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ML and common noise price models

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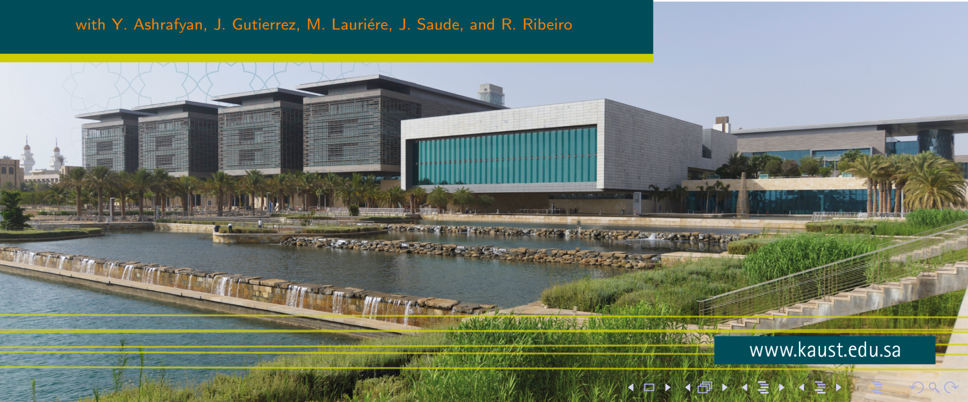


Outline

Overview

We consider the following price model:

- The model involves numerous agents trading a commodity (such as energy stored in batteries) continuously.
- Agents aim to maximize profit by trading at price $\varpi(t)$, determined by supply-demand balance.
- the supply, $\mathbf{Q}(t)$, is exogenous (and possibly stochastic).

Related references

- Basar and Srikant - revenue maximizing Stackelberg games
- Kizilkale and Malhamé - load adaptive pricing (see also Alasseur, Ben Taher, and Matoussi)
- Fujii and Takahashi - market clearing conditions with common noise
- Shrivats, Firoozi and Jaimungal - equilibrium pricing in solar renewable energy certificates

Deterministic Framework

The model involves three key variables:

- a price $\varpi \in C([0, T])$
- a value function $u \in C(\mathbb{R} \times [0, T])$
- a path describing the statistical distribution of the agents, $m \in C([0, T], \mathcal{P})$.

NOTE: $\mathcal{P}$ is the set of probabilities on $\mathbb{R}$ with finite second-moment endowed with the 1-Wasserstein distance.

Deterministic problem

Given $\epsilon \geq 0$, $H \in C^\infty$, a supply rate $\mathbf{Q} : [0, T] \to \mathbb{R}$, $\mathbf{Q} \in C^\infty$, solve

$$\begin{cases} -u_t + H(x, \varpi(t) + u_x) = \epsilon u_{xx} \\ m_t - (D_p H(x, \varpi(t) + u_x) m)_x = \epsilon m_{xx} \\ \int_\Omega D_p H(x, \varpi(t) + u_x) dm = -\mathbf{Q}(t), \end{cases}$$

with the initial-terminal conditions

$$\begin{cases} u(x, T) = \bar{u}(x), \\ m(x, 0) = \bar{m}(x), \end{cases}$$

where where $\bar{u}$, $\bar{m}$ are smooth, and $\bar{m}$ is a probability.

Main Result - deterministic case

Theorem (G. and Saúde)

Under natural assumptions, there exists a solution (u, m, ϖ):

- *u is a viscosity solution, Lipschitz and semiconcave in x, and differentiable almost everywhere with respect to m*
- *$m \in C([0, T], \mathcal{P})$*
- *ϖ is Lipschitz continuous on $[0, T]$.*

If $\epsilon > 0$ or if $\epsilon = 0$, under additional convexity assumptions, the solution is unique.

A stochastic PDE for common noise

Let Ω be a probability space and consider the supply

$$\begin{cases} \mathrm{d}Q(t) = b^S(Q(t), t)\mathrm{d}t + \sigma^S(Q(t), t)\mathrm{d}W(t), \\ Q(0) = q_0. \end{cases}$$

Find $m : [0, T] \times \mathbb{R} \to \mathbb{R}$, $u, Z : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$, and $\varpi : [0, T] \times \Omega \to \mathbb{R}$ progressively measurable, satisfying $m \geqslant 0$ and

$$\begin{cases} -\mathrm{d}u + H(x, \varpi + u_x)\mathrm{d}t = Z(t, x)\mathrm{d}W(t), \\ u(T, x) = u_T(x), \\ m_t - (H_p(x, \varpi + u_x)m)_x = 0, \\ m(0, x) = m_0(x), \\ -\int_{\mathbb{R}} H_p(x, \varpi + u_x)m\mathrm{d}x = Q(t). \end{cases}$$

The model - deterministic case

To simplify, we set $\epsilon = 0$.

- Each consumer has a storage device connected to the network.
- Consumers trade electricity, charging the batteries when price is low and selling electricity when price is high.
- Consumers take into account price and battery wear.

The control problem

- Each agent battery's charge $\mathbf{x}(t)$ changes according to

$$\dot{\mathbf{x}}(t) = \alpha(t).$$

- Each agent selects α to minimize the cost

$$J(x,t,\alpha) = \int_t^T \ell(\mathbf{x}(t), \alpha(s), t)ds + \bar{u}(\mathbf{x}(T)),$$

where the Lagrangian is

$$\ell(x,\alpha,t) = \ell_0(x,\alpha) + \varpi(t)\alpha(t).$$

and the terminal cost, $\bar{u}$ is given.

Running cost as a price impact

For example, if

$$\ell_0(x, \alpha, t) = \frac{c}{2}\alpha^2(t) + V(x).$$

the running term $\frac{c}{2}\alpha^2(t)$ can also be seen as a (temporary) price impact:

- Agents trading at a rate α pay an effective price

$$\varpi + \frac{c}{2}\alpha.$$

Value function

The *value function*, u, is the infimum of J over all bounded measurable controls:

$$u(x,t) = \inf_{\alpha} J(x,t,\alpha).$$

The corresponding *Hamiltonian*, H, is

$$H(x,p) = \sup_{a\in\mathbb{R}} \left(-pa - \ell_0(x,a)\right).$$

The Hamilton–Jacobi equation

Then, u is a viscosity solution of

$$\begin{cases} -u_t + H(x, \varpi(t) + u_x) = 0 \\ u(x, T) = \bar{u}(x). \end{cases}$$

At points of differentiability of u,

$$\alpha^*(t) = -D_p H(\mathbf{x}(t), \varpi(t) + u_x(\mathbf{x}(t), t)).$$

The transport equation

The associated *transport equation* is:

$$\begin{cases} m_t - (D_pH(x, u_x + \varpi(t))m)_x = 0, \\ m(x, 0) = \bar{m}(x), \end{cases}$$

where $\bar{m}$ is the initial distribution of the agents.

Balance condition

We require that demand matches the *energy production function* $\mathbf{Q}(t)$:

$$\int_{\mathbb{R}} \alpha^*(t) m(x,t) dx = \mathbf{Q}(t);$$

that is,

$$\int_{\mathbb{R}} D_p H(x, u_x + \varpi(t)) m(x,t) dx = -\mathbf{Q}(t).$$

This constraint determines the price, $\varpi(t)$.

Linear-quadratic model - deterministic

Let $\ell(t,\alpha) = \frac{c}{2}\alpha^2 + \alpha\varpi(t)$. Then

$$\begin{cases} -u_t + \frac{(\varpi(t)+u_x)^2}{2c} = 0 \\ m_t - \frac{1}{c}(m(\varpi(t)+u_x))_x = 0 \\ \frac{1}{c}\int_{\mathbb{R}}(\varpi(t)+u_x)m\,dx = -\mathbf{Q}(t). \end{cases}$$

The preceding system implies the **linear price-supply relation**

$$\varpi = \Theta - c\mathbf{Q}(t).$$

Outline

Finitely many agents

- Let Q, be an L^2 adapted stochastic process with respect to a filtration $\mathbb{F}$
- Each agent controls their trading rate according to

$$dX_t = v_t dt, \ \ t \in [0, T],$$

  where $v \in \mathbb{H}_{\mathbb{F}}$
- $\mathbb{H}_{\mathbb{F}}$ is the set of processes $v : [0, T] \times \Omega \to \mathbb{R}$, that are measurable and adapted w.r.t. $\mathbb{F}$, and satisfy $\|v\|^2_{\mathbb{H}_{\mathbb{F}}} < \infty$, where

$$\langle v, w\rangle_{\mathbb{H}_{\mathbb{F}}} := \mathbb{E}\left[\int_0^T v_t w_t dt\right], \quad \|v\|^2_{\mathbb{H}_{\mathbb{F}}} := \langle v, v\rangle_{\mathbb{H}_{\mathbb{F}}}$$

Problem formulation

Find a price ϖ and control v^i, all adapted to $\mathbb{F}$, such that for $1 \leqslant i \leqslant N$, X^i solves $dX_t^i = v_t^i dt$, with $X_0^i = x_0^i$, and minimizes the

$$\mathbb{E}\left[\int_0^T L(X_t^i, v_t^i) + \varpi_t v_t^i \, dt + \bar{u}(X_T^i)\right],$$

subject to

$$\frac{1}{N}\sum_{i=1}^{N} v_t^i = Q_t, \qquad \text{for } 0 \leqslant t \leqslant T.$$

Here, ϖ is the Lagrange multiplier for this balance constraint.

Existence of a price

Theorem

Under natural growth and convexity assumptions:

- *There exists a unique minimizer* $v^* \in \mathbb{H}_{\mathbb{F}}^N$
- *consider the corresponding trajectory* X^*. *For* $1 \leqslant i \leqslant N$, *let* $P^i, Z^i \in \mathbb{H}_{\mathbb{F}}$ *solve, on* $[0, T]$,

$$\begin{cases} dP_t^i = -L_x(X_t^{*i}, v_t^{*i})dt + Z_t^i dW_t \\ P_T^i = \bar{u}'(X_T^{*i}). \end{cases}$$

- *There exists a unique* $\Pi \in \mathbb{H}_{\mathbb{F}}$ *such that*

$$\Pi = P^i + L_v(X^{*i}, v^{*i}) \quad \text{for } 1 \leqslant i \leqslant N.$$

- *Further,* $\varpi = -\Pi$.

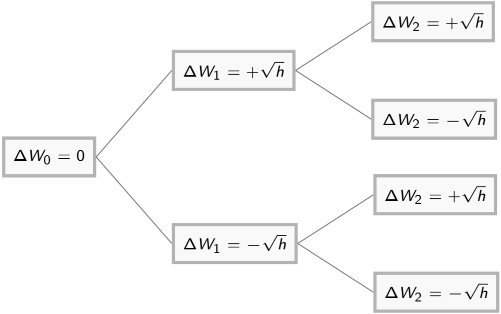
Outline

Numerical approximations

- Except for quadratic problems, there are no other known solutions.
- Numerical methods are needed and a binomial tree is perhaps one of the easiest ways to do so.

Binomial aproximation

To obtain numerical approximations, we consider a binomial discretization of the driving Brownian motion.



$$\{0, +\sqrt{h}, \ +\sqrt{h}\}$$

$$\{0, +\sqrt{h}, \ -\sqrt{h}\}$$

$$\{0, -\sqrt{h}, \ +\sqrt{h}\}$$

$$\{0, -\sqrt{h}, \ -\sqrt{h}\}$$

Fig. Binomial Tree diagram for $M = 2$ time steps (left) and list of realizations of the noise (right).

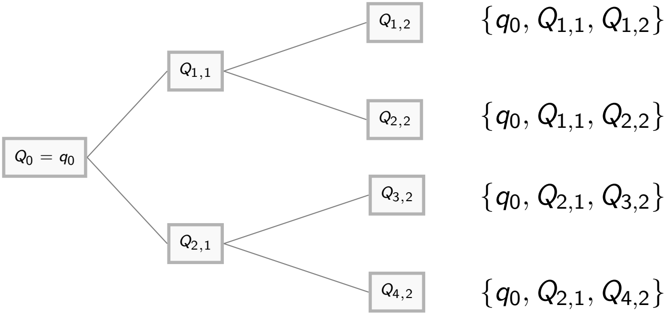


Fig. Binomial Tree diagram for $M = 2$ time steps (left) and list of realizations of the supply (right).

- At time t_k, the discrete price process ϖ takes the value ϖ_k, and the measurability condition w.r.t. $\mathcal{F}_k$ means that $\varpi_k \in \left\{\varpi_{1,k}, \ldots, \varpi_{2^k,k}\right\}$, where the values $\varpi_{j,k}$ are unknown
- The controls for each player are also a function of the tree
- At each node, we imposed the balance condition constraint
- We discretize the objective functional in the natural way

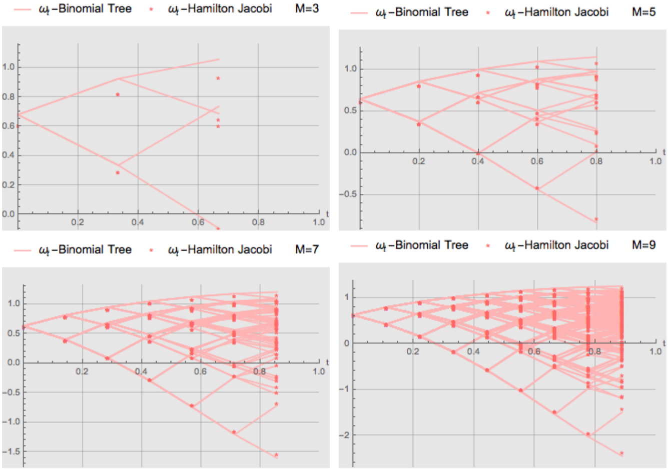


Fig. Binomial Tree and Hamilton-Jacobi approximations for $\eta = 0$ and 3, 5, 7, and 9 time steps.

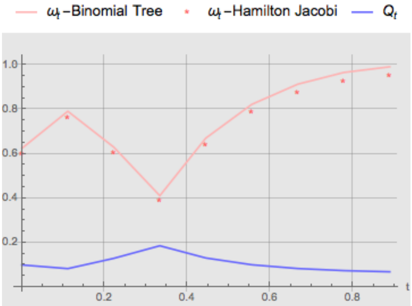


Fig. Sample path of the supply and the corresponding Binomial Tree and Hamilton-Jacobi approximations of the price for $M = 9$ time steps. The L^2 distance between price approximations is $9.16618 * 10^{-2}$.

Outline

Motivation for ML approaches

- Stochastic supply price can be approximated numerically by a binomial tree
- Good agreement between numerical results and exact solutions
- However, dimensionality curse limits accuracy.
- Machine learning can improve resolution as we show here.

RNN architecture - trading rate

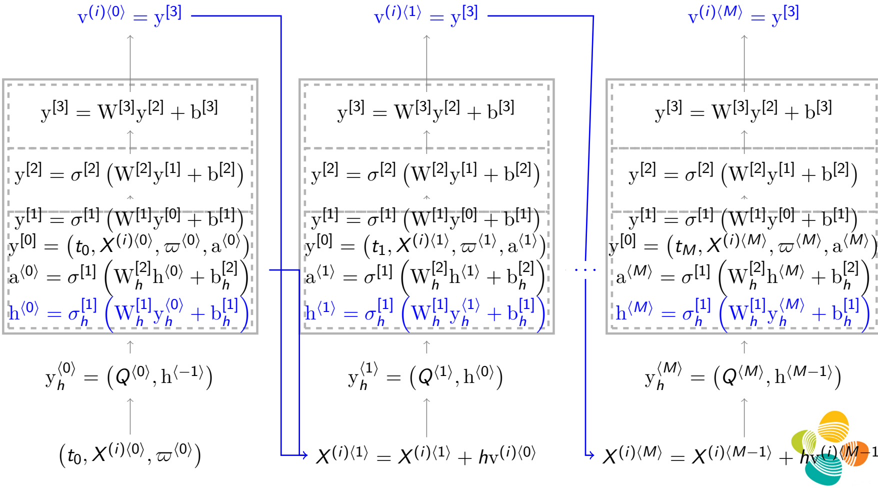


Fig. Iteration of the RNN for v^*, RNN_v, with supply history dependence

RNN price

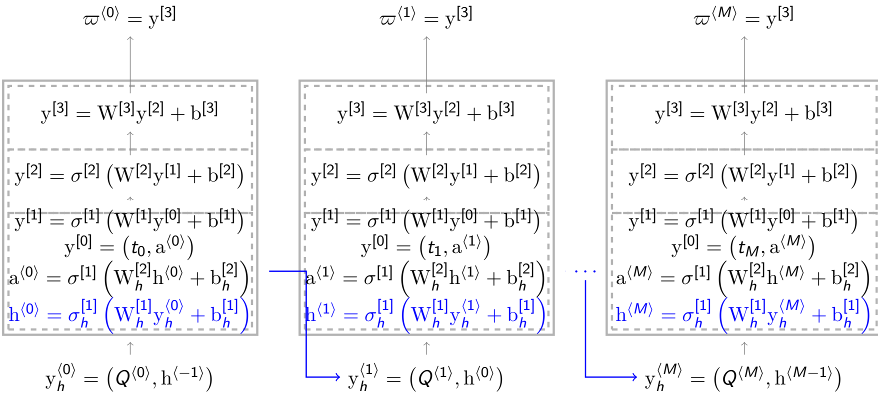


Fig. Iteration of the RNN for ϖ, RNN_{ϖ}, with supply history dependence

Loss function

We consider the adversarial loss function

$$\mathcal{L}\left(\Theta_v, \Theta_\varpi\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{k=0}^{M-1} h\Big(L(X^{(i)\langle k\rangle}, \mathrm{v}^{(i)\langle k\rangle}(\Theta_v)) \right.$$
$$+ \varpi^{\langle k\rangle}(\Theta_\varpi)\left(\mathrm{v}^{(i)\langle k\rangle}(\Theta_v) - Q^{\langle k\rangle}\right)\Big)$$
$$\left. + u_T(X^{(i)\langle M\rangle})\right).$$

Using $\mathcal{L}$, we train NN_v and NN_ϖ using an adversarial approach.

Arrow-Hurwicz-Uzawa like iteration

Key idea:

- Perform a descent step in Θ_v
- Perform a ascent step in Θ_{ϖ}.

(a) Supply and price (b) Optimal feedback v^* (c) m and characteristics

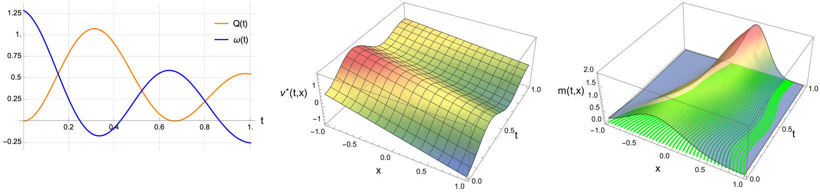
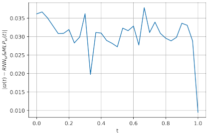


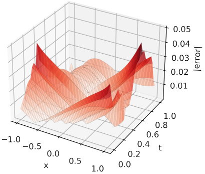
Fig. Analytical solutions for $\overline{Q} = 7e^{-t}\sin(3\pi t)$

Results

(a) Absolute error (ϖ)



(b) Absolute error (v^*)



(c) m-weighted error (v^*)

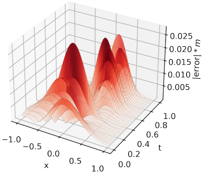


Fig. Approximation errors for ϖ and v^* using RNN_{ϖ} and RNN_{v}, respectively, for $\overline{Q}(t) = 7e^{-t}\sin(3\pi t)$

Common noise RRN training

- We discretize the SDE for the price.
- The loss functional is identical to the deterministic setting.
- To train the RNN, we use a new sample for Q at each SGD step.
- The RNN preserves progressive measurability.

Common noise - approximate optimality conditions

In general, the ML framework returns an approximate solution of the optimality conditions

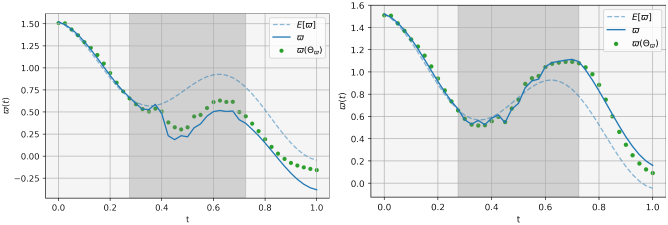
$$\begin{cases} \mathrm{d}\tilde{P}^n(t) = \Big(H_x(\tilde{X}^n(t), \tilde{P}^n(t) + \varpi^N(t)) + \epsilon^n(t) \Big)\, \mathrm{d}t \\ \qquad\qquad + \tilde{Z}^n(t) \mathrm{d}W(t), \\ \tilde{P}^n(T) = u_T'(\tilde{X}^n(T)) - \epsilon_T^n, \\ \mathrm{d}\tilde{X}^n(t) = -H_p(\tilde{X}^n(t), \tilde{P}^n(t) + \tilde{\varpi}^N(t))\mathrm{d}t, \\ \tilde{X}^n(0) = x_0^n, \\ \frac{1}{N}\sum_{n=1}^{N} -H_p(\tilde{X}^n(t), \tilde{P}^n(t) + \tilde{\varpi}^N(t)) = Q(t) + \epsilon_B(t), \end{cases}$$

A posteriori estimates - common noise

Theorem

Let H be uniformly concave-convex in (x, p), separable, with Lipschitz continuous derivatives, u_T is convex with Du_T Lipschitz. Let $(\mathbf{X}, \mathbf{P})$ and ϖ^N solve the N-player price problem with a common noise. Let $(\tilde{\mathbf{X}}, \tilde{\mathbf{P}})$ and $\tilde{\varpi}^N$ be a corresponding approximate solution. Then

$$\|\varpi^N - \tilde{\varpi}^N\| \leqslant C\left(\|\epsilon_H\| + \|\epsilon_B\|\right).$$



(a) Price realization (b) Price realization.

Fig. Exact price and RNN approximation. The grey window highlights the times where noise operates.

Outline

Integrating the transport equation

- Let (u, m, ϖ) solve the price problem.
- The transport equation can be written as $\mathrm{div}_{(t,x)}\left(m, -H_p(x, \varpi + u_x)m\right) = 0$.
- Hence, by Poincaré lemma, there exists φ such that

$$m = \varphi_x, \quad H_p(x, \varpi + u_x)m = \varphi_t.$$

Perspective function

- Consider the perspective function of L

$$F(x,j,m)=\begin{cases} L\left(x,\frac{j}{m}\right)m, & m>0 \\ +\infty, & j\neq 0,\ m=0 \\ 0, & j=0,\ m=0. \end{cases}$$

Deterministic potential problem

Find φ minimizing

$$\int F(x, -\varphi_t, \varphi_x) - u_T'(x)\varphi_t \,\mathrm{d}x\mathrm{d}t,$$

over φ s.t. $\varphi(0,x) = \int_{-\infty}^{x} m_0(y)\mathrm{d}y$, and, for all $t \in [0,T]$, $\varphi_x(t,\cdot) \in \mathcal{P}(\mathbb{R})$ and

$$\int_{\mathbb{R}} \varphi(t,x) - \varphi(0,x)\mathrm{d}x = -\int_0^t Q(s)\mathrm{d}s.$$

The Euler-Lagrange equations for

$$\int F(x, -\varphi_t, \varphi_x) - \varpi\,(\varphi_t + Q\varphi_x) - u'_T\varphi_t \,\mathrm{d}x\mathrm{d}t.$$

are equivalent to the Hamilton-Jacobi equation

$$(-u_x)_t + (H(x, \varpi + u_x))_x = 0.$$

Stochastic potential problem

Find a progressively measurable function φ minimizing

$$E\int F(x,-\varphi_t,\varphi_x)-\varpi\left(\varphi_t+Q\varphi_x\right)-u_T'\varphi_t \,\mathrm{d}x\mathrm{d}t$$

over the set of PMP φ such that $\varphi(0,x)=\int_{-\infty}^{x} m_0(y)\mathrm{d}y$, and for all $t\in[0,T]$ satisfy $\varphi_x(t,\cdot)\in\mathcal{P}(\mathbb{R})$ and

$$\int_{\mathbb{R}}\varphi(t,x)-\varphi(0,x)\mathrm{d}x=-\int_0^t Q(s)\mathrm{d}s.$$

The optimality conditions for this problem are also equivalent to HJ equations.

Outline

Machine learning framework

- To approximate φ, we consider a RNN.
- A hidden state, h, carries information about the path history. Thus, the outputs of this architecture depend on the path history and this guarantees their progressive measurability.
- Following several numerical experiments, we select a RNN with a hidden layer and three dense layers, all layers with dimension 32.

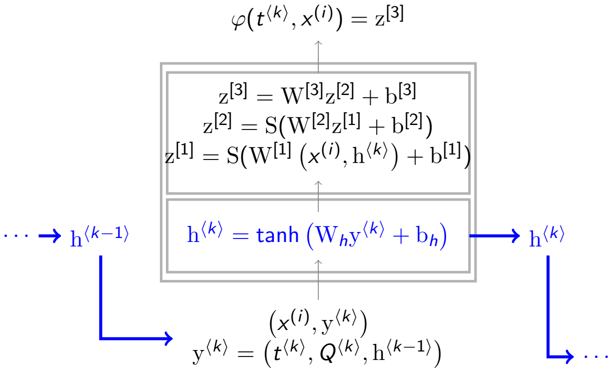


Fig. RNN cell computation at time k. S is the sigmoid function, W denotes a weight matrix, and b denotes a bias.

Loss components

$$
\begin{aligned}
\mathcal{L}_{\mathcal{V}}(\varphi_{\mathcal{T}}) &= h_x h_t \sum_{k=1}^{M_t} \sum_{i=1}^{M_x} \left\{ F\left(x^{(i)}, -\partial_t \varphi_{\mathcal{T}}^{\langle k \rangle (i)}, \partial_x \varphi_{\mathcal{T}}^{\langle k \rangle (i)} \right) \right. \\
&\qquad \left. - u_T'(x^{(i)}) \partial_t \varphi_{\mathcal{T}}^{\langle k \rangle (i)} \right\}, \\
\mathcal{L}_0(\varphi_{\mathcal{T}}) &= \sum_{k=1}^{M_t} \sum_{i=1}^{M_x} \max \left\{ -\partial_x \varphi_{\mathcal{T}}^{\langle k \rangle (i)}, 0 \right\}, \\
\mathcal{L}_{\mathcal{B}}(\varphi_{\mathcal{T}}) &= \sum_{k=1}^{M_t} \left(h_x \sum_{i=1}^{M_x} \partial_t \varphi_{\mathcal{T}}^{\langle k \rangle (i)} + Q^{\langle k \rangle} \right)^2, \\
\mathcal{L}_{M_0}(\varphi_{\mathcal{T}}) &= \sum_{i=1}^{M_x} \left(\varphi_{\mathcal{T}}^{\langle 0 \rangle (i)} - M_0(x^{(i)}) \right)^2, \\
\mathcal{L}_{\mathcal{P}}(\varphi_{\mathcal{T}}) &= \sum_{k=1}^{M_t} \left(1 - h_x \sum_{i=1}^{M_x} \partial_x \varphi_{\mathcal{T}}^{\langle k \rangle (i)} \right)^2.
\end{aligned}
$$

$\mathcal{L}_{\mathcal{V}}$ corresponds to the discretization of the variational functional. Deviations from the balance constraint are penalized by $\mathcal{L}_{\mathcal{B}}$, and the initial condition is enforced in $\mathcal{L}_{M_0}$. Moreover, $\mathcal{L}_0$ and $\mathcal{L}_{\mathcal{P}}$ guarantee that $\varphi_x(t, \cdot)$ is a density function.

Consider the loss function that aggregates the terms

$$\mathcal{L} = \mathcal{L}_{\mathcal{V}} + \mathcal{L}_0 + \mathcal{L}_{\mathcal{B}} + \mathcal{L}_{M_0} + \mathcal{L}_{\mathcal{P}}.$$

Table Stochastic training step j.

Input: M_0, dQ, τ^{j-1}.

compute sample Q_j
for $k = 1, \ldots, M_t$ do
  compute $\mathrm{h}^{\langle k \rangle}$
  for $i = 1, \ldots, M_x$ do
    compute $\varphi_{\tau^{j-1}}(t^{\langle k \rangle}, x^{(i)})$
compute $\mathcal{L}(\varphi_{\tau^{j-1}})$
compute τ^j by gradient descend

Output: τ^j

Stochastic examples

- We consider a mean-reverting supply

$$dQ(t) = \theta \left(\overline{Q} - Q(t) \right) dt + \sigma dW(t),$$

on $[0, T]$, where $Q(0) = -0.5$, $\theta = 2$, $\overline{Q} = 1$, and $\sigma = 0.2$.

- The theoretical formula for the price is

$$d\varpi = -dQ$$

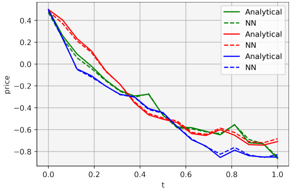


Fig. Stochastic price approx. (3 samples)

Outline

Conclusions and future work

- We developed two variational approaches to price formation with common noise.
- Our formulations, combined with machine learning techniques, provide a way for solving certain infinite-dimensional MFGs without using the master equation.
- Future work should identify better network architectures and convergence results.

The end

Thanks a lot for your attention! Questions?