Decaying sensitivity and separable optimal value functions

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based on joint work with
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Nonlinear partial differential equations: theory, numerics and applications
A conference in memory of Maurizio Falcone
Rome, 24–26 May 2023

Setting

We consider nonlinear control systems in continuous time

$$\dot{x}(t) := \frac{d}{dt}x(t) = f(x(t), u(t)), \quad x(0) = x_0,$$

or in discrete time

$$x^+(t) := x(t+1) = f(x(t), u(t)), \quad x(0) = x_0,$$

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Objective:

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For this problem, an (approximately) optimal feedback control can be computed from (an approximation of) the optimal value function

$$V(x_0) := \inf_{u\in\mathcal{U}} J(x_0,u)$$

via the associated Hamilton-Jacobi-Bellman equation

$$\sup_{u\in U}\{-DV(x)f(x,u) - \ell(x,u)\} = 0$$

or the Bellman equation

$$\sup_{u\in U}\{V(x) - V(f(x,u)) - \ell(x,u)\} = 0$$

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- Known fact: Deep Neural Networks (DNNs) are capable of overcoming the curse of dimensionality for functions with certain beneficial structures
- Goal: Detect and exploit such structures for approximating control Lyapunov functions and optimal value functions

Simplified setting

Instead of looking for solutions of the equations

$$\sup_{u\in U}\{-DV(x)f(x,u)-\ell(x,u)\} = 0 \quad \text{or} \quad \sup_{u\in U}\{V(x)-V(f(x,u))-\ell(x,u)\} = 0$$

we start by computing supersolutions

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These are interesting in their own right, because they describe control Lyapunov functions and, in addition, upper bounds for the value functions

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A continuously differentiable $V : \mathbb{R}^n \to \mathbb{R}_0^+$ is a Lyapunov function, if there are functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \alpha_1(\|x\|) \;&\leq\; V(x) \;\leq\; \alpha_2(\|x\|) \\ DV(x)f(x) \;&\leq\; -\alpha_3(\|x\|) \end{aligned}$$

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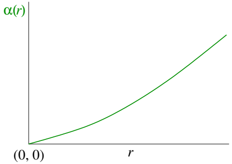
$\alpha \in \mathcal{K}_\infty$: $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, continuous, strictly increasing, $\alpha(0) = 0$, unbounded



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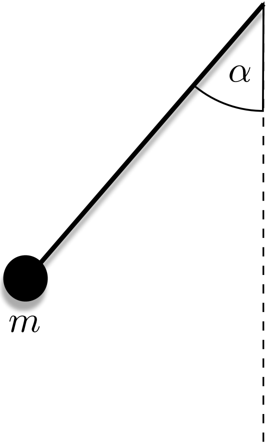
Example: Mathematical Pendulum



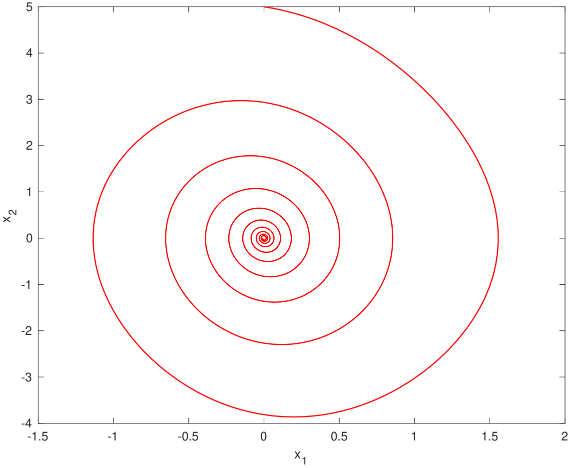
$x_1 = \alpha =$ angle

$x_2 =$ angular velocity

$\rightsquigarrow$ ordinary differential equation

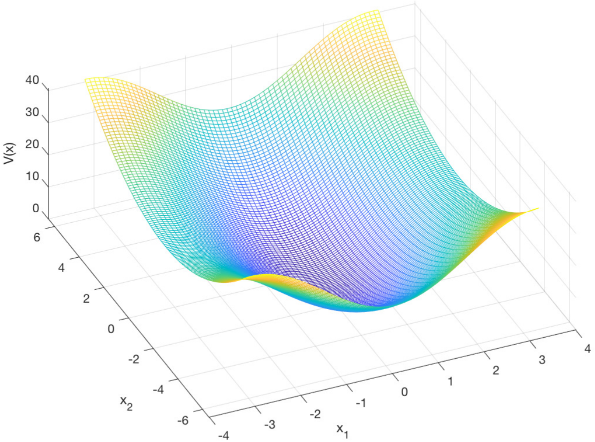
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -g\sin(x_1) - \frac{k}{m}x_2 \end{aligned}$$

Pendulum solution and Lyapunov function



Solution of pendulum equation

Pendulum solution and Lyapunov function



Lyapunov function $V(x) = x_2^2/2 + g(1 - \cos x_1) + 0.1 x_2 \sin(x_1)$

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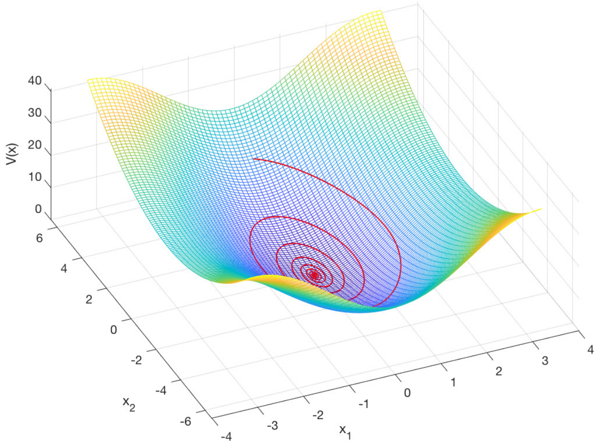
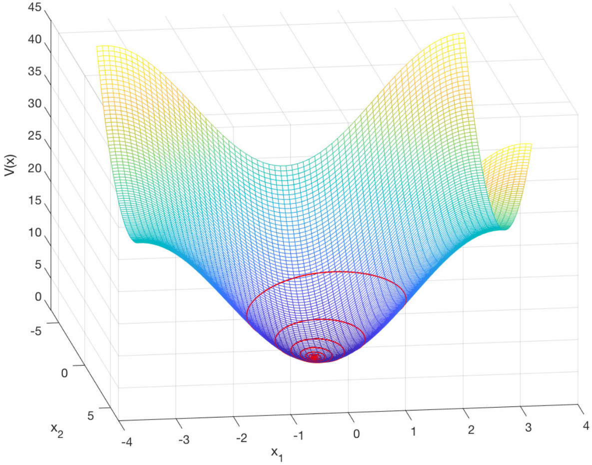


Lyapunov function with solution superimposed

Pendulum solution and Lyapunov function



Lyapunov function with solution superimposed

Numerical computation of Lyapunov functions

Various numerical approaches for computing (control) Lyapunov functions have been developed over the years:

- Series expansion [Kirin et al. '82]
- Semi-Lagrangian schemes [Camilli/Gr./Wirth '00, Falcone/Gr./Wirth '00]
- Finite elements and linear programming [Hafstein '02ff]
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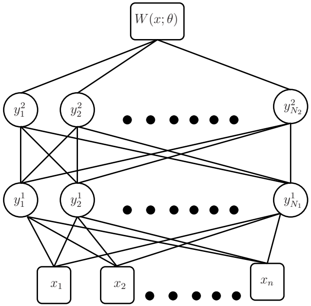
Can deep neural networks do better?

Deep neural networks

Deep neural network with 2 hidden layers



output $W(x;\theta) = a \cdot y_k^2 + c$

$\ell = 2$ $y_k^2 = \sigma^2(w_k^2 \cdot y^1 + b_k^2)$

$\ell = 1$ $y_k^1 = \sigma^1(w_k^1 \cdot x + b_k^1)$

input

w_k^1, w_k^2, a = vectors of weights, " $\cdot$ " = scalar product

$b_k^1,\ b_k^2,\ c$ = scalar parameters, $\sigma^1,\ \sigma^2 : \mathbb{R} \to \mathbb{R}$ = activation functions

Examples: $\sigma(r) = r$, $\sigma(r) = \max\{r, 0\}$, $\sigma(r) = \ln(e^r + 1)$

θ = vector of all parameters $(w_k^\ell, b_k^\ell, a, c)$

$W(x;\theta^*) \approx V(x)$ approx. Lyapunov function for "trained" θ^*

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  $\rightsquigarrow$ Unlikely to work for deterministic problems

Beneficial structures

- Compositional functions, cf. e.g. [Poggio/Mhaskar/Rosasco/Miranda/Liao '17, Kang/Gong '22, Dahmen '23]

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$$g(x) = g_1(g_2(x_{i_1}, x_{i_2}), g_3(x_{i_3}) + g_4(g_5(g_6(x_{i_4}, x_{i_5}))) + \dots$$

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A particular example are separable functions

$$V(x) = \sum_{j=1}^{s} V_j(z_j), \quad z_j = \begin{pmatrix} x_{i_{j,1}} \\ \vdots \\ x_{i_{j,d_j}} \end{pmatrix}$$

with m bounded independent of n and $s \leq n$

Why are separable functions beneficial?

Separable function: $V(x) = \sum_{j=1}^{s} V_j(z_j), \quad z_j = \begin{pmatrix} x_{i_{j,1}} \\ \vdots \\ x_{i_{j,d_j}} \end{pmatrix}$

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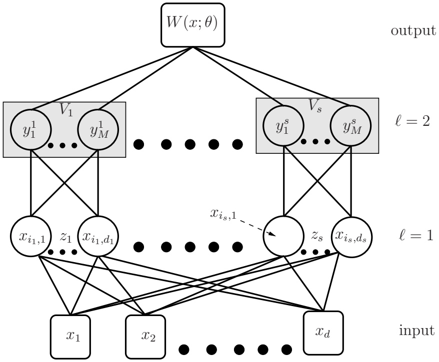

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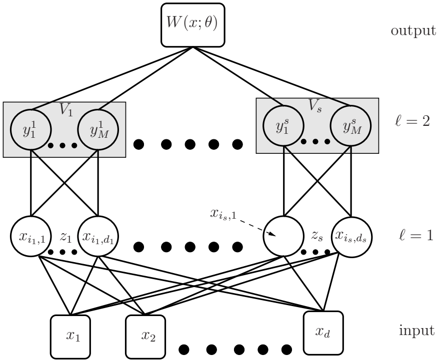

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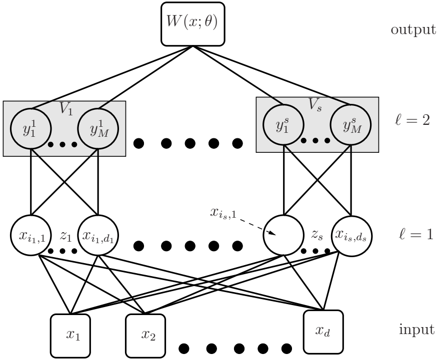

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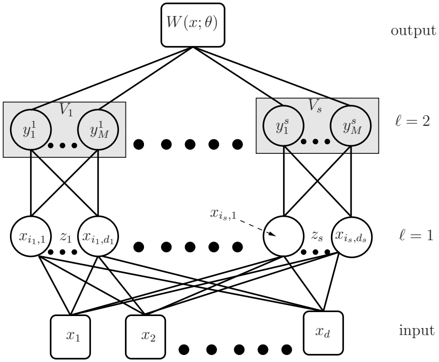

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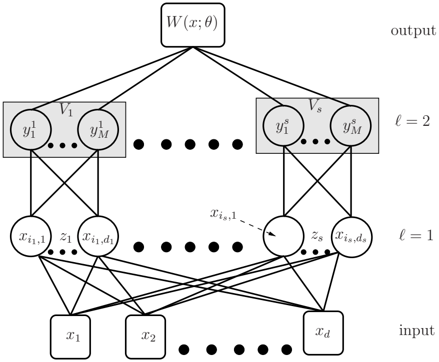

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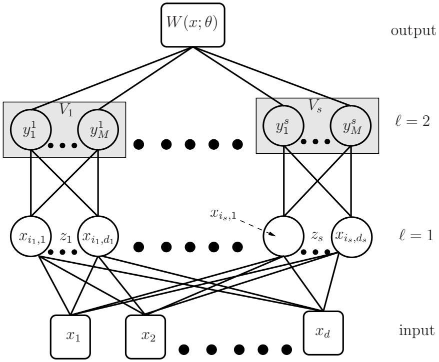

In the first layer we can even implement more complex transformations than merely splitting up x into the z_j

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then a separable Lyapunov function $V(x) = \sum_{j=1}^{s} V_j(z_j)$ exists

[Dashkovskiy/Rüffer/Wirth '10, Dashkovskiy/Ito/Wirth '11]

See also [Jiang/Teel/Praly '94, Jiang/Mareels/Wang '96, Rüffer '07ff, ...]

Complexity theorem

Theorem [Gr. 21]: Lyapunov functions $V(x) = \sum_{j=1}^{s} V_j(T_j x)$ with $d_j = \text{rank} T_j \leq d_{\max}$ independent of n can be approximated with any accuracy $\varepsilon > 0$ with a number of neurons growing only polynomially in n

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Using an appropriate training algorithm, the network will "learn" this structure during the training process

10d Numerical Example

$$\dot{x}(t) = T^{-1}\hat{f}(x)(Tx)$$

with

$$\hat{f}(x) = \begin{pmatrix} -x_1 + 0.5x_2 - 0.1x_9^2 \\ -0.5x_1 - x_2 \\ -x_3 + 0.5x_4 - 0.1x_1^2 \\ -0.5x_3 - x_4 \\ -x_5 + 0.5x_6 + 0.1x_7^2 \\ -0.5x_5 - x_6 \\ -x_7 + 0.5x_8 \\ -0.5x_7 - x_8 \\ -x_9 + 0.5x_{10} \\ -0.5x_9 - x_{10} + 0.1x_2^2 \end{pmatrix}, \qquad T = \begin{pmatrix} -\frac{1}{5} & -\frac{3}{10} & \frac{1}{2} & -\frac{4}{5} & \frac{4}{5} & \frac{2}{5} & \frac{7}{10} & \frac{7}{10} & -1 & \frac{4}{5} \\ \frac{1}{5} & 1 & \frac{9}{10} & \frac{4}{5} & -\frac{1}{10} & \frac{3}{5} & -\frac{3}{10} & \frac{1}{2} & \frac{4}{5} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{3}{10} & \frac{2}{5} & -\frac{2}{5} & 0 & -\frac{3}{5} & \frac{3}{10} & \frac{3}{5} & 1 & -\frac{1}{2} \\ -\frac{7}{10} & -\frac{1}{10} & -\frac{3}{5} & -\frac{1}{5} & -\frac{3}{5} & \frac{2}{5} & \frac{1}{10} & -\frac{1}{10} & \frac{1}{10} & -\frac{3}{5} \\ \frac{1}{10} & -\frac{3}{5} & -\frac{9}{10} & -\frac{7}{10} & -\frac{1}{5} & -\frac{1}{10} & \frac{1}{10} & \frac{1}{5} & 0 & -\frac{4}{5} \\ \frac{3}{5} & \frac{9}{10} & -\frac{1}{5} & 1 & \frac{2}{5} & \frac{1}{2} & 0 & -\frac{1}{10} & -\frac{2}{5} & 0 \\ -1 & 1 & \frac{7}{10} & \frac{3}{5} & -\frac{4}{5} & -\frac{4}{5} & 0 & -\frac{1}{5} & -\frac{1}{5} & \frac{7}{10} \\ -\frac{9}{10} & \frac{4}{5} & \frac{1}{5} & 1 & -\frac{4}{5} & \frac{2}{5} & -\frac{3}{10} & \frac{7}{10} & \frac{1}{5} & -\frac{4}{5} \\ \frac{3}{5} & -\frac{1}{10} & -\frac{2}{5} & -\frac{1}{2} & -\frac{3}{10} & -\frac{1}{10} & -\frac{7}{10} & 1 & \frac{4}{5} & -\frac{3}{10} \\ 0 & -1 & -\frac{1}{10} & \frac{2}{5} & -\frac{3}{10} & -\frac{1}{10} & -\frac{1}{5} & \frac{7}{10} & -\frac{1}{10} & \frac{4}{5} \end{pmatrix}$$

10d Numerical Example

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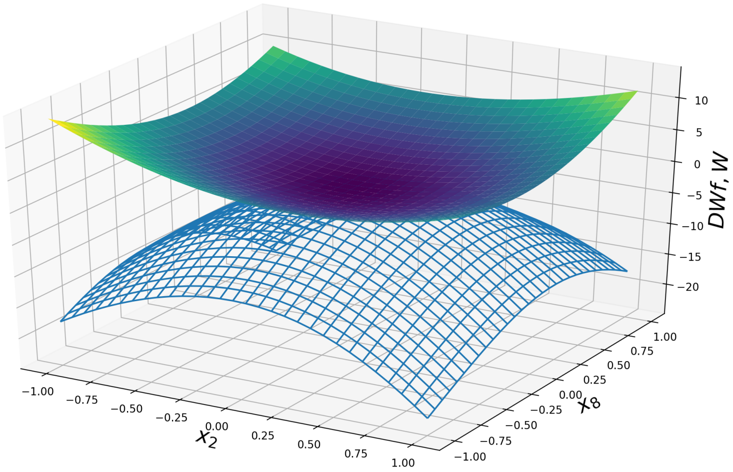
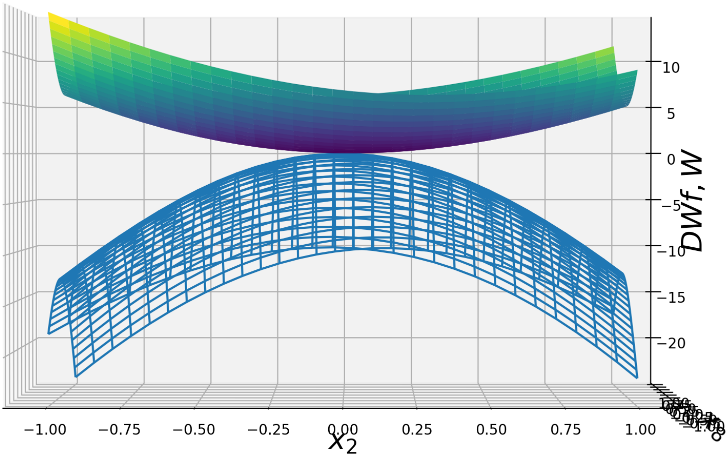
We perform the training with a network with 5 sublayers with dimension $d_{max} = 2$ ($\rightsquigarrow$ 2 671 parameters), and $m = 400\,000$ test points

10d Example



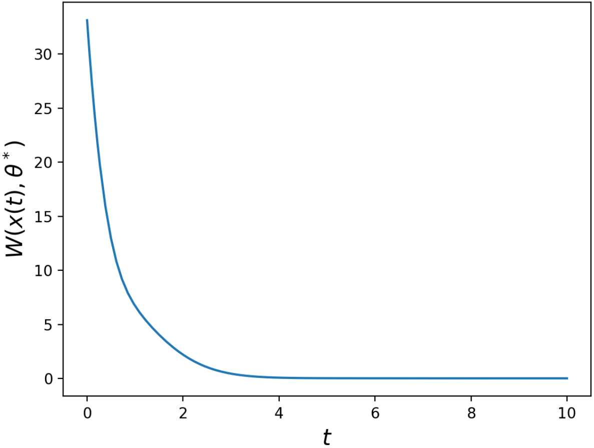
10
5
0
−5
−10
−15
−20
DWf, W
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00
x_2
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
x_8

10d Example



Computation time: 266s

10d Example – Evaluation along trajectories



Initial value $x_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$

Control Lyapunov functions

Control Lyapunov functions

If we assume smoothness, a control Lyapunov function (clf) is characterised by

$$\begin{aligned} \alpha_1(\|x\|) \;&\le\; V(x) \;\le\; \alpha_2(\|x\|) \\ \inf_{u\in U} DV(x)f(x,u) \;&\le\; -\alpha_3(\|x\|) \end{aligned}$$

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Question: When can we employ small gain techniques here?

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Recall the sufficient condition

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \ldots \circ \gamma_{i_m i_1} < \mathrm{id} \qquad \text{for each cycle in the graph}$$

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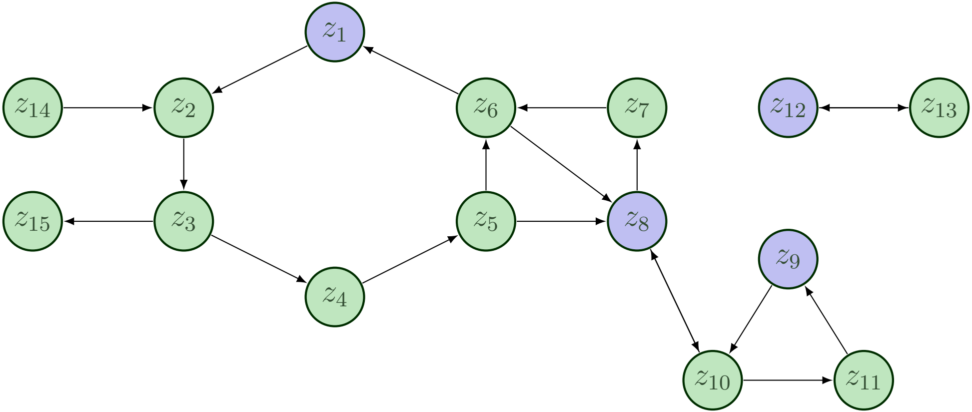
This implies:

If in each cycle of the graph there is at least one subsystem for which the γ_{ij} can be made arbitrarily "flat" ("active nodes"), then there exists a clf of the separable form $V(x) = \sum_{j=1}^{s} V_j(z_j)$ [Chen/Astolfi '20]

Example for a suitable graph structure



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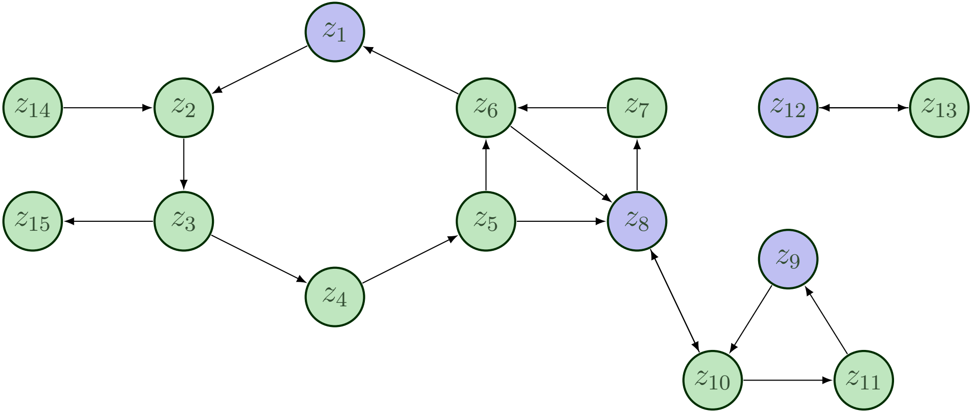
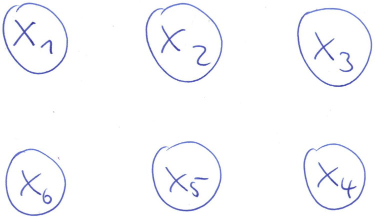


$$\rightsquigarrow \quad V(x) = \sum_{j=1}^{15} V_j(z_j)$$

Computation with DNN

Example:

$$\begin{aligned}\dot{x}_1 &= x_3 + u\\ \dot{x}_2 &= x_1 - x_2 + x_1^2\\ \dot{x}_3 &= x_2 - x_3\\ \dot{x}_4 &= x_3 - x_4\\ \dot{x}_5 &= x_4 - x_5\\ \dot{x}_6 &= x_5 - x_6\end{aligned}$$



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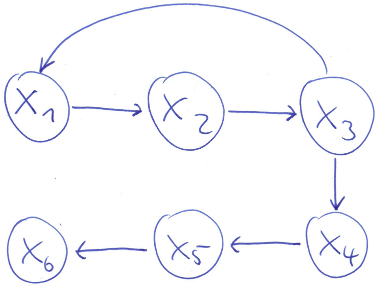
$$



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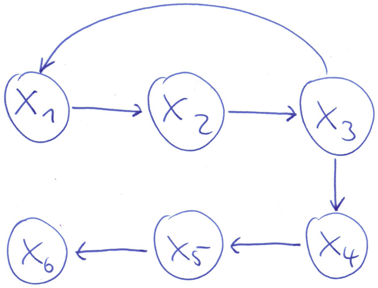

$\rightsquigarrow$ $$V(x) = \sum_{j=1}^{6} V(x_j)$$

Computation with DNN

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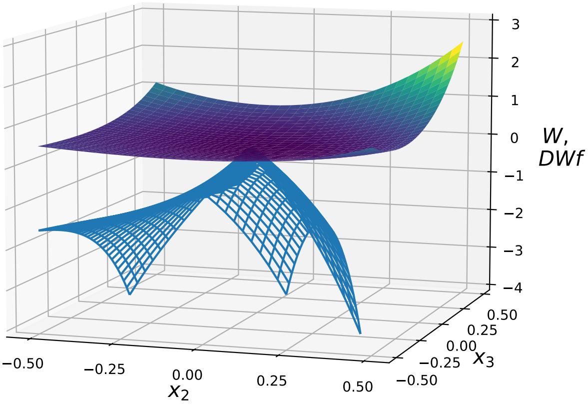

Computation time: 820s

(Approximately) Optimal Value Functions

Optimal Value Functions

In optimal control, we want to solve

$$\sup_{u \in U} \{-DV(x)f(x,u) - \ell(x,u)\} = 0$$

in continuous time or

$$\sup_{u \in U} \{V(x) - V(f(x,u)) - \ell(x,u)\} = 0$$

in discrete time

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Remedy: Overlapping decompositions offer more flexibility

Decaying sensitivity

Consider, as before, a decomposition of x into subvectors z_j corresponding to subsystems connected via a graph

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$$z_k \mapsto V(z_1, \ldots, z_{l-1}, z_l, z_{l+1}, \ldots, z_s) - V(z_1, \ldots, z_{l-1}, 0, z_{l+1}, \ldots, z_s) \quad (*)$$

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- Related to [Shin/Anitescu/Zavala '22, Zhang/Li/Li '22]

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Example: convoy of vehicles

It is known that a perturbation in the first vehicle (e.g., a braking manoeuvre) may amplify while propagating through the convoy

However, the perturbation will decrease quickly, if the vehicles are controlled optimally

Decaying sensitivity: Example

Consider a convoy of $i = 1, \ldots, N$ vehicles on a road with state $z_i = (x_i, v_i)^T$ and dynamics

$$\dot{x}_i = v_i, \quad \dot{v}_i = u_i$$

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In the simulation: $N = 100$, shown $i = 1, \dots, 5$, $x_{ref} \equiv 0$, $v_{ref} \equiv 0$

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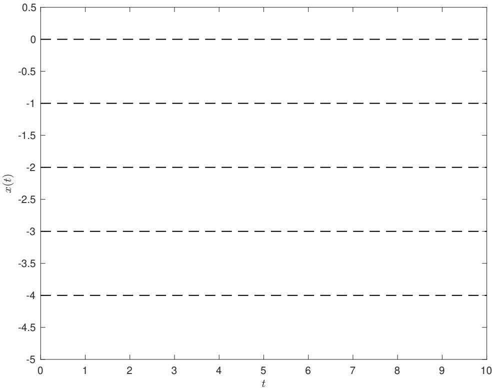

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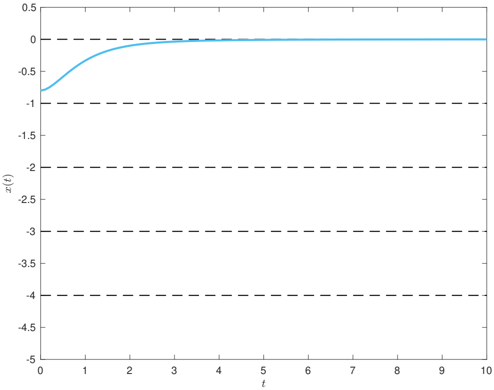

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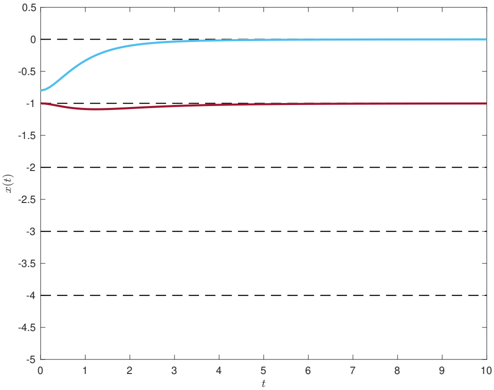

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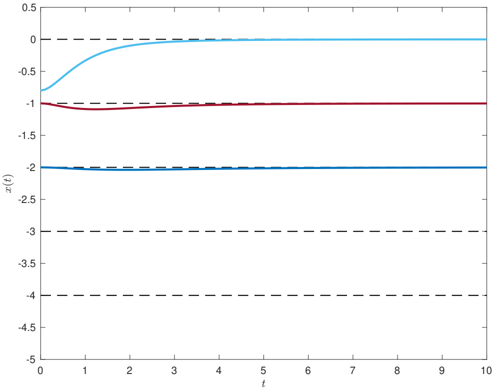

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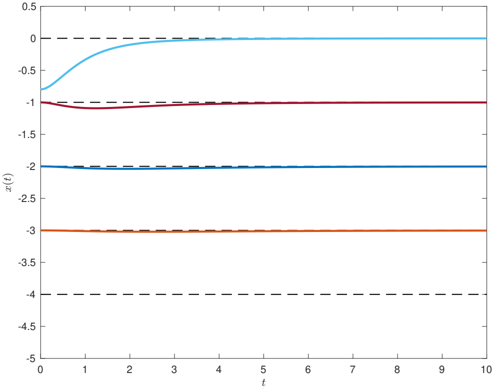

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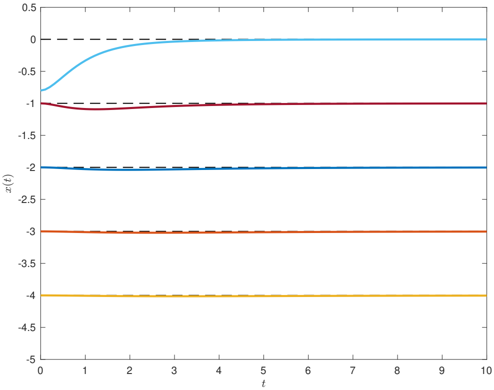

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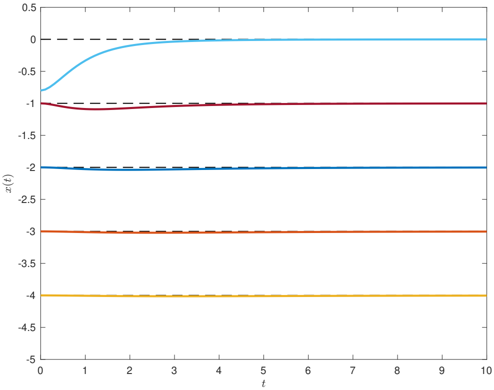

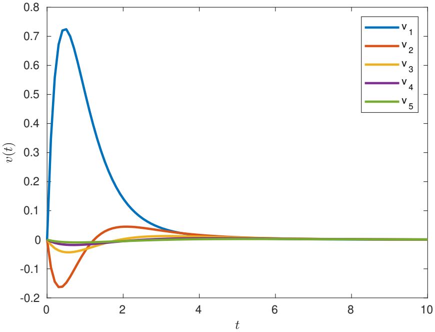

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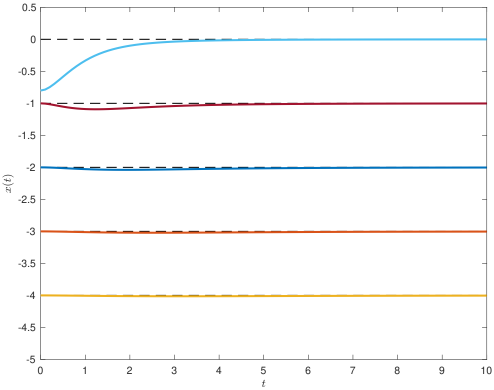

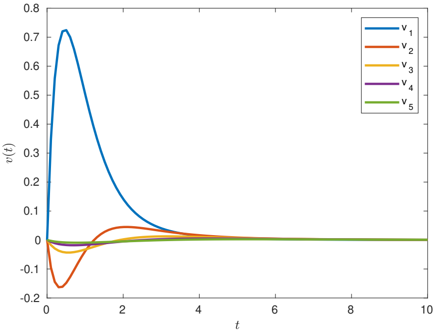

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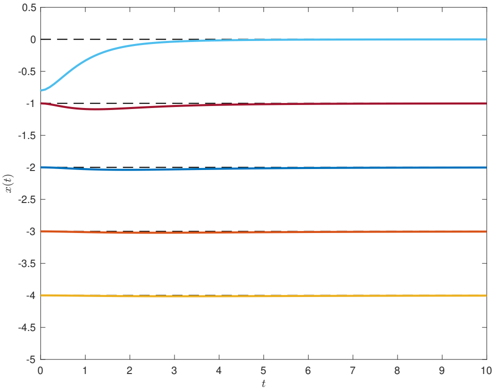

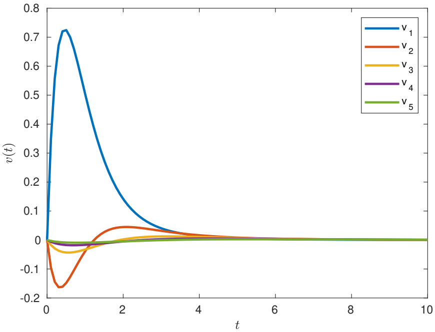

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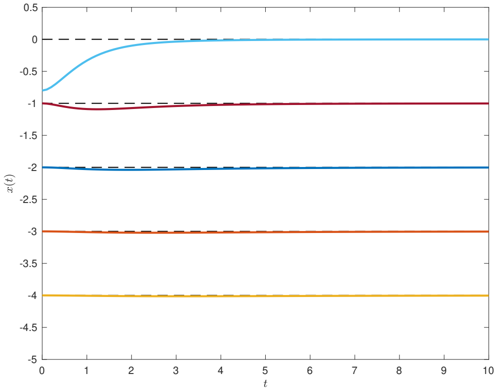

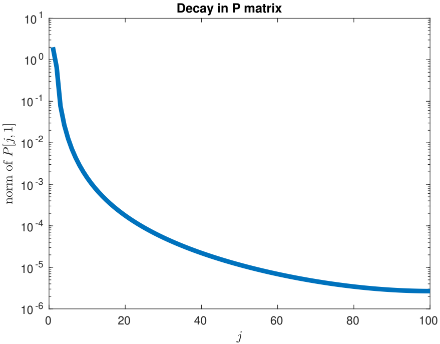

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Construction of overlapping approximation

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$$z_k \mapsto V(z_1, \ldots, z_{l-1}, z_l, z_{l+1}, \ldots, z_s) - V(z_1, \ldots, z_{l-1}, 0, z_{l+1}, \ldots, z_s) \qquad (*)$$

implies

$$(*) \approx V(0, \ldots, 0, z_j, z_{j+1}, \ldots, z_{j+l}, 0, \ldots, 0) - V(0, \ldots, 0, 0, z_{j+1}, \ldots, z_{j+l}, 0, \ldots, 0)$$

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Concrete estimates in [Sperl/Saluzzi/Gr./Kalise '23] using exponentially decaying sensitivity, i.e., $|(*) - \psi_l^j| \le c\rho^j$ for some $\rho \in (0,1)$, yield

$$\left| V(x) - V(0) - \sum_{j=1}^{s} \Psi_l^j(z_j,\ldots,z_{j+l}) \right| \le c(s-1)\rho^j$$

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- Topics of current and future research: separable approximate supersolutions, nonsmoothness, efficient training, relation to low-rank approximations

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