Principal eigenvalues and eigenfunctions for fully nonlinear equations in punctured balls

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We focus on existence, multiplicity and regularity results for the eigenvalue problem associated with a fully nonlinear uniformly elliptic operator, in presence of a a singular potential defined in the punctured unit ball. Precisely, we look for non trivial solutions $(\lambda_\gamma, u_\gamma)$ of the fully nonlinear Dirichlet problem

$$\begin{cases} -F(D^2 u_\gamma) = \lambda_\gamma \dfrac{u_\gamma}{|x|^\gamma} & \text{in } B \setminus \{0\} \\ u_\gamma = 0 & \text{on } \partial B \end{cases} \tag{1}$$

where

- $B = B_1(0)$ is the unit ball in $\mathbb{R}^n$;
- $\gamma > 0$ is the exponent of the singular potential.

On the operator F, we assume that $F : \mathcal{S}_N \to \mathbb{R}$ is a continuous uniformly elliptic function defined on the set $\mathcal{S}_N$ of symmetric $N \times N$ matrices, that is, for positive constants $\Lambda \geq \lambda > 0$,

$$\lambda \, \mathrm{tr}(M') \leq F(M + M') - F(M) \leq \Lambda \, \mathrm{tr}(M') \, ,$$

for all $M, M' \in \mathcal{S}_N$, with M' positive semidefinite.
We suppose also that F is rotationally invariant, that is

$$F(O^t M O) = F(M)$$

for every orthogonal matrix O and for all $M \in \mathcal{S}_N$, and that F is positively homogeneous of degree 1, i.e.

$$F(tM) = tF(M)$$

for any $M \in \mathcal{S}_N$ and for all $t > 0$.

In the model cases, F is one of Pucci's extremal operators $\mathcal{M}^-_{\lambda,\Lambda}$ and $\mathcal{M}^+_{\lambda,\Lambda}$, respectively defined as

$$\mathcal{M}^-_{\lambda,\Lambda}(X) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{tr}(AX) = \lambda \sum_{\mu_i>0} \mu_i + \Lambda \sum_{\mu_i<0} \mu_i$$

$$\mathcal{M}^+_{\lambda,\Lambda}(X) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{tr}(AX) = \Lambda \sum_{\mu_i>0} \mu_i + \lambda \sum_{\mu_i<0} \mu_i$$

where $\mathcal{A}_{\lambda,\Lambda} = \{A \in \mathcal{S}_n \,:\, \lambda\, I_n \leq A \leq \Lambda\, I_n\}$, ($I_n$ identity matrix), and $\mu_1, \ldots, \mu_n$ are the eigenvalues of the matrix $X \in \mathcal{S}_n$.

[C. Pucci, 1966]

Let us recall that

- Pucci's extremal operators act as barriers in the whole class of uniformly elliptic operators, that is

$$\mathcal{M}^-_{\lambda,\Lambda}(X) \leq F(X) \leq \mathcal{M}^+_{\lambda,\Lambda}(X)$$

  for any uniformly elliptic F having ellipticity constants Λ, λ.

- They play a crucial role in the regularity theory for fully nonlinear elliptic equations [Caffarelli, Cabré, AMS book 1995]

- Pucci's extremal operators appear in the context of stochastic control [Bensoussan, Lions, book 1982]

- They can be seen as a generalization of the Laplace operator

$$\mathcal{M}^-_{\lambda,\lambda}(X) = \mathcal{M}^+_{\lambda,\lambda}(X) = \lambda \, \mathrm{tr}(X)$$

- One has

$$\mathcal{M}^-_{\lambda,\Lambda}(-X) = -\mathcal{M}^+_{\lambda,\Lambda}(X)$$

Let us emphasize that, for $\lambda < \Lambda$, the operators $\mathcal{M}^{\pm}_{\lambda,\lambda}$ are neither linear nor in divergence form. As an example, observe that

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) = 0 \quad \text{in } \mathbb{R}^2$$

$$\Longleftrightarrow$$

$$\Delta u = \left(\sqrt{\frac{\Lambda}{\lambda}} - \sqrt{\frac{\lambda}{\Lambda}}\right)\sqrt{-\det(D^2u)}$$

For smooth, radial functions $u(x) = u(|x|)$, Pucci's operators take the form

$$\begin{aligned}\mathcal{M}^+(D^2u) = \ & \Lambda(u''(r))^+ - \lambda(u''(r))^- \\ & + \Lambda(N-1)\left(\frac{u'(r)}{r}\right)^+ - \lambda(N-1)\left(\frac{u'(r)}{r}\right)^- ,\end{aligned}$$

as well as

$$\begin{aligned}\mathcal{M}^-(D^2u) = \ & \lambda(u''(r))^+ - \Lambda(u''(r))^- \\ & + \lambda(N-1)\left(\frac{u'(r)}{r}\right)^+ - \Lambda(N-1)\left(\frac{u'(r)}{r}\right)^- .\end{aligned}$$

Thus, the ODEs satisfied by radial solutions u of Pucci's extremal equations have coefficients jumping at the points where u changes monotonicity and/or convexity. For fixed monotonicity/convexity regime, the obtained ODEs are analogous to the ones satisfied by solutions of Laplace operator, with coefficients replaced by new ones depending on the dimension like parameters

$$\tilde{N}_+ = \frac{\lambda}{\Lambda}(N-1)+1\,, \quad \tilde{N}_- = \frac{\Lambda}{\lambda}(N-1)+1\,.$$

We will refer to $\tilde{N}_+$ as the effective dimension associated with $\mathcal{M}^+$, as well as to $\tilde{N}_-$ as the effective dimension associated with $\mathcal{M}^-$. Note that one has always

$$\tilde{N}_- \geq N \geq \tilde{N}_+ \,,$$

with equalities holding true if and only if $\Lambda = \lambda$. We will assume always that $\tilde{N}_+ > 2$.

In the linear case $\Lambda = \lambda$, up to a constant factor the problem becomes

$$\begin{cases} -\Delta u_\gamma = \lambda_\gamma \dfrac{u_\gamma}{|x|^\gamma} & \text{in } B \setminus \{0\} \\ \qquad\qquad u_\gamma = 0 & \text{on } \partial B \end{cases}$$

which can be solved by considering the minimum problem

$$\lambda_\gamma := \inf_{u \in H_0^1(B), \int_B \frac{|u(x)|^2}{|x|^\gamma} = 1} \int_B |\nabla u|^2 \, dx$$

i.e. by minimizing the normalized related Rayleigh quotient.

When $\gamma < 2$, by the compact embedding of $H_0^1(B)$ into the weighted space $L^2\left(B, \frac{1}{|x|^\gamma}\right)$, the positivity of λ_γ and the existence of a related positive, radial, smooth eigenfunction u_γ can be easily proved by standard arguments of the direct methods in calculus of variations.

If $\gamma = 2$, one still has the continuous inclusion of $H_0^1(B)$ into $L^2\left(B, \frac{1}{|x|^2}\right)$ and λ_2 is nothing but the inverse of the best constant in Hardy's inequality

$$\lambda_2 = \frac{(N-2)^2}{4}\,.$$

Due to loss of compactness, the problem has no finite energy solutions, and an explicit radial solution (not belonging to $H_0^1(B)$) is

$$u_2(x) = \frac{(-\ln|x|)}{|x|^{\frac{N-2}{2}}}\,.$$

Finally, if $\gamma > 2$, there is no embedding of $H_0^1(B)$ into $L^2\left(B, \frac{1}{|x|^\gamma}\right)$, and

$$\lambda_\gamma = 0\,.$$

The relevant literature in the linear case is extremely extensive. Let us quote only the monographs [I. Peral, F. Soria, 2021] and [F.C. Cirstea, 2014] for Hardy type inequalities and related elliptic (and parabolic) problems in the critical case.

In the fully nonlinear case, we can follow the same approach used to study the principal eigenvalue problem for regular potentials (i.e. $\gamma = 0$).
Let us define the eigenvalue on the model of [H. Berestycki, L. Nirenberg, S.R.S. Varadhan, 1994], i.e. by the optimization formula

$$\lambda_\gamma := \sup\{\mu : \exists\, u \in C(B \setminus \{0\}), \\ u > 0 \text{ in } B \setminus \{0\},\ F(D^2u) + \mu \frac{u}{|x|^\gamma} \leq 0\}$$

This is the standard approach to study eigenvalues problems in the fully nonlinear framework, see [I. Birindelli, F. Demengel 2007 and 2010], [J. Busca, M. Esteban, A. Quaas, 2005], [H. Ishii, Y. Yoshimura, unpublished preprint 2005], [A. Quaas, B. Sirakov, 2008].

As in the linear case, we obtain different results according to the cases $\gamma < 2$, $\gamma = 2$ and $\gamma > 2$.

The essential feature characterizing the case $\gamma < 2$ is the existence of bounded barriers:
for any $0 < \tau \leq 2 - \gamma$ the function

$$w(r) = 1 - r^\tau$$

satisfies

$$\mathcal{M}^+(D^2 w) \leq -c\, r^{-\gamma} \qquad \text{in } B \setminus \{0\}$$

for a universal constant $c = c(\lambda, \Lambda, N, \gamma)$.

Moreover, we have in this case a comparison principle for smooth, radial bounded sub and super-solutions which does not require any condition at the origin.

Proposition

Let $f \in C(B)$ and $u, v \in C(\overline{B}) \cap C^2(B \setminus \{0\})$ be radial functions satisfying

$$F(D^2 u) \geq f(r) r^{-\gamma} \geq F(D^2 v) \qquad \text{in } B \setminus \{0\}\,.$$

Then, $u \leq v$ on ∂B implies $u \leq v$ in $\overline{B}$.

These basic ingredients lead to the following result.

Theorem

Suppose that $\gamma < 2$. *Then:*

(i) $\lambda_\gamma > 0$ *and there exists a function* u, *continuous in* $\overline{B} \setminus \{0\}$, *radial, strictly positive in* B, *such that*

$$\begin{cases} -F(D^2 u) = \lambda_\gamma \frac{u}{r^\gamma} & \text{in } B \setminus \{0\} \\ u = 0 & \text{on } \partial B \end{cases}$$

Furthermore u *is* $C^2(B \setminus \{0\})$ *and it can be extended on* B *as a Lipschitz continuous function if* $\gamma \leq 1$, *as a function of class* $C^1(B)$ *when* $\gamma < 1$, *and as an Hölder continuous function with exponent* $2 - \gamma$ *if* $\gamma > 1$.

Theorem

(ii) λ_γ *is stable under regular approximations both of the potential and the domain:*

$$\lambda_\gamma = \lim_{\epsilon \to 0} \lambda \left(F, \frac{1}{(r^2 + \epsilon^2)^{\frac{\gamma}{2}}}, B \right),$$

$$\lambda_\gamma = \lim_{\delta \to 0} \lambda \left(F, \frac{1}{r^\gamma}, B \setminus \overline{B_\delta} \right).$$

Remark

The existence and uniqueness results established "below" the eigenvalue are limited to smooth, radial solutions and they actually allow to prove that the value

$$\lambda'_\gamma := \sup\left\{\mu \,:\, \exists\, u \in C^2(B\setminus\{0\}),\ u \text{ radial}, \right.$$
$$\left. u>0 \text{ in } B\setminus\{0\},\ F(D^2u)+\mu\frac{u}{|x|^\gamma}\le 0\right\}$$

is an eigenvalue for which there exist positive radial eigenfunctions. On the other hand, the full result is recovered thanks to the stability properties. Indeed, one has

$$\lambda'_\gamma \le \lambda_\gamma \le \lambda_\gamma(B\setminus\overline{B_\delta})$$
$$\le \lambda_\gamma\left(\tfrac{1}{(r^2+\epsilon^2)^{\gamma/2}}, B\setminus\overline{B_\delta}\right) \stackrel{\delta\to 0}{\to} \lambda_\gamma\left(\tfrac{1}{(r^2+\epsilon^2)^{\gamma/2}}, B\right) \stackrel{\epsilon\to 0}{\to} \lambda'_\gamma$$

Remark

We proved also that the eigenvalue λ'_γ is simple, meaning that the radial eigenfunction is unique up to multiplicative constants. However, due to the singularity of the potential, we cannot prove that all the eigenfunctions are radial.

Remark

For the Pucci's operators $\mathcal{M}^\pm$ the existence of the eigenfunctions can be proved also by a direct approach, that is by solving the Cauchy problem for the associated ODE, in which we impose a priori the expected monotonicity and convexity properties of the eigenfunctions.

In the case $\gamma = 2$, repeating the computations for radial solutions of the linear equation, one immediately realizes that also for the operators $\mathcal{M}^\pm$ there are explicit solutions of the eigenvalue problem. Namely,

$$u(r) = \frac{-\ln r}{r^{\frac{\tilde{N}_+ - 2}{2}}} \Longrightarrow -\mathcal{M}^+(D^2u) = \Lambda \left(\frac{\tilde{N}_+ - 2}{2}\right)^2 \frac{u}{r^2}$$

$$u(r) = \frac{-\ln r}{r^{\frac{\tilde{N}_- - 2}{2}}} \Longrightarrow -\mathcal{M}^-(D^2u) = \lambda \left(\frac{\tilde{N}_- - 2}{2}\right)^2 \frac{u}{r^2}$$

Then, by definition, one has

$$\Lambda \left(\frac{\tilde{N}_+ - 2}{2}\right)^2 \leq \lambda_2(\mathcal{M}^+), \quad \lambda \left(\frac{\tilde{N}_- - 2}{2}\right)^2 \leq \lambda_2(\mathcal{M}^-)$$

In order to establish equalities, we pursue the analogy with the linear case, fully taking advantage of the radial symmetry. One easily proves that

$$\left(\frac{\tilde{N}-2}{2}\right)^2 = \inf_{\substack{u\in\mathcal{H}^1_0,\\ \int_0^1 u^2 r^{\tilde{N}-3}dr=1}} \int_0^1 |u'|^2 r^{\tilde{N}-1} dr$$

Now, the variational formulation is clearly stable with respect to the exponent γ, so that

$$\inf_{\substack{u\in\mathcal{H}^1_0,\\ \int_0^1 u^2 r^{\tilde{N}-3}dr=1}} \int_0^1 |u'|^2 r^{\tilde{N}-1} dr \overset{\gamma\to 2}{\longleftarrow} \inf_{\substack{u\in\mathcal{H}^1_0,\\ \int_0^1 u^2 r^{\tilde{N}-1-\gamma}dr=1}} \int_0^1 |u'|^2 r^{\tilde{N}-1} dr =: \lambda_{\gamma,var}$$

On the other hand, for $1 < \gamma < 2$ the variational formulation produces a solution of the ODE which is always decreasing and convex, so that it is a radial solution for Pucci's operator and, therefore, an eigenfunction. In other words, we have

$$1 < \gamma < 2 \Longrightarrow \lambda_\gamma = \Lambda\, \lambda_{\gamma, var}\,.$$

Thus, by using also the monotonicity of the eigenvalue with respect to the potential, we conclude

$$\lambda_2 \leq \lambda_\gamma = \Lambda\, \lambda_{\gamma, var} \overset{\gamma\to 2}{\to} \Lambda\, \lambda_{2, var} = \Lambda \left(\frac{\tilde{N}_+ - 2}{2} \right)^2 \leq \lambda_2\,.$$

Theorem

Assume that $\gamma = 2$*. Then:*

(i) $\lambda_2(\mathcal{M}^+) = \Lambda \left(\frac{\tilde{N}_+ - 2}{2}\right)^2$, $\quad \lambda_2(\mathcal{M}^-) = \lambda \left(\frac{\tilde{N}_- - 2}{2}\right)^2$ *and the functions*

$$u^\pm(x) = \frac{-\ln r}{r^{\frac{\tilde{N}_\pm - 2}{2}}}$$

are related explicit eigenfunctions;

(ii) $\lambda_2(\mathcal{M}^\pm)$ *are stable with respect to potential and domain approximations:*

$$\lambda_2(\mathcal{M}^\pm) = \lim_{\gamma \to 2} \lambda_\gamma(\mathcal{M}^\pm)$$

$$\lambda_2(\mathcal{M}^\pm) = \lim_{\delta \to 0} \lambda(\mathcal{M}^\pm, B \setminus \overline{B_\delta})$$

$$\lambda_2(\mathcal{M}^\pm) = \lim_{\epsilon \to 0} \lambda \left(\mathcal{M}^\pm, \frac{1}{r^2 + \epsilon^2}\right)$$

Theorem

(iii) *for any operator F one has*

$$\Lambda\left(\frac{\tilde{N}_+ - 2}{2}\right)^2 \leq \lambda_2(F) \leq \lambda\left(\frac{\tilde{N}_- - 2}{2}\right)^2 .$$

Let us remark that, in the case $\gamma = 2$, for a general F we cannot prove up to now the existence of eigenfunctions.

Finally, if $\gamma > 2$, the too strong singularity of the potential prevents the existence of positive supersolutions, at least in the radial case.

Theorem

If $\gamma > 2$, then $\lambda'_\gamma = 0$.