu} \text{ closed},\ \int_{TM} L\, d\tilde{\mu} = -c(H) \right\}.$$

Proof of the asymptotic Convergence

$$\lambda u_H^\lambda(x) + H(x, d_x u_H^\lambda) = c(H) \quad \text{in } M \qquad (EH_\lambda)$$

$$H(x, d_x u) = c(H) \quad \text{in } M \qquad (EH_0)$$

$$u_H^0(x) := \sup \left\{ v(x) : v \text{ critical subsolution}, \int_{TM} v \, d\tilde{\mu} \leq 0 \ \ \forall \tilde{\mu} \in \tilde{\mathfrak{M}}(H) \right\}, \ x \in M$$

Let $u_H^{\lambda_n} \rightrightarrows u$ in M for some $\lambda_n \to 0^+$

Step 1 : $u \leq u_H^0$

Proposition: $\int_M u_H^\lambda \, d\tilde{\mu} \leq 0 \quad \forall \tilde{\mu} \in \mathfrak{M}(\tilde{H}), \ \forall \lambda > 0$.

Step 2: $u \geq u_H^0$

$$u_H^\lambda(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} e^{\lambda s}\left(L_H(\gamma(s),\dot\gamma(s)) + c(H)\right) ds$$

Thm: $\forall x \in M$ and $\lambda > 0$, the above inf is attained by a Lipschitz curve $\gamma_x^\lambda : (-\infty, 0] \to M$ with $\gamma_x^\lambda(0) = x$. Furthermore, $\|\dot\gamma_x^\lambda\|_\infty \leq \alpha$ for some constant α independent of λ and x.

Idea: $\forall x \in M$ and $\lambda > 0$, define $\tilde\mu_x^\lambda \in \mathcal{P}(TM)$ as

$$\int_{TM} f(y,q)\, d\tilde\mu_x^\lambda(y,q) := \int_{-\infty}^{0} \lambda e^{\lambda s} f(\gamma_x^\lambda(s), \dot\gamma_x^\lambda(s))\, ds \quad \forall f \in C_b(TM).$$

Proposition: Let v be a critical subsolution. Then $\forall x \in M$ and $\lambda > 0$

$$u_H^\lambda(x) \geq v(x) - \int_{TM} v(y)\, d\tilde{\mu}_x^\lambda(y,q) \,. \qquad (*)$$

Proposition: Let v be a critical subsolution. Then $\forall x \in M$ and $\lambda > 0$

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Proposition: Let $x \in M$ and $\lambda_n \to 0^+$. Then, up to subsequences,

$$\tilde{\mu}_x^{\lambda_n} \rightharpoonup \tilde{\mu}_x \qquad \text{in } \mathcal{P}(TM)$$

for some Mather measure $\tilde{\mu}_x \in \tilde{\mathfrak{M}}(H)$.

$\operatorname{spt}(\tilde{\mu}_x^\lambda) \subseteq \mathbb{T}^d \times \bar{B}_\alpha$
which is compact

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Proof of Step 2: If $u_H^{\lambda_n} \rightrightarrows u$, by taking the limit of $(*)$ we get

$$u(x) \geq v(x) - \int_{TM} v\, d\tilde{\mu}_x \quad \text{for some } \tilde{\mu}_x \in \tilde{\mathfrak{M}}(H).$$

Hence

$$u(x) \geq u_H^0(x) = \sup\left\{ v(x) : \begin{array}{l} v \text{ critical subsolution} \\ \int_{TM} v\, d\tilde{\mu} \leq 0 \ \forall \tilde{\mu} \in \tilde{\mathfrak{M}}(H) \end{array} \right\}.$$

$\square$

The vanishing discount problem for $G(x,p) = H(x,p) - V(x)$

$H \in C(\mathbb{R}\times\mathbb{R})$ 1-periodic in x as before, $V \in C_c(\mathbb{R})$

What are the difficulties?

1. Extension of Mather theory to a non-compact setting

Definition: A measure $\tilde{\mu} \in \mathcal{P}(\mathbb{R}\times\mathbb{R})$ is termed closed if

(a) $\displaystyle\int_{\mathbb{R}\times\mathbb{R}} |q| \, d\tilde{\mu}(x,q) < +\infty$;

(b) $\displaystyle\int_{\mathbb{R}\times\mathbb{R}} \varphi'(x)\, q \, d\tilde{\mu}(x,q) = 0 \qquad \forall \varphi \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$

Thm: Let $L_G(x,q) = L_H(x,q) + V(x)$. Then

$$\inf_{\substack{\tilde{\mu}\in\mathcal{P}(\mathbb{R}\times\mathbb{R})\\ \tilde{\mu}\text{ closed}}} \int_{\mathbb{R}\times\mathbb{R}} L_G(x,q)\, d\tilde{\mu}(x,q) = -c_f(G)$$

it holds in any dimension

So minimizing Mather measures may exist at level $c_f(G)$ only.

So minimizing Mather measures may exist at level $c_f(G)$ only.

In dimension 1, Mather measures always exist.

Thm: The following holds:

$$c_f(G) = \max_{x\in\mathbb{R}} \min_{p\in\mathbb{R}} G(x,p).$$

Set $\mathcal{E}(G) := \{ y : \min_p G(y,p) = c_f(G) \}$ set of equilibria

Corollary: $\{ \delta_{(y,0)} \in \mathcal{P}(\mathbb{R}\times\mathbb{R}) : y \in \mathcal{E}(G) \} \subseteq \widetilde{\mathcal{M}}(G)$.

Proof: In fact, let $\tilde{\mu} := \delta_{(y,0)}$ for some $y \in \mathcal{E}(G)$. Then

$$\int_{\mathbb{R}\times\mathbb{R}} L_G \, d\tilde{\mu} = L_G(y,0) = \max_p \{ p\cdot 0 - G(x,p) \} = -c_f(G). \quad \square$$

2. Dispersion of mass at infinity

$$\lambda u_G^\lambda(x) + G(x, d_x u_G^\lambda) = c(G) \qquad \text{in } \mathbb{R} \qquad (EG_\lambda)$$

The following variational formula holds:

$$u_G^\lambda(x) = \min_{\gamma(0)=x} \int_{-\infty}^{0} e^{\lambda s}\big(L_G(\gamma(s), \dot\gamma(s)) + c(G) \big)\, ds$$

Fact: $\exists$ a constant $\alpha > 0$, independent of x and $\lambda > 0$, such that any minimizing $\gamma_x^\lambda : (-\infty, 0] \to \mathbb{R}$ for $u_G^\lambda(x)$ enjoys $\|\dot\gamma_x^\lambda\|_\infty \leq \alpha$.

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We can still define discounted measures $\tilde\mu_x^\lambda \in \mathcal{P}(\mathbb{R}\times\mathbb{R})$ via

$$\int_{\mathbb{R}\times\mathbb{R}} f(y,q)\, d\tilde\mu_x^\lambda(y,q) := \int_{-\infty}^{0} \lambda e^{\lambda s} f\left(\gamma_x^\lambda(s), \dot\gamma_x^\lambda(s)\right) ds \quad \forall f \in C_b(\mathbb{R}\times\mathbb{R})$$

and still have that

$$u_G^\lambda(x) \geq v(x) - \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde\mu_x^\lambda$$

for every (bounded) subsolution v of $G(x, v') = c(G)$ in $\mathbb{R}$.

2. Dispersion of mass at infinity

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Problem: Since the base manifold $\mathbb{R}$ is not compact, each family of measures $\{\tilde\mu_x^\lambda : \lambda > 0\}$ is not relatively compact in $\mathcal{P}(\mathbb{R}\times\mathbb{R})$.

Proof of the asymptotic Convergence

$$\lambda u_G^\lambda(x) + G(x, d_x u_G^\lambda) = c(G) \quad \text{in } \mathbb{R} \qquad (EG_\lambda)$$

$$G(x, d_x u) = c(G) \quad \text{in } \mathbb{R} \qquad (EG_0)$$

There are 3 cases :

(I) $c(G) > c(H)$

(II) $c(G) = c(H) > c_f(H)$

(III) $c(G) = c(H) = c_f(H)$

Case (I) : $c(G) > c(H)$

$$G(x, u') = c(G) \qquad \text{in } \mathbb{R} \qquad (EG_0)$$

Theorem: $c(G) = c_f(G)$.

Case (I) : $c(G) > c(H)$

$$G(x, u') = c(G) \quad \text{in } \mathbb{R} \qquad (EG_0)$$

Theorem: $\exists$ compact set $K \supseteq \operatorname{supp}(V)$ and a bounded subsol. v_G to (EG_0) which is uniformly strict in $\mathbb{R} \setminus K$, i.e.,

$$G(x, v_G'(x)) \leq c(G) - \delta \qquad \text{for a.e. } x \in \mathbb{R} \setminus K$$

for some $\delta > 0$.

Case (I) : $c(G) > c(H)$

$G(x, u') = c(G)$ in $\mathbb{R}$ $\quad (EG_0)$

<u>Theorem</u>: $\exists$ compact set $K \supseteq \mathrm{supp}(V)$ and a bounded subsol. v_G to (EG_0) which is uniformly strict in $\mathbb{R} \setminus K$, i.e.,

$$G(x, v_G'(x)) \leq c(G) - \delta \qquad \text{for a.e. } x \in \mathbb{R} \setminus K$$

for some $\delta > 0$.

<u>Consequence</u>: $\forall x \in \mathbb{R}$, $\tilde{\mu}_x^\lambda\big((\mathbb{R} \setminus K) \times \mathbb{R}\big) \to 0$ as $\lambda \to 0^+$.

In particular, $\{\tilde{\mu}_x^\lambda : \lambda > 0\}$ is pre-compact in $\mathcal{P}(\mathbb{R} \times \mathbb{R})$

We can proceed as in the compact (periodic) case.

Theorem: $u_G^\lambda \rightrightarrows u_G^0$, where u_G^0 is the critical solution defined as

$$u_G^0(x) = \sup\left\{ v(x) : v \text{ critical subsolution}, \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu} \leq 0 \ \forall \tilde{\mu} \in \tilde{\mathfrak{M}}(G) \right\}.$$

Furthermore, u_G^0 is coercive.

Remarks:

- The result (and the proof) is analogous to the corresponding one in the compact setting
- Our proof works in $\mathbb{R}^d$ for any $d \geq 1$ (with minor modifications).
- This case is similar (in spirit) to the one considered by H. Ishii and A. Siconolfi in [IS20].

Case II & III : $c(G) = c(H)$

set
$\boxed{c := c(G) = c(H)}$

$$\lambda u_H^\lambda(x) + H(x, d_x u_H^\lambda) = c \quad \text{in } \mathbb{R} \qquad (EH_\lambda)$$

$$\lambda u_G^\lambda(x) + G(x, d_x u_G^\lambda) = c \quad \text{in } \mathbb{R} \qquad (EG_\lambda)$$

$$u_H^\lambda(x) = \min_{\gamma(0)=x} \int_{-\infty}^0 e^{\lambda s} \left(L_H(\gamma(s), \dot\gamma(s)) + c \right) ds$$

$\boxed{L_G(x,q) = L_H(x,q) + V(x)}$

$$u_G^\lambda(x) = \min_{\gamma(0)=x} \int_{-\infty}^0 e^{\lambda s} \left(L_G(\gamma(s), \dot\gamma(s)) + c \right) ds$$

Fact: $u_H^\lambda \rightrightarrows u_H^0$

Remark: $u_G^\lambda(x) \cong u_H^\lambda(x)$ for $|x| >> 1$

Notation: Let us denote by $\mathfrak{S}_{b-}(G)$ the family of bounded subsolutions v of $G(x,u')=c$ in $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}\times\mathbb{R}} v(y)\,d\tilde{\mu}(y,q) \leq \mathbb{1}_0\big(c(G)-c_f(G)\big) \qquad \forall\, \tilde{\mu}\in\widetilde{\mathfrak{M}}(G)$$

where

$$\mathbb{1}_0(t)=\begin{cases} 0 & \text{if } t=0 \\ +\infty & \text{if } t\neq 0. \end{cases}$$

No constraint when $c(G)>c_f(G)$.

Case II: $c(G) = c(H) > c_f(H)$

$$H(x, u') = c \qquad \text{in } \mathbb{R} \qquad (EH_0)$$

Set

$$Z_H(x) := \{p : H(x,p) \leq c\} = [p_H^-(x), p_H^+(x)]$$

There are 2 subcases

(A) $\int_0^1 p_H^+(x)\,dx = 0 > \int_0^1 p_H^-(x)\,dx$

1-periodic solutions to (EH_0) are of the form $\int_0^x p_H^+(z)\,dz + \text{Const.}$

(B) $\int_0^1 p_H^+(x)\,dx > 0 = \int_0^1 p_H^-(x)\,dx$

1-periodic solutions to (EH_0) are of the form $\int_0^x p_H^-(z)\,dz + \text{Const}$

$\boxed{\text{Let us focus on case II-(A)}}$

__Prop:__ There exists $\lambda_0 > 0$ such that $\forall \lambda \in (0, \lambda_0)$ we have:

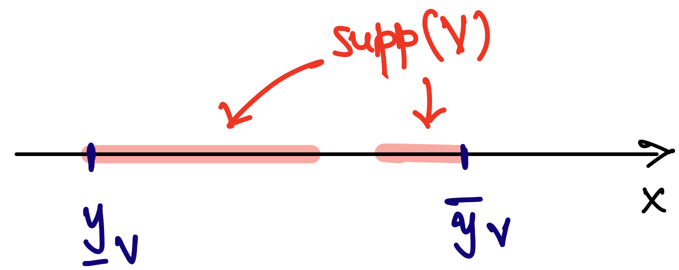
(a) If $\gamma : (-\infty, 0] \to \mathbb{R}$ is optimal for $u_H^\lambda(x)$, then

$$\gamma(t) < x \qquad \forall t < 0;$$

(b) If $\gamma : (-\infty, 0] \to \mathbb{R}$ is optimal for $u_G^\lambda(x)$, then

$$\gamma(t) \leq \max\{x, \bar{y}_V\} \qquad \forall t < 0,$$

where $\bar{y}_V = \max(\operatorname{supp}(V))$.



__Corollary:__ $\forall \lambda \in (0, \lambda_0)$ we have

$$u_G^\lambda \leq u_H^\lambda \quad \text{in} \quad (-\infty, \underline{y}_V) \qquad \text{where} \quad \underline{y}_V = \min(\operatorname{supp}(V)).$$

$\boxed{\text{Let us focus on case II-(A)}}$

<u>Prop</u>: There exists $\lambda_0 > 0$ such that $\forall \lambda \in (0, \lambda_0)$ we have:

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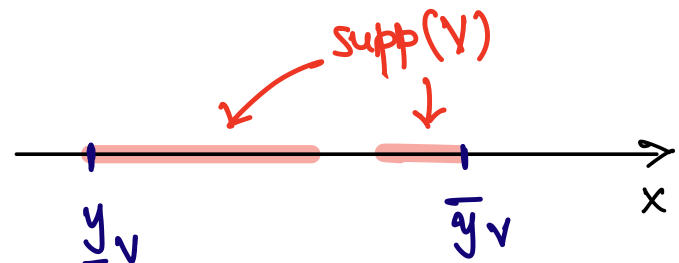
(b) If $\gamma : (-\infty, 0] \to \mathbb{R}$ is optimal for $u_G^\lambda(x)$, then

$$\gamma(t) \leq \max\{x, \bar{y}_\nu\} \qquad \forall t < 0,$$

where $\bar{y}_\nu = \max(\operatorname{supp}(\nu))$.



<u>Corollary</u>: Let $\lambda \in (0, \lambda_0)$ and γ optimal for $u_G^\lambda(x)$.

If γ is not bounded, then $\lim_{t \to -\infty} \gamma(t) = -\infty$.

Thm: $u_G^\lambda \rightrightarrows u_G^0$ in $\mathbb{R}$ as $\lambda \to 0^+$, where

$$u_G^0(x) := \sup\{ v(x) : v \in \mathcal{S}_{b-}(G),\ v \leq u_H^0 \text{ in } (-\infty, \underline{y}_v) \}.$$

Proof: Let $u_G^{\lambda_n} \rightrightarrows u$ for some $\lambda_n \to 0^+$

Step 1: $u \leq u_G^0$.

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Proof: Let $u_G^{\lambda_n} \rightrightarrows u$ for some $\lambda_n \to 0^+$

Step 1: $u \leq u_G^0$.

Step 2: $u \geq u_G^0$

Fix $\lambda \in (0, \lambda_0)$, $x \in \mathbb{R}$ and $v \in \mathcal{S}_{b-}(G)$ with $v \leq u_H^0$ in $(-\infty, \underline{y}_v)$.

Then

$$u_G^\lambda(x) \geq v(x) - \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}_x^\lambda$$

Idea: Choose a ball B_r containing $\{x\} \cup \mathrm{supp}(V)$ and write

$$\tilde{\mu}^{\lambda}_{x} = \theta^{\lambda}_{x} \tilde{\mu}^{\lambda}_{1,x} + (1 - \theta^{\lambda}_{x}) \tilde{\mu}^{\lambda}_{2,x} \quad \text{with } \tilde{\mu}^{\lambda}_{1,x}, \mu^{\lambda}_{2,x} \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$$

satisfying $\mathrm{supp}(\tilde{\mu}^{\lambda}_{1,x}) \subseteq \overline{B_r} \times \overline{B_\alpha}$, $\mathrm{supp}(\tilde{\mu}^{\lambda}_{2,x}) \subseteq (\mathbb{R} \setminus \overline{B_r}) \times \overline{B_\alpha}$.

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This is done by setting

$$\int_{\mathbb{R}\times\mathbb{R}} f \, d\tilde{\mu}_{1,x}^\lambda := \frac{1}{\theta_x^\lambda} \int_{-\infty}^0 \chi_{\overline{B}_r}(\gamma_x^\lambda(s))\, \lambda e^{\lambda s} f(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s))\, ds$$

(if $\theta_x^\lambda \in (0,1]$)

$$\int_{\mathbb{R}\times\mathbb{R}} f \, d\tilde{\mu}_{2,x}^\lambda := \frac{1}{1-\theta_x^\lambda} \int_{-\infty}^0 \chi_{\mathbb{R}\setminus\overline{B}_r}(\gamma_x^\lambda(s))\, \lambda e^{\lambda s} f(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s))\, ds$$

(if $\theta_x^\lambda \in [0,1)$)

for every $f \in C_b(\mathbb{R}\times\mathbb{R})$.

Proposition: $\{\tilde{\mu}^{\lambda}_{1,\alpha} : \lambda > 0\}$ is pre-compact in $\mathcal{P}(\mathbb{R} \times \mathbb{R})$.

Proof: $\operatorname{supp}(\tilde{\mu}^{\lambda}_{1,\alpha}) \subseteq \overline{B}_r \times \overline{B}_\alpha \quad \forall \lambda > 0.$ $\square$

Proposition: $\{\tilde{\mu}^{\lambda}_{1,x} : \lambda > 0\}$ is pre-compact in $\mathcal{P}(\mathbb{R}\times\mathbb{R})$.

Proof: $\operatorname{supp}(\tilde{\mu}^{\lambda}_{1,x}) \subseteq \overline{B}_r \times \overline{B}_\alpha \quad \forall \lambda > 0.$ $\square$

The probability measures $\tilde{\mu}^{\lambda}_{2,x}$ do not see the perturbation of H by the potential V, so we can exploit the compactness hidden in the model.

$$\pi : R \times R \longrightarrow \mathbb{T}^1 \times R \qquad \text{projection}$$

Proposition: $\{\pi_{\#}\tilde{\mu}^{\lambda}_{2,x} : \lambda > 0\}$ is pre-compact in $\mathcal{P}(\mathbb{T}^1 \times \mathbb{R})$.

Thm.: Let $\lambda_n \searrow 0$.

(a) If $\theta_x^{\lambda_n} \to \theta > 0$ and $\tilde{\mu}_{1,x}^{\lambda_n} \to \tilde{\mu}_1$ in $\mathcal{P}(\mathbb{R}\times\mathbb{R})$, then $c(G) = c_f(G)$ and $\tilde{\mu}_1 \in \widetilde{\mathcal{M}}(G)$.

(b) If $\theta_x^{\lambda_n} \to \theta < 1$ and $\pi_{\#}\tilde{\mu}_{2,x}^{\lambda_n} \to \tilde{\mu}_2$ in $\mathcal{P}(\mathbb{T}^1 \times \mathbb{R})$, then $c(G) = c(H)$ and $\tilde{\mu}_2 \in \widetilde{\mathcal{M}}(H)$.

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(a) If $\theta_x^{\lambda_n} \to \theta > 0$ and $\tilde{\mu}_{1,x}^{\lambda_n} \to \tilde{\mu}_1$ in $\mathcal{P}(\mathbb{R}\times\mathbb{R})$, then $c(G) = c_f(G)$ and $\tilde{\mu}_1 \in \widetilde{\mathcal{M}}(G)$.

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Crucial point: prove that $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are closed

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Answer: OK if $\#\{t<0 : \gamma_x^\lambda(t) \in \partial B\} \cdot \lambda \to 0$ as $\lambda \to 0^+$.

Thm.: Let $\lambda_n \searrow 0$.

(a) If $\theta_x^{\lambda_n} \to \theta > 0$ and $\tilde{\mu}_{1,x}^{\lambda_n} \to \tilde{\mu}_1$ in $\mathcal{P}(\mathbb{R}\times\mathbb{R})$, then $c(G) = c_f(G)$ and $\tilde{\mu}_1 \in \widetilde{\mathcal{M}}(G)$.

(b) If $\theta_x^{\lambda_n} \to \theta < 1$ and $\pi_{\#}\tilde{\mu}_{2,x}^{\lambda_n} \to \tilde{\mu}_2$ in $\mathcal{P}(\mathbb{T}^1 \times \mathbb{R})$, then $c(G) = c(H)$ and $\tilde{\mu}_2 \in \widetilde{\mathcal{M}}(H)$.

Crucial point: prove that $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are closed

Answer: OK if $\#\{t<0 : \gamma_x^\lambda(t) \in \partial B\} \cdot \lambda \to 0$ as $\lambda \to 0^+$.

In dimension 1 this is true due to the following

Thm: For every $x \in \mathbb{R}$ and $\lambda > 0$, the optimal curve γ_x^λ for $u_G^\lambda(x)$ is monotone.

$$u^\lambda_G(x) \geq v(x) - \left(\theta^\lambda_x \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}^\lambda_{1,x} + (1-\theta^\lambda_x) \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}^\lambda_{2,x} \right)$$

$$u_G^\lambda(x) \geq v(x) - \left(\theta_x^\lambda \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}_{1,x}^\lambda + (1-\theta_x^\lambda) \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}_{2,x}^\lambda \right)$$

$v \leq u_H^0$ in $(-\infty, \underline{y}_Y)$

$$\geq v(x) - \left(\theta_x^\lambda \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}_{1,x}^\lambda + (1-\theta_x^\lambda) \int_{\mathbb{R}\times\mathbb{R}} u_H^0 \, d\tilde{\mu}_{2,x}^\lambda \right)$$

$$u^\lambda_G(x) \geq v(x) - \left(\theta^\lambda_x \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}^\lambda_{1,x} + (1-\theta^\lambda_x)\int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}^\lambda_{2,x}\right)$$

$v \leq u^0_H$ in $(-\infty, \underline{y}_\lambda)$

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$$= v(x) - \left(\theta^\lambda_x \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}^\lambda_{1,x} + (1-\theta^\lambda_x)\int_{\pi^1\times\mathbb{R}} u^0_H \, d\pi_\#\tilde{\mu}^\lambda_{2,x}\right)$$

$$u^\lambda_G(x) \geq v(x) - \left(\theta^\lambda_x \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}^\lambda_{1,x} + (1-\theta^\lambda_x) \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}^\lambda_{2,x} \right)$$

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$$\geq v(x) - \left(\theta^\lambda_x \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}^\lambda_{1,x} + (1-\theta^\lambda_x) \int_{\mathbb{R}\times\mathbb{R}} u^0_H \, d\tilde{\mu}^\lambda_{2,x} \right)$$

$$= v(x) - \left(\theta^\lambda_x \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}^\lambda_{1,x} + (1-\theta^\lambda_x) \int_{\Pi^1\times\mathbb{R}} u^0_H \, d\,\pi_\# \tilde{\mu}^\lambda_{2,x} \right)$$

Now set $\lambda := \lambda_n$. Up to subsequences, $\theta^{\lambda_n}_x \to \theta \in [0,1]$,

$\tilde{\mu}^\lambda_{1,x} \xrightarrow{*} \tilde{\mu}^\lambda_1$ in $\mathcal{P}(\mathbb{R}\times\mathbb{R})$, $\pi_\#\tilde{\mu}^\lambda_{2,x} \xrightarrow{*} \tilde{\mu}_2$ in $\mathcal{P}(\Pi^1\times\mathbb{R})$

and

$$u(x) \geq v(x) - \left(\theta \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}_1 + (1-\theta) \int_{\Pi^1\times\mathbb{R}} u^0_H \, d\tilde{\mu}_2 \right).$$

$$u(x) \geq v(x) - \left(\theta \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}_1 + (1-\theta) \int_{\mathbb{T}^1\times\mathbb{R}} u^\circ_H \, d\tilde{\mu}_2 \right).$$

Furthermore

$$\underset{\displaystyle\downarrow}{v \in \mathcal{S}_{b,-}(G)}$$

- $\theta \neq 0 \Longrightarrow c(G) = \mathfrak{f}(G)$, $\tilde{\mu}_1 \in \widetilde{\mathfrak{M}}(G)$ and $\int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu}_1 \leq 0$

- $\theta \neq 1 \Longrightarrow \tilde{\mu}_2 \in \widetilde{\mathfrak{M}}(H)$ and $\int_{\mathbb{T}^1\times\mathbb{R}} u^\circ_H \, d\tilde{\mu}_2 \leq 0$.

by definition of u°_H

In any case, we get $u(x) \geq v(x)$, hence

$$u(x) \geq u^\circ_G(x) := \sup \{ v(x) : v \in \mathcal{S}_{b,-}(G),\ v \leq u^\circ_H \text{ in } (-\infty, \underline{y}_v) \}.$$

$\square$

Case III: $c(G) = c(H) = c_f(H)$

Since $c(G) \geq c_f(G) \geq c(H)$, we also have $c(G) = c_f(G)$.

$\underline{\text{Thm}}$: $u_G^\lambda \rightrightarrows u_G^0$ in $\mathbb{R}$ as $\lambda \to 0^+$, where

$$u_G^0(x) := \sup \left\{ v(x) \;\middle|\; \begin{array}{l} v \text{ critical subsolution} \\ \displaystyle\int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu} \leq 0 \quad \forall \tilde{\mu} \in \tilde{\mathfrak{M}}(G) \end{array} \right\}$$

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Prop : $\forall\, x \in \mathbb{R}$ we have

$$u_G^0(x) = \sup \left\{ v(x) : v \text{ critical subsolution for } G,\ v \leq 0 \text{ on } \mathcal{E}(G) \right\}.$$

We assume that a similar property holds for u^0_H as well, i.e.,

$$u^0_H(x) = \sup\left\{ v(x) : \begin{array}{l} v \text{ critical 1-periodic subsol. for } H \\ v \leq 0 \text{ on } \mathcal{E}(H) \end{array} \right\} \qquad (U)$$

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<u>**Remark:**</u> Condition (U) is fullfilled when H is Tonelli.

M. Zavidovique provided us an example of $H \in C^0(\mathbb{R}\times\mathbb{R})$ but $H \notin C^1(\mathbb{R}\times\mathbb{R})$ for which condition (U) is violated.

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Remark: Condition (U) is fulfilled when H is Tonelli.

M. Zavidovique provided us an example of $H \in C^0(\mathbb{R}\times\mathbb{R})$ but $H \notin C^1(\mathbb{R}\times\mathbb{R})$ for which condition (U) is violated.

Prop.: Assume condition (U) holds. Then

$$u^0_H \geq u^0_G \quad \text{in } \mathbb{R} \setminus [\underline{y}_V - 1, \overline{y}_V + 1],$$

where $\underline{y}_V = \min(\operatorname{supp}(V))$ and $\overline{y}_V = \max(\operatorname{supp}(V))$.

Proof of Thm: Let $\lambda_n \to 0$ such that $u_G^{\lambda_n} \rightrightarrows u$ in $\mathbb{R}$

Step 1: $u \leq u_G^0$ easy

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Step 2: $u \geq u_G^0$

We have

$$u_G^\lambda(x) \geq u_G^0(x) - \int_{\mathbb{R}\times\mathbb{R}} u_G^0 \, d\tilde{\mu}_x^\lambda$$

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We have

$$u_G^\lambda(x) \geq u_G^0(x) - \int_{\mathbb{R}\times\mathbb{R}} u_G^0 \, d\tilde{\mu}_x^\lambda$$

Pick $r > 0$ such that $\{x\} \cup [\underline{y}_V - 1, \overline{y}_V + 1] \subseteq B_r$ and write $\tilde{\mu}_x^\lambda$ as

$$\tilde{\mu}_x^\lambda = \theta_x^\lambda \tilde{\mu}_{1,x}^\lambda + (1 - \theta_x^\lambda) \tilde{\mu}_{2,x}^\lambda$$

with $\operatorname{supp}(\tilde{\mu}_{1,x}^\lambda) \subseteq \overline{B}_r \times \overline{B}_\alpha$ and $\operatorname{supp}(\tilde{\mu}_{2,x}^\lambda) \subseteq (\mathbb{R} \setminus \overline{B}_r) \times \overline{B}_\alpha$.

We get

$$u_G^\lambda(x) \geq u_G^0(x) - \left(\theta_x^\lambda \int_{\mathbb{R}\times\mathbb{R}} u_G^0 \, d\tilde{\mu}_{1,x}^\lambda + (1-\theta_x^\lambda) \int_{\mathbb{R}\times\mathbb{R}} u_G^0 \, d\tilde{\mu}_{2,x}^\lambda \right)$$

We get

$$u_G^\lambda(x) \geq u_G^0(x) - \left(\theta_x^\lambda \int_{R\times R} u_G^0 \, d\tilde{\mu}_{1,x}^\lambda + (1-\theta_x^\lambda) \int_{R\times R} u_G^0 \, d\tilde{\mu}_{2,x}^\lambda \right)$$

$u_G^0 \leq u_H^0$ on $\mathrm{supp}(\tilde{\mu}_{2,x}^\lambda)$

$$\geq u_G^0(x) - \left(\theta_x^\lambda \int_{R\times R} u_G^0 \, d\tilde{\mu}_{1,x}^\lambda + (1-\theta_x^\lambda) \int_{R\times R} u_H^0 \, d\tilde{\mu}_{2,x}^\lambda \right)$$

We get

$$u^\lambda_G(x) \geqslant u^0_G(x) - \left(\theta^\lambda_x \int_{R\times R} u^0_G \, d\tilde{\mu}^\lambda_{1,x} + (1-\theta^\lambda_x) \int_{R\times R} u^0_G \, d\tilde{\mu}^\lambda_{2,x} \right)$$

$u^0_G \leq u^0_H$ on $\mathrm{supp}(\tilde{\mu}^\lambda_{2,x})$

$$\geqslant u^0_G(x) - \left(\theta^\lambda_x \int_{R\times R} u^0_G \, d\tilde{\mu}^\lambda_{1,x} + (1-\theta^\lambda_x) \int_{R\times R} u^0_H \, d\tilde{\mu}^\lambda_{2,x} \right)$$

$$= u^0_G(x) - \left(\theta^\lambda_x \int_{R\times R} u^0_G \, d\tilde{\mu}^\lambda_{1,x} + (1-\theta^\lambda_x) \int_{T^1_x R} u^0_H \, d\pi_\# \tilde{\mu}^\lambda_{2,x} \right)$$

We conclude by arguing as in case II. $\square$

CONCLUSION: the limit solution u_G^0 is identified as:

(I) if $c(G) > c_f(G)$, then

$$u_G^0(x) = \sup \left\{ v(x) : v \text{ bdd critical subsol.}, \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu} \leq 0 \ \ \forall \mu \in \tilde{\mathcal{U}}(G) \right\}$$

(II-A) if $c(G) = c(H) > c_f(H)$ and $\int_0^1 p_H^+(x)\,dx = 0$

$$u_G^0(x) = \sup \left\{ v(x) \ \middle| \ \begin{array}{l} v \text{ bdd critical subsol.}, \quad v \leq u_H^0 \text{ in } (-\infty, \underline{y}_v) \\ \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu} \leq \mathbb{1}_0 \left(c(G) - c_f(G) \right) \end{array} \right\}$$

(III) if $c(G) = c(H) = c_f(H)$, then $c(G) = c_f(G)$ and

$$u_G^0(x) = \sup \left\{ v(x) : v \text{ bdd critical subsol.}, \int_{\mathbb{R}\times\mathbb{R}} v \, d\tilde{\mu} \leq 0 \ \ \forall \mu \in \tilde{\mathcal{U}}(G) \right\}$$

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Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations

Martino Bardi
Italo Capuzzo-Dolcetta

APPENDIX A

Numerical Solution of Dynamic Programming Equations

by Maurizio Falcone

As shown in the book, the Dynamic Programming approach to the solution of deterministic optimal control problems is essentially based on the characterization of the value function in terms of a partial differential equation, the Hamilton-Jacobi-Bellman equation. In the case of deterministic optimal control problems, which we consider here, this is a nonlinear first order equation $H(x, u(x), Du(x)) = 0$ where H is convex with respect to Du. However, this approach is very flexible since a similar characterization can be obtained for the value function of stochastic optimal control problems (in which case the equation will be of the second order and/or an integro-differential equation depending on the stochastic process describing the dynamics) and of differential games (in which case we still have a first order equation but we lose the convexity with respect to Du).

One feature of this approach is particularly interesting in many applications: it permits computation of approximate optimal controls in feedback form and, as a consequence, approximate optimal trajectories.

Although these are very appealing properties if compared to the results given by the open-loop approach based on the Pontryagin Maximum Principle, the Dynamic Programming approach suffers the problem of the "rise of dimension". In fact, to compute the value function we need to solve a Hamilton-Jacobi-Bellman equation in the domain where the initial condition for the dynamics is taken. On the other hand, the necessary conditions of the Pontryagin Maximum Principle often correspond to the solution of a two-point boundary value problem for a system of ordinary differential equations involving the state and the adjoint variables. This simple observation implies that the number of computations needed by the Dynamic Programming approach can be huge when the state variable belongs to a space of high dimension ($\mathbb{R}^N$ with $N = 4, 6$ or more). This is one of the main motivations to develop efficient algorithms to solve equations of this type. Of course, when comparing the Dynamic Programming approach with the Pontryagin

We will make the following assumptions on the data

(1.3) $\quad f : \mathbb{R}^N \times A \to \mathbb{R}^N$ and $\ell : \mathbb{R}^N \times A \to \mathbb{R}$ are continuous;

(1.4) $\quad |f(x_1, a) - f(x_2, a)| \le L_f |x_1 - x_2|$ for any $a \in A$ and $\|f\|_\infty \le M_f$;

(1.5) $\quad |\ell(x_1, a) - \ell(x_2, a)| \le L_\ell |x_1 - x_2|$ for any $a \in A$ and $\|\ell\|_\infty \le M_\ell$.

Assumptions (1.4) guarantee that there exists a unique solution trajectory y defined in $[0, +\infty[$ for any fixed control α. The value function v, defined for any initial state x, is

$$v(x) \equiv \inf_{\alpha \in \mathcal{A}} J_x(\alpha) \, .$$

1.1. The Dynamic Programming equation

Let us recall from Chapter III that the value function v of the *unconstrained* problem above is the unique viscosity solution of

$$\text{(HJ)} \qquad \lambda u(x) + \sup_{a \in A}\{-f(x, a) Du(x) - \ell(x, a)\} = 0, \qquad \text{for } x \in \mathbb{R}^N \, .$$

Making a discretization in time of the original control problem, which consists in replacing the dynamics (1.1) by a one-step scheme (e.g., by the Euler method) and the cost functional (1.2) by its discretization by a quadrature formula (e.g., the rectangle rule), one can get a new control problem in discrete time. The value function v_h for this problem (as we have seen in Chapter VI) satisfies a discrete dynamic programming principle which gives the following approximation scheme,

$$\text{(HJ)}_h \qquad u_h(x) = \min_{a \in A}\{(1 - \lambda h)\, u_h(x + h f(x, a)) + h\, \ell(x, a)\}, \text{ for } x \in \mathbb{R}^N \, .$$

Under our assumptions (1.3)–(1.5) and for $\lambda > L_f$, the family of functions v_h is equibounded (by M_f/λ) and equicontinuous

$$\text{(1.6)} \qquad |v_h(x) - v_h(y)| \le \frac{L_\ell}{\lambda - L_f} |x - y| \, .$$

Then, by the Ascoli-Arzelà theorem we can pass to the limit and prove that it converges locally uniformly to v for h going to 0. Moreover, the following estimate holds true,

$$\text{(1.7)} \qquad \|v - v_h\|_\infty \le C\, h^{1/2}$$

(see §VI.1). In order to compute an approximate value function and solve (HJ)_h we have to make a further step: a discretization in space.

We start building a grid in the state space and, to simplify our presentation, we will assume that there exists a bounded polyhedron $\Omega \subset \mathbb{R}^N$ such that, for h sufficiently small

$$\text{(1.8)} \qquad x + h f(x, a) \in \overline{\Omega} \qquad \forall x \in \overline{\Omega} \quad \forall a \in A$$

Thank you for your attention!