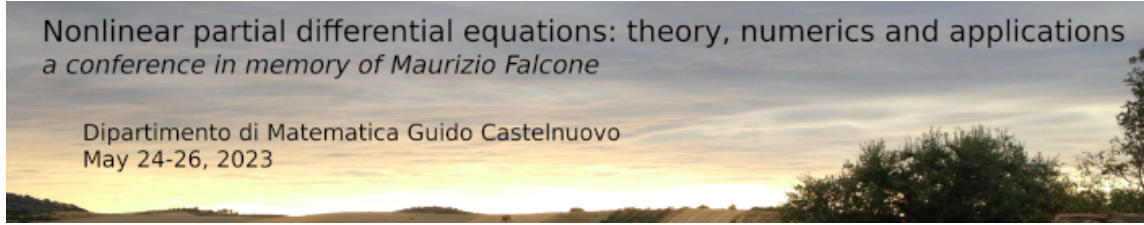

Multiscale models of cell movements and their numerical approximation

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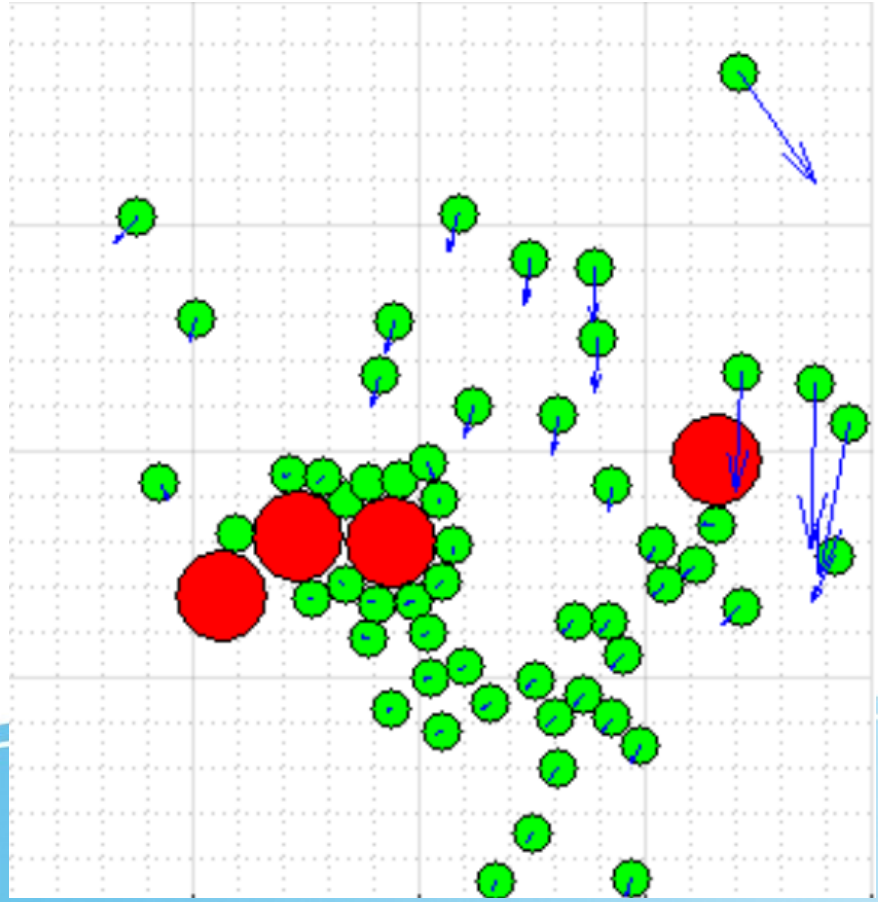
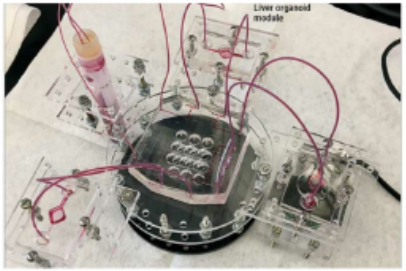
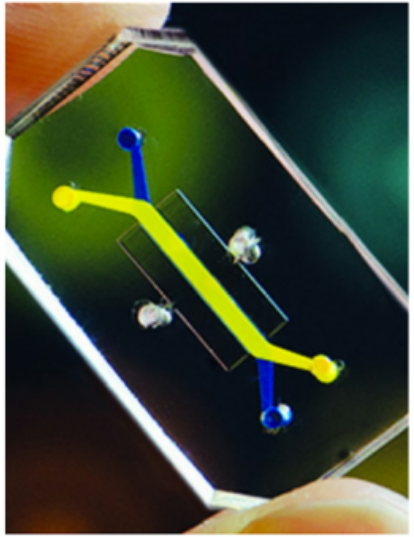
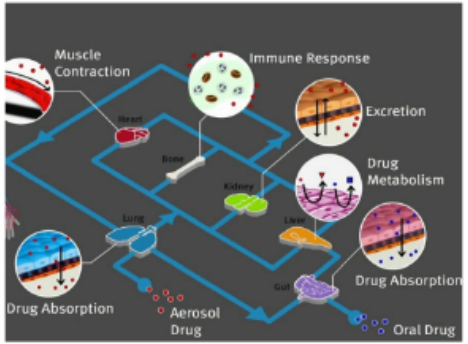
In collaboration with
Gabriella Bretti (IAC-Cnr)
Thierry Paul (Cnrs, Paris Sorbonne)
Marta Menci (Uni. Campus-Biomedico)

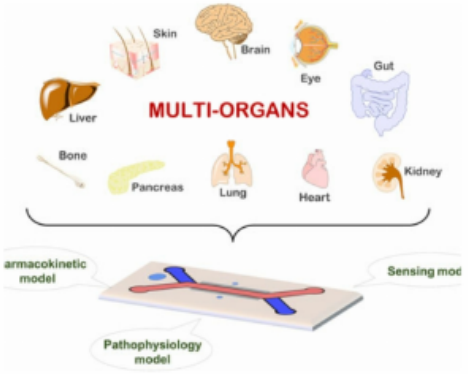


Organs-on-chip and the new medicine

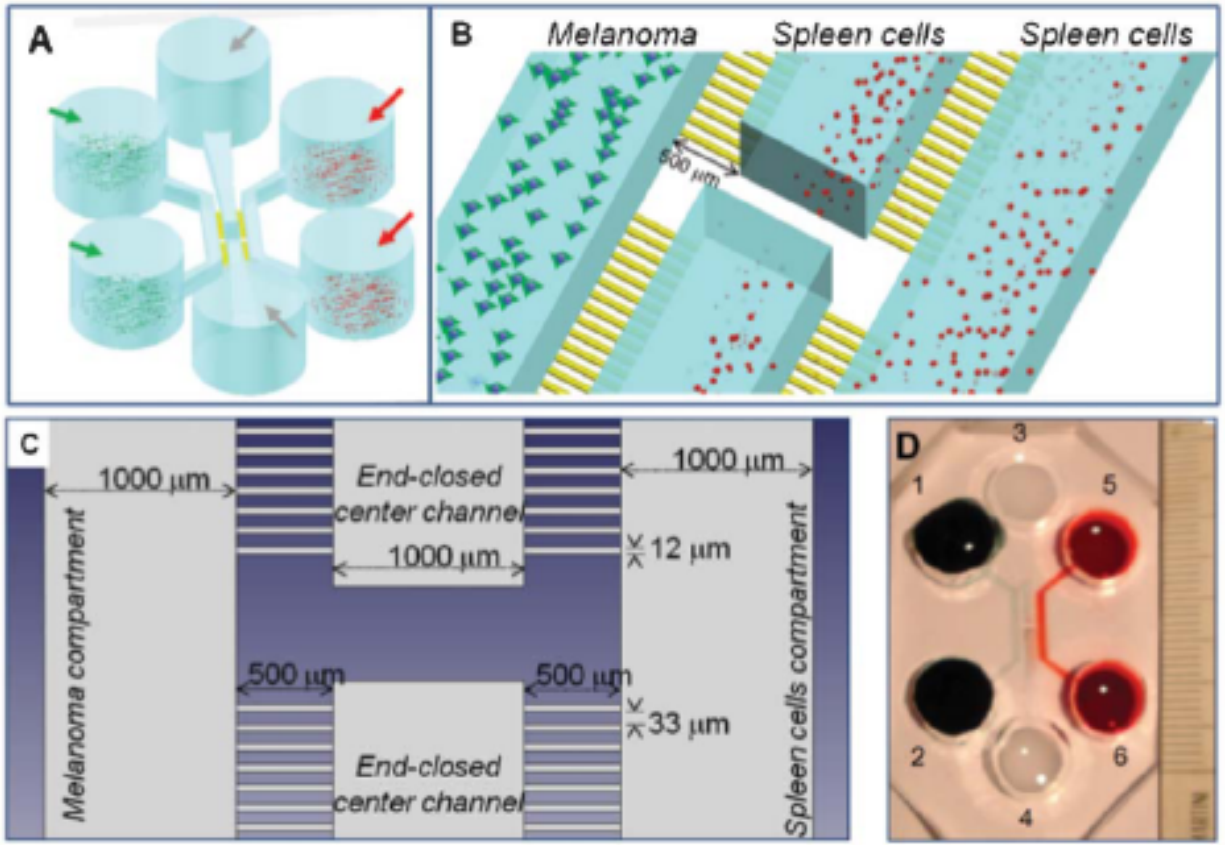




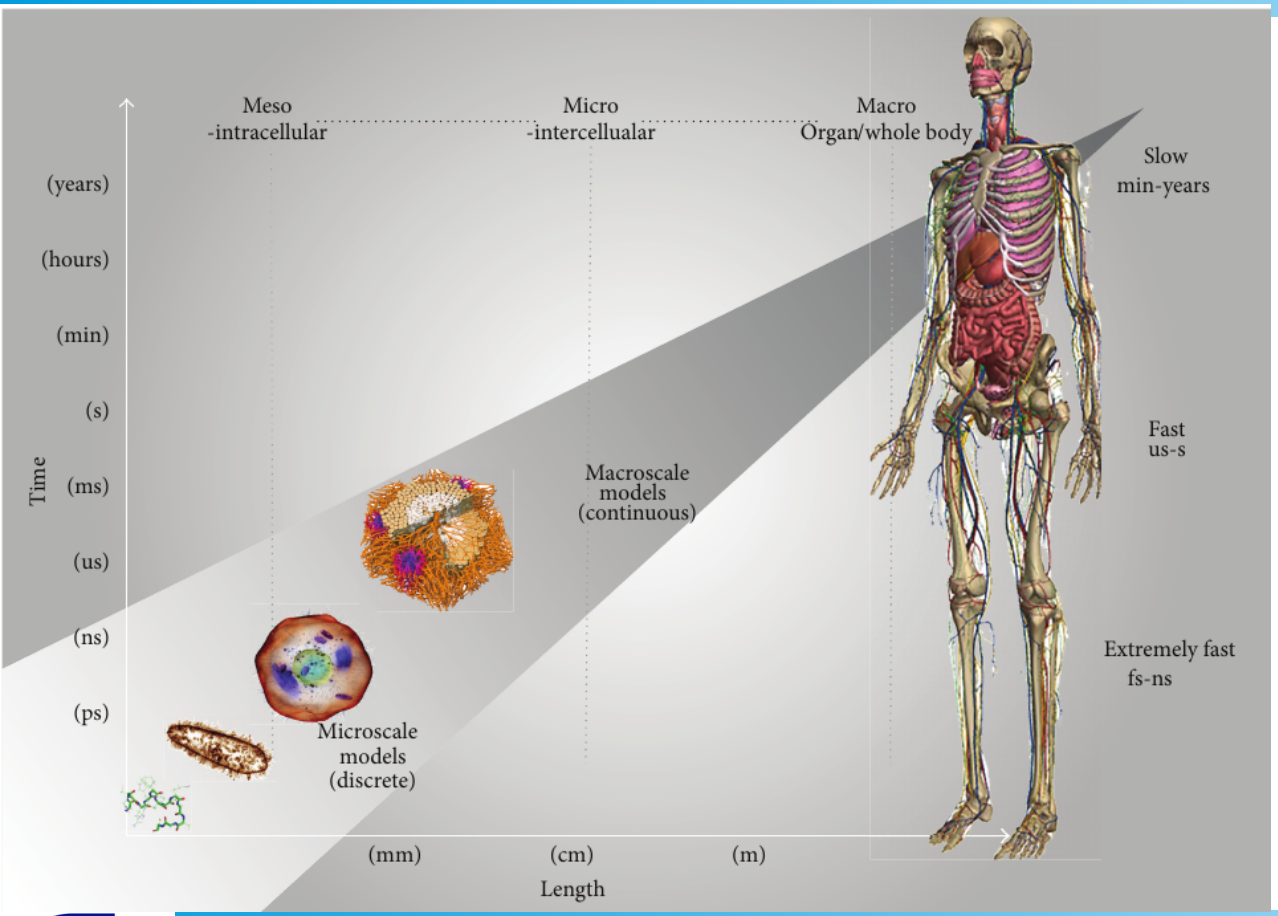




Organs-on-chip and the new medicine



The multiscale approach



- The real goal of these models is to use Organs-on-chip to predict the behaviour of real organs, so on the scale of billions of cells, using the data from the experiments at the microscale
- Create more realistic digital twins of our body
- It is crucial to keep the micro information in the memory of the macro models

A crop of the full chip

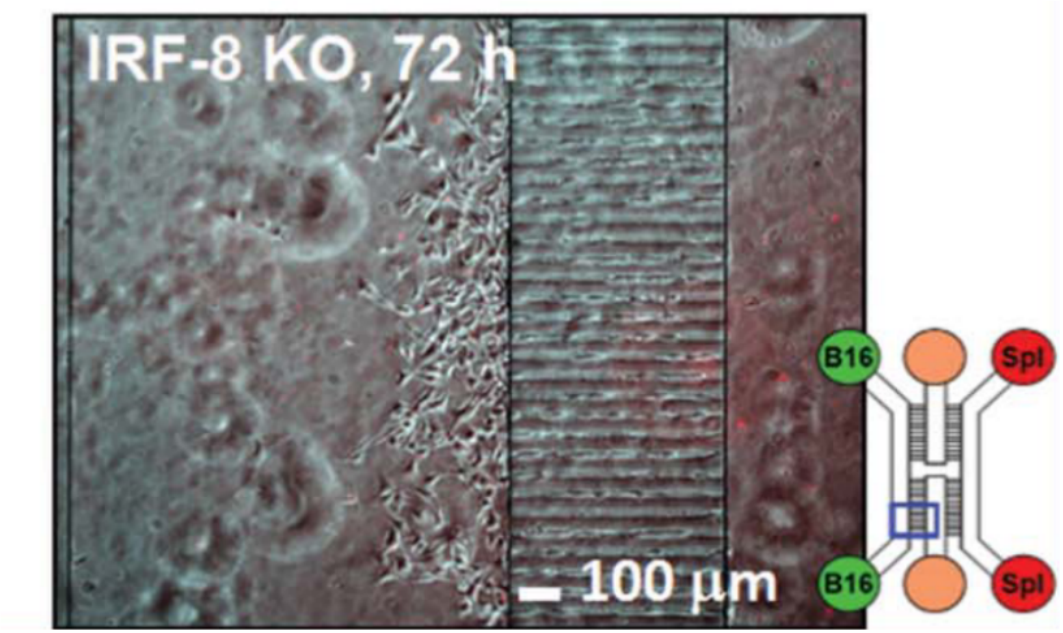
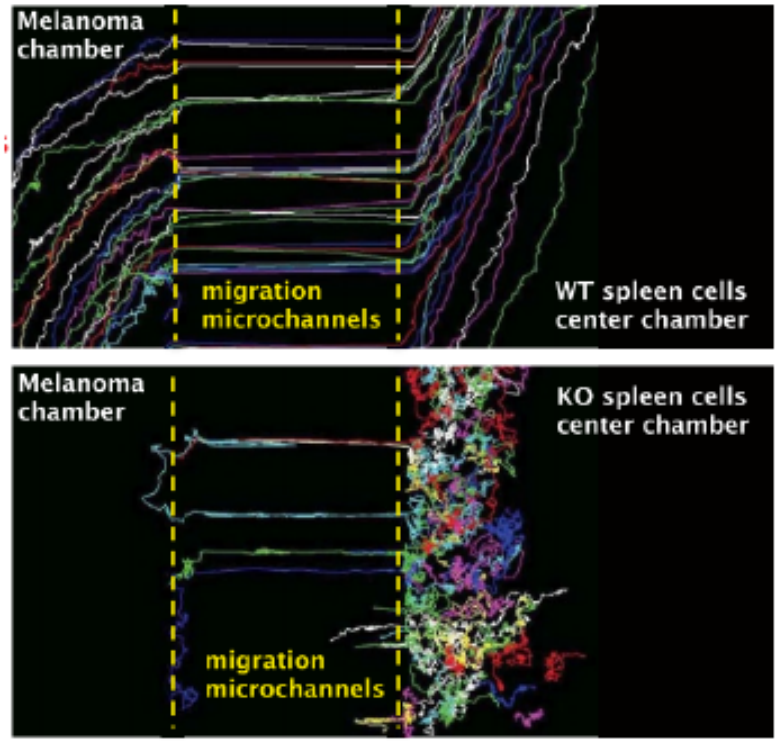


Figure: Crop of the microfluid chip after 72 hours

De Ninno A., Bertani F.R., Gerardino A., Schiavoni G., Musella M., Galassi C., Mattei F., Sistigu A., Businaro L. Microfluidic Co-Culture Models for Dissecting the Immune Response in in vitro Tumor Microenvironments. J. Vis. Exp. 170, e61895, doi:10.3791/61895, 2021

Cell trajectories



Cell trajectories performed by WT-cells and KO-type cells

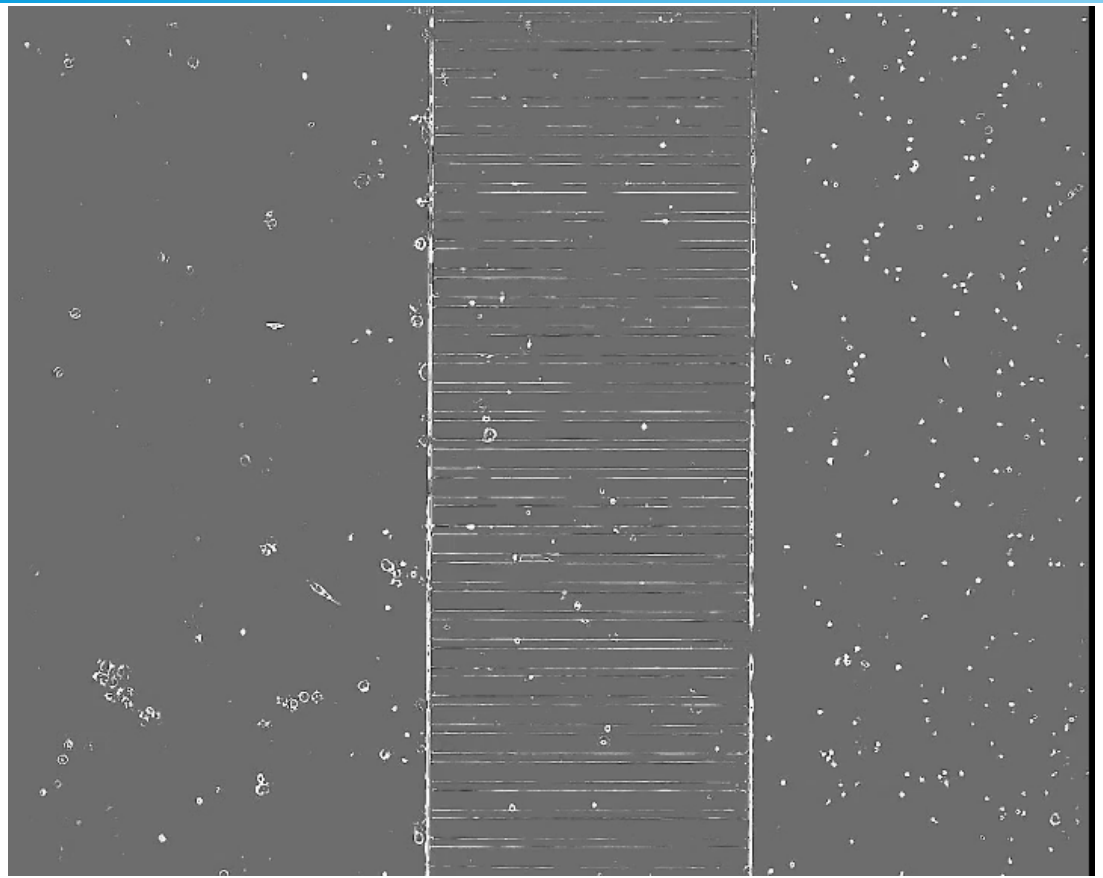
Experiments (see Vacchelli et al: Science. (2015), Businaro et al: Lab on Chip rep.(2014))

- Tumour and KT/WT-type immune cells
- KO-type: uncorrelated random walk
- WT-type: strongly directed movement ⇒ Chemotaxis
- Tumour cells local sources of chemoattractan
- WT-type: Generation of tumour killing agent
- cytokine
- Chemical diffusion

Available Data:

- Tumour and immune cell trajectories
- No information about chemical distribution

The time-lapse of the experiment



- 1702μm × 1361μm
- Duration: 48h-72h
- Cell movement: Diffusion and Chemotaxis
- Micro-channels allow only one cell to move through each.

A first macroscopic model

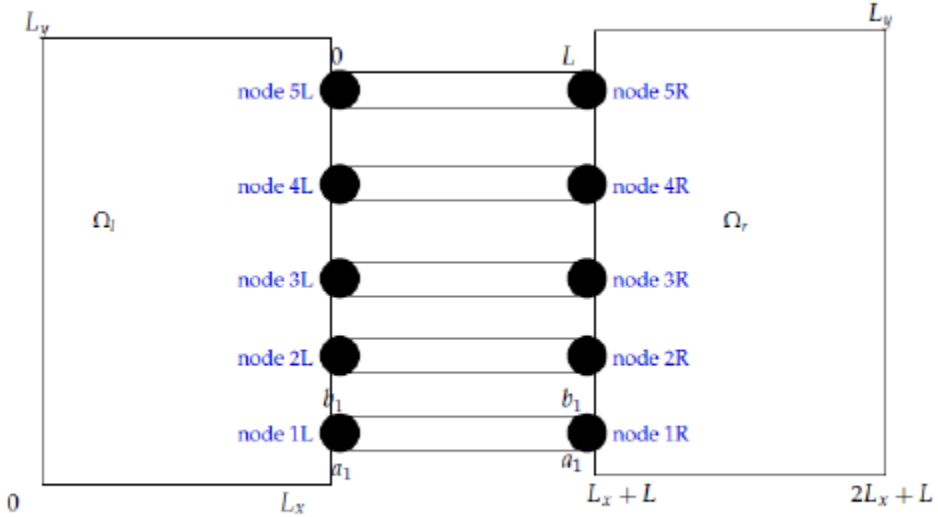


Figure: Numerical domain

Model(2D chamber) modified Keller-Segel model

$$\frac{\partial}{\partial t} T = D_T \Delta T - \lambda_T(\omega) T - k_T(t) T,$$

$$\frac{\partial}{\partial t} M = D_M \Delta M - div(M \cdot \chi \nabla) - k_M(t) M,$$

$$\frac{\partial}{\partial t} \varphi = D_\varphi \Delta \varphi + \alpha_\phi T - \beta_\varphi \varphi,$$

$$\frac{\partial}{\partial t} \omega = D_\omega \Delta \omega + \alpha_\omega M - \beta_\omega \omega,$$

Braun E.C., Bretti G., Natalini R. Mass-Preserving Approximation of a Chemotaxis Multi-Domain Transmission Model for Microfluidic Chips. Mathematics, 9(6)688, 1–34, 2021

A first macroscopic model

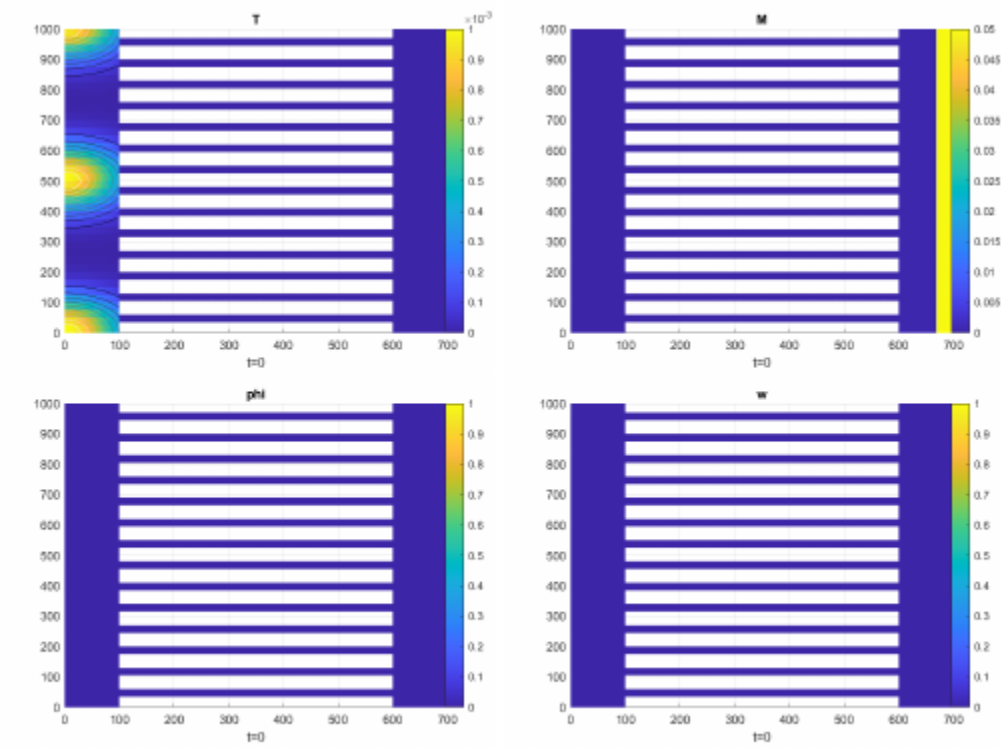


Figure: Initial distribution of T_0, M_0, φ_0 and ω_0

- Implicit Crank-Nicolson method for diffusion term and source term
- Explicit central discretization for chemotactical convection term
- Articial viscosity for improved stability in presence of high Péclet numbers
- Asymptotic Higher Order schemes(AHO) for the hyperbolic partial dierential equation.
- Numerical boundary conditions that preserve the mass (for no-ux BC).
- Trasmission condition of Kedem-Katchalski type (jump for density at interface)

A first macroscopic model

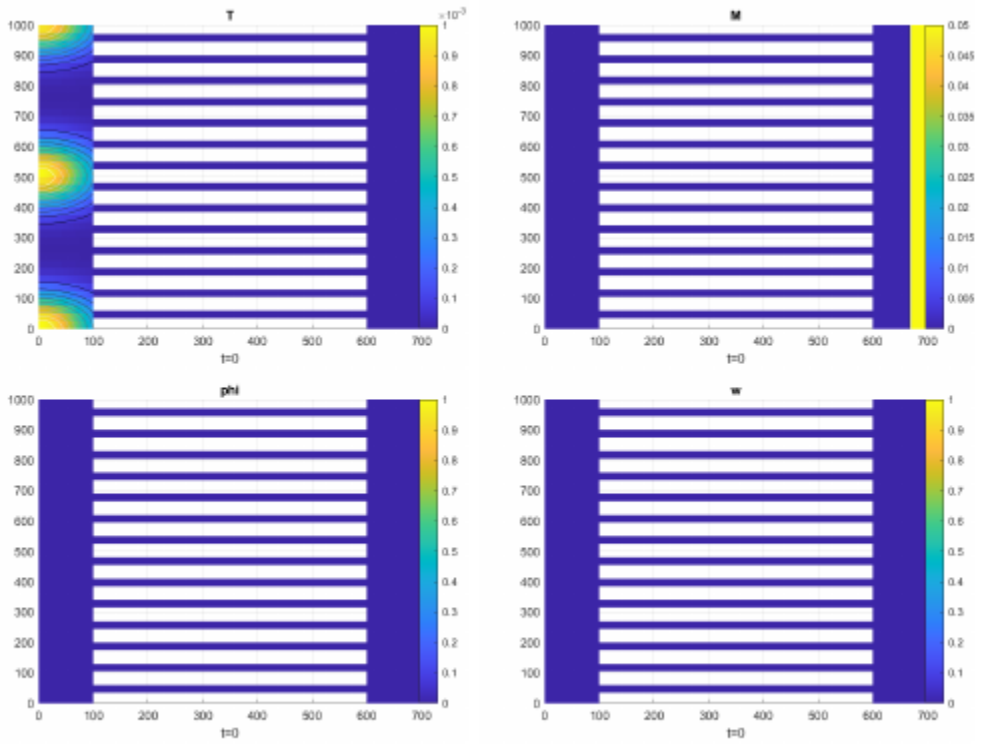
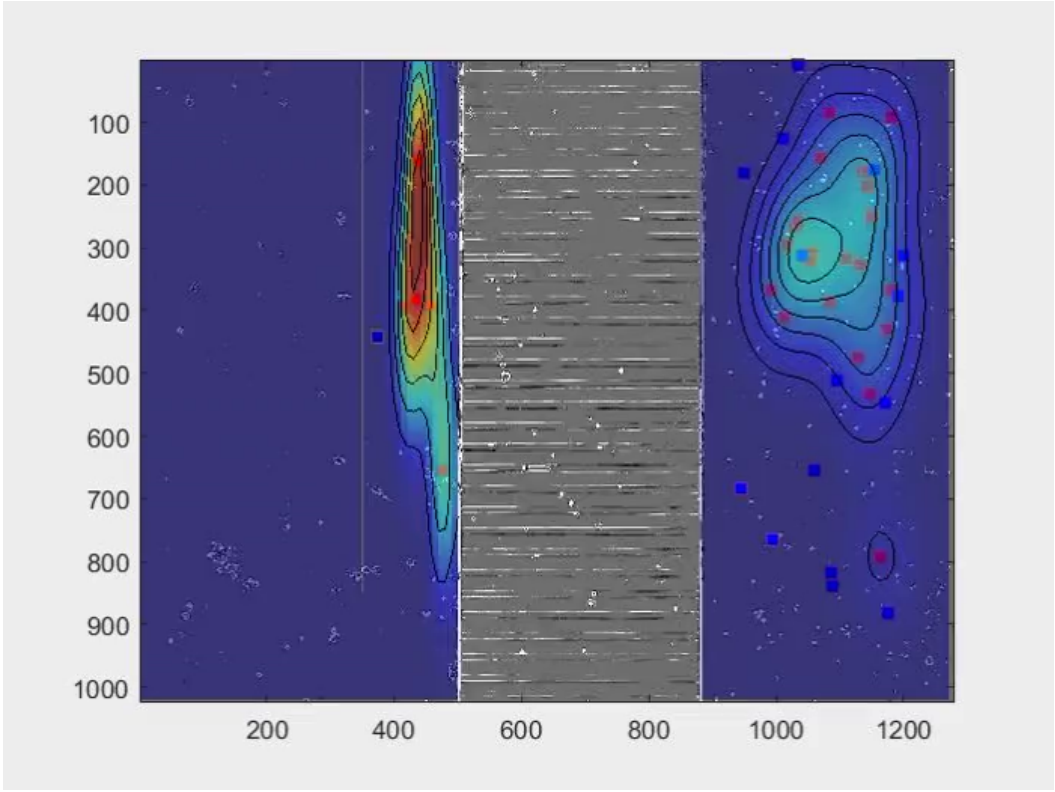
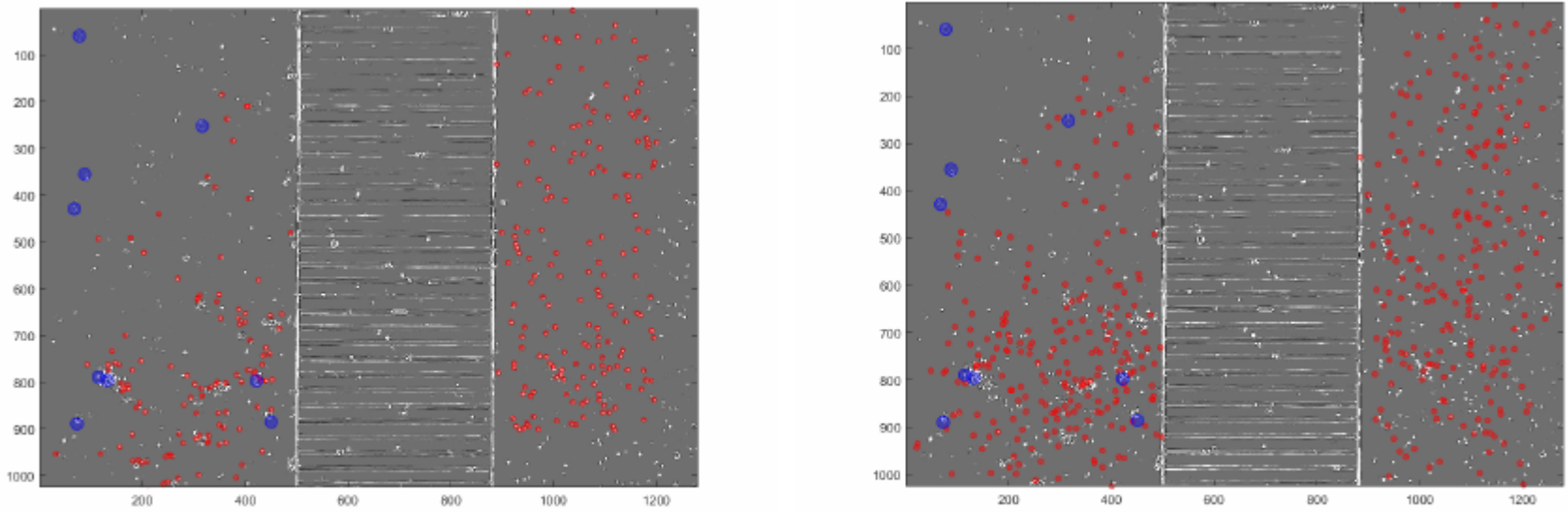


Figure: Initial distribution of T_0, M_0, φ_0 and ω_0



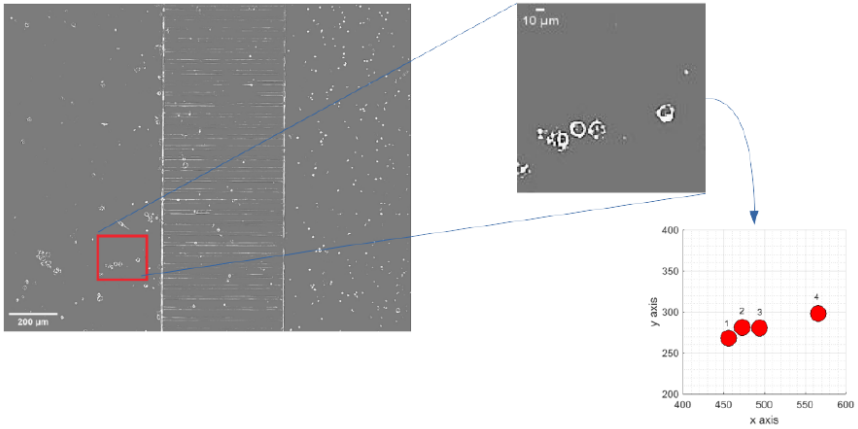
Back to the experimental data

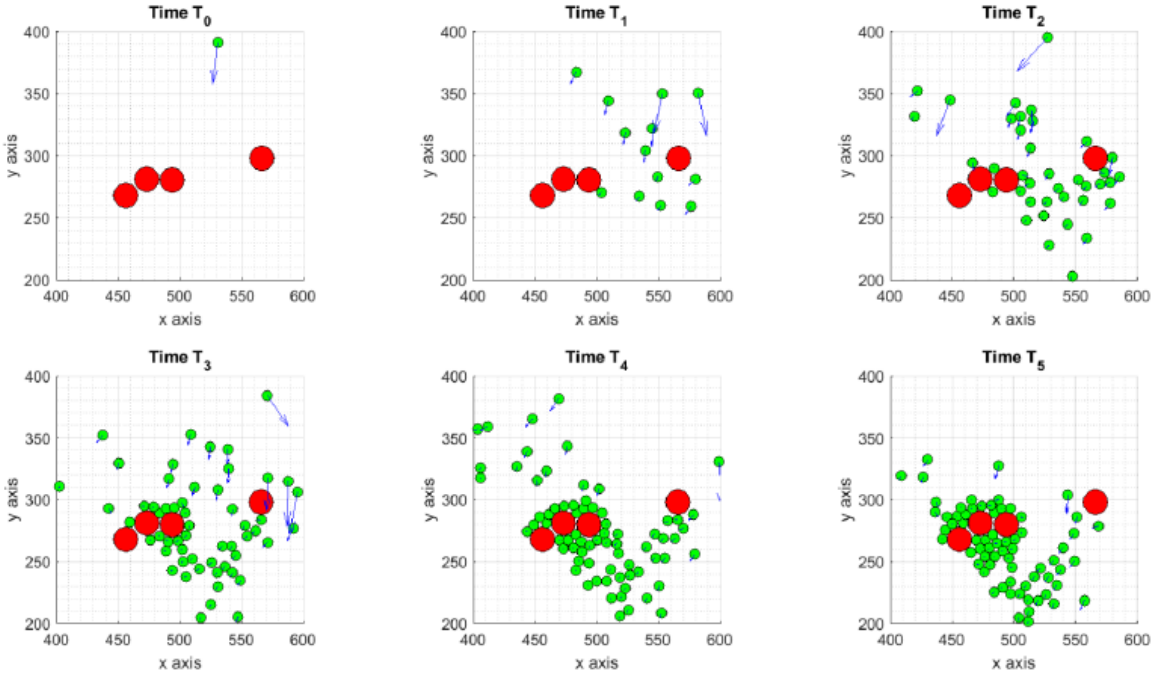


(a) (b)

Figure: left: Tracked cell . Right: Artificially created cells.

Beyond Keller-Segel: a micro-hybrid model



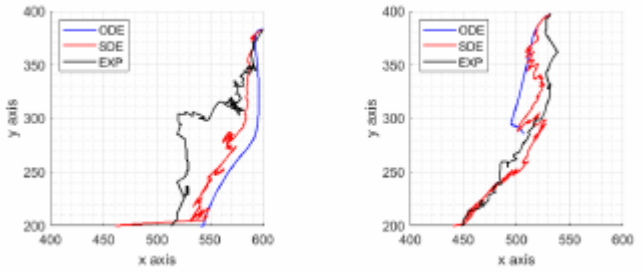


Gabriella Bretti, Adele De Ninno, Roberto Natalini, Daniele Peri and Nicole Roselli, Estimation Algorithm for a Hybrid PDE–ODE Model Inspired by Immunocompetent Cancer-on-Chip Experiment, Axioms 2021, 10(4), 243

A hybrid model

$X_i(t)$: position of the i-th cell
$\phi(x,t)$: concentration of the chemoattractant

$$\begin{cases} \overbrace{\ddot{X}_i}^{\text{acceleration i-th cell}} = \overbrace{\gamma F_1\left(\chi(\phi)\nabla\phi\right)}^{\text{chemotaxis}} + \overbrace{F_2(X,Y)}^{\text{immune/cancer}} + \overbrace{F_3(X)}^{\text{attraction/repulsion}} - \overbrace{\mu\left(\dot{X}_i - \sigma\psi\right)}^{\text{damping with noise}} \\ \overbrace{\partial_t\phi}^{\text{chemo rate in time}} = \overbrace{D\Delta\phi}^{\text{diffusion}} + \overbrace{\xi F_4(Y)}^{\text{production cancer}} - \overbrace{\eta\phi}^{\text{molecular degradation}} \end{cases}$$



A basic discrete model with chemotaxis

$$(1)\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \gamma(v_i - v_j, x_i - x_j) + \eta \nabla_x \varphi^t(x_i) \\ \partial_t \varphi = D\Delta\varphi - \kappa\varphi + f(x, X(t)) \end{cases}$$

- x_i Position at time t of the i-th cell
- v_i Velocity at time t of the i-th cell
- φ Concentration of a chemoattractant produced by the cells
- $\gamma(v, x)$ Interaction function among the cells
- $f(x, X)$ Production of chemoattractant

Main goals

- Study of the mascroscopic behavior of the dynamical system
- Kinetic approach (Vlasov-type model): density of particles
- Hydrodynamic limit (Euler-type model): density of cells + velocity field: a numerical study

  - without chemotaxis
  - with chemotaxis

The Cucker-Smale model (no chemistry)

$$(2)\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi(x_i - x_j)(v_j - v_i) \end{cases}$$

* F. Cucker & S. Smale, Ieee T. Automat. Contr (2007)
* S.Y. Ha & J.G. Liu, Commun Math Sci (2009)

$$\psi(\xi) = \left(1 + \frac{\|\xi\|^2}{R^2}\right)^{-\sigma}$$

time-asymptotic flocking

$$\lim_{t\to+\infty} \sum_{i=1}^{N} \|v_i(t) - v_{\mathrm{CM}}(t)\|^2 = 0;$$

$$\sup_{0\le t\le+\infty} \sum_{i=1}^{N} \|x_i(t) - x_{\mathrm{CM}}(t)\|^2 < +\infty$$

The flow

The system (2) $\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \psi(x_i - x_j)(v_j - v_i) \end{cases}$

Generates a flow Φ^t which is, given the initial data $(x^{in}, v^{in}) = (x_1^{in}, \dots, x_N^{in}, v_1^{in}, \dots, v_N^{in})$

$$\Phi^t(x^{in}, v^{in}) = (x(t), v(t))$$

A first mean field approach: the empirical measures

Given a solution (x(t), v(t)) to system (2). The so-called empirical measure

$$\Pi_N^t(x,v) := \frac{1}{N}\sum_{i=1}^{N}\delta(x - x_i(t))\delta(v - v_i(t))$$

is a special solution to the Vlasov equation:

$$(3)\partial_t\rho^t(x,v) + v\cdot\nabla_x\rho^t(x,v) + \nabla_v\cdot\left(\rho^t(x,v)\int_{2d}\psi(x-y)(v-w)\rho^t(y,w)dydw\right) = 0$$

* S.-Y. Ha, E. Tadmor, From particle to kinetic and hydrodynamic descriptions of flocking, Kinet. Relat. Models 1 (2008) 415-435
* Carrillo, José A.; Choi, Young-Pil Mean-field limits: from particle descriptions to macroscopic equations. Arch. Ration. Mech. Anal. 24 (3) (2021) 1529-1573.

The mean field limit: the empirical measures approach

Theorem: given an initial measure ρ^{in} such that $\lim_{N\to\infty} W(\Pi_N^{in}, \rho^{in}) = 0$

then there exist a unique solution ρ^t to

$$\partial_t \rho^t(x,v) + v \cdot \nabla_x \rho^t(x,v) + \nabla_v \cdot \left(\rho^t(x,v) \int_{2d} \psi(x-y)(v-w)\rho^t(y,w)dydw \right) = 0$$

this solution belongs to $L^\infty([0,\infty); \mathcal{P}(\mathbb{R}^{\in \mathrm{d}}))$

and

$$\overline{\lim}_{N\to\infty} \sup_{t\in[0,\infty)} W(\Pi_N^t, \rho^t) = 0$$

Seung-Yeal Ha,Jeongho Kim,Xiongtao Zhang, Kinetic & Related Models (2018)

A different approach: Mean field limit by marginals

Idea: considering the movement of a generic particle solution to the Liouville equation associated to an Hamiltonian system

Consider the pushforward by the flow Φ^t of a compactly supported (symmetric by permutations of the variables N-particles) probability density ρ_N^{in} on the phase space

$$\Phi^t \# \rho_N^{in} := \rho_N^t$$

which is the solution to the Liouville's equations associated to the system

$$\partial_t \rho_N^t + V \cdot \nabla_X \rho_N^t + \sum_{i=1}^{N} \nabla_{v_i} . (G_i \rho_N^t) = 0, \ \rho_N^{t=0} = \rho_N^{in}$$

with $$G_i = \frac{1}{N} \sum_{j=1}^{N} \psi(x_i - x_j)(v_j - v_i)$$

Mean field limit by marginals

To describe the movement of a “generic” particle in the limit of diverging number of particles, we want to perform an average on the N particles but one. Then, we take the first marginal to the solution to the Liouville equation

$$\rho^t_{N;1}(x,\xi) := \int_{\mathbb{R}^{2(d-1)N}} \rho^t_N(x,\xi,x_2,\xi_2,\dots,x_N,\xi_N)dx_2 d\xi_2 \dots dx_N d\xi_N$$

The marginal approaches the solution to the Vlasov eq.

$$\rho^t_{N;1}(x,\xi) \approx \rho^t$$

A mean field limit with estimate

Theorem 1. R.N. - T. Paul, DCDS-B (2022)

Let Φ^t be the flow generated by the system (2), ψ bounded positive nonincreasing Lipschitz continuous, and let ρ^t be the solution to the Vlasov equation (3) with an initial condition $\rho^{in} \in L^1(\mathbb{R}^{2d})$ compactly supported. Let moreover $\rho^t_{N;1}$ be the first marginal of $\rho^t_N := \Phi^t \#(\rho^{in})^{\otimes N}$.

Then, for all $N > 1,\ t \geq 0$,

$$W_2(\rho^t_{N;1}, \rho^t) \leq 4\|\psi\|^2_\infty (2|\bar{v}| + |supp[\rho^{in}]|) \left(\frac{e^{Lt} - 1}{L}\right)^{\frac{1}{2}} N^{-\frac{1}{2}}$$

with

$$L := 2(1 + 8\|\psi\|^2_\infty \|v\|^2_{L^\infty(supp[\rho^{in}])}) \text{ and } \bar{v} = \int v\rho^{in} dxdv$$

where $\|v\|_{L^\infty(supp[\rho^{in}])} := \sup_{(x,v)\in supp[\rho^{in}]} |v|$.

Comparison between microscopic and Vlasov

Flocking case (Simulation by Marta Menci)

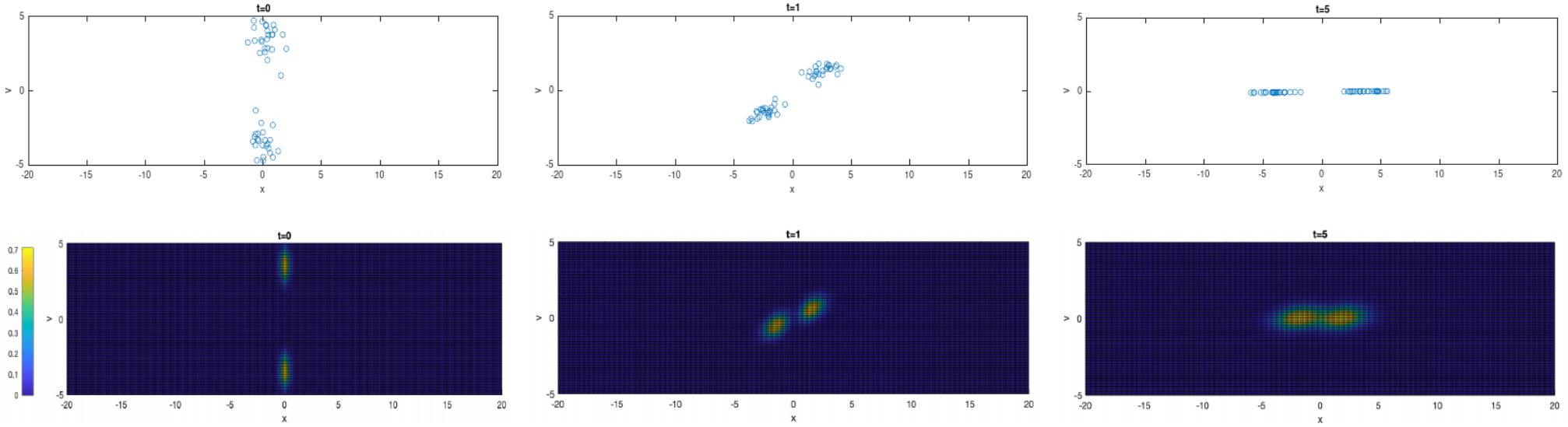


FIGURE 1. Test1: Numerical simulation of Cucker-Smale model with $\beta = 0.05$, at particle level (first line) and kinetic (second line) level.

Comparison between microscopic and Vlasov

Non flocking case (Simulation by Marta Menci)

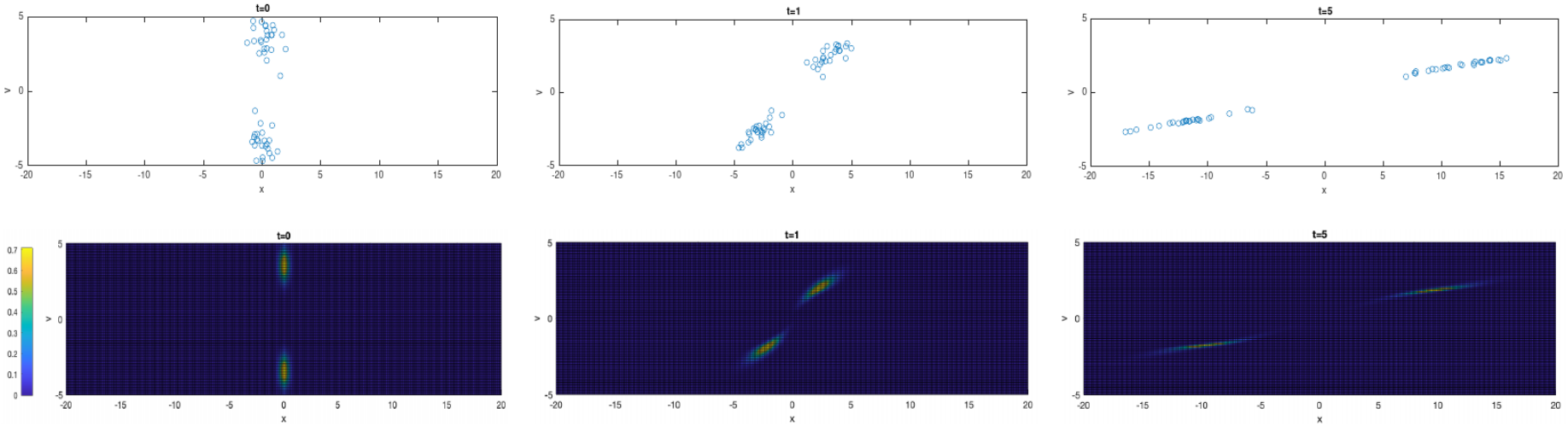


FIGURE 2. Test2: Numerical simulation of Cucker-Smale model with $\beta = 0.95$, at particle level (first line) and kinetic (second line) level.

Come back to chemotaxis

$$(1)\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \gamma(v_i - v_j, x_i - x_j) + \eta \nabla_x \varphi^t(x_i) \\ \partial_t \varphi = D\Delta\varphi - \kappa\varphi + f(x, X(t)) \end{cases}$$

- Non local space-time interaction in the Force field
- The associated solution is no more a flow!
- $\rho_N^t = \Phi_N^t \# \rho_N^{in}$ does not satisfy a closed equation (pseudo-Liouville eq.)

The nonlinear nonlocal Vlasov equation

$$(V)\quad \begin{cases} \partial_t \rho^t + v \cdot \nabla_x \rho^t = \nabla_v(\nu(t,x,v)\rho^t), \ \ \rho^0 = \rho^{in} \in \mathcal{P}(\mathbf{R}^{2d}) \\ \nu(t,x,v) = \gamma * \rho^t(x,v) + \eta \nabla_x \psi^t(x) + F_{ext}(x), \\ \partial_s \psi^s(x) = D\Delta_x \psi - \kappa\psi + g(x,\rho^s), \ \ \psi^0 = \varphi^{in}. \\ g(x,\rho^s) = \chi * \rho^s(x). \end{cases}$$

The main convergence theorem

Natalini, Roberto; Paul, Thierry The mean-field limit for hybrid models of collective motions with chemotaxis.
SIAM J. Math. Anal. 55, No. 2, 900-928 (2023)

Theorem 2.1. *Let ρ^{in} be a compactly supported probability on $\mathbf{R}^{2dN}$, let Φ_N^t be the mapping generated by the particles system $(3,4,5,6)$ as defined by (7), and let $\tau_{\rho^{in}}$ be the function defined in formula (41) below.*

Then, for any $t \geq 0$,

$$W_2\left((\Phi_N^t \#(\rho^{in})^{\otimes N})_{N;1}, \rho^t\right)^2 \leq \tau_{\rho^{in}}(t) \begin{cases} N^{-\frac{1}{2}} & d=1 \\ N^{-\frac{1}{2}} \log N & d=2 \\ N^{-\frac{1}{d}} & d>2 \end{cases}$$

Moreover, let us denote by $\varphi^t_{Z^{in}}$ the chemical density solution of $(3,4,5,6)$ with initial data (Z^{in}, φ^{in}) and by $\psi^t_{\rho^{in}}$ the one solution of $(14,15,16,17)$ with initial data $(\rho^{in}, \varphi^{in})$.

Then

$$\int_{\mathbf{R}^{2dN}} \|\nabla \varphi^t_{Z^{in}} - \nabla \psi^t_{\rho^{in}}\|^2_\infty (\rho^{in})^{\otimes N}(dZ^{in}) \leq \tau_c(t) \begin{cases} N^{-\frac{1}{2}} & d=1 \\ N^{-\frac{1}{2}} \log N & d=2 \\ N^{-\frac{1}{d}} & d>2 \end{cases}$$

Comparison between microscopic and Vlasov with chemotaxis

Non flocking case (Simulation by Marta Menci)

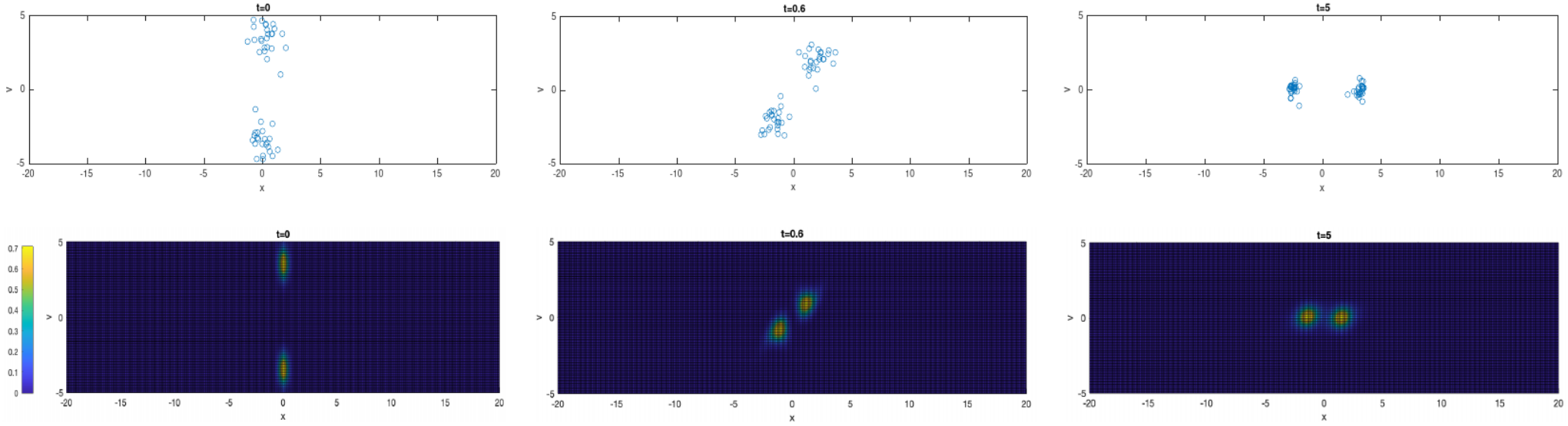


FIGURE 3. Test3: Numerical simulation of Cucker-Smale model with chemotaxis at particle level (first line) and kinetic (second line) level, with $\beta = 0.95$ and $\eta = 1.4$.

The monokinetic situation

Assume now the initial density is monokinetic $\rho^{in}(x,v) = \mu^{in}(x)\delta(v - u^{in}(x))$

Theorem 6.1. *Let μ^t, u^t, ψ^t be a solution to the following system*

$$\begin{cases} \partial_t \mu^t + \nabla(u^t \mu^t) = 0 \\ \partial_t(\mu^t u^t) + \nabla(\mu^t (u^t)^{\otimes 2}) = \mu^t \int \gamma(\cdot - y, u^t(\cdot) - u^t(y))\mu^t(y)dy + \eta\mu^t \nabla\psi^t + \mu^t F \\ \partial_s \psi^s = D\Delta\psi - \kappa\psi + \chi * \mu^s, \ \ s \in [0,t], \\ (\mu^0, u^0, \psi^0) = (\mu^{in}, u^{in}, \psi^{in}) \in H^s, \ \ s > \frac{d}{2} + 1. \end{cases}$$

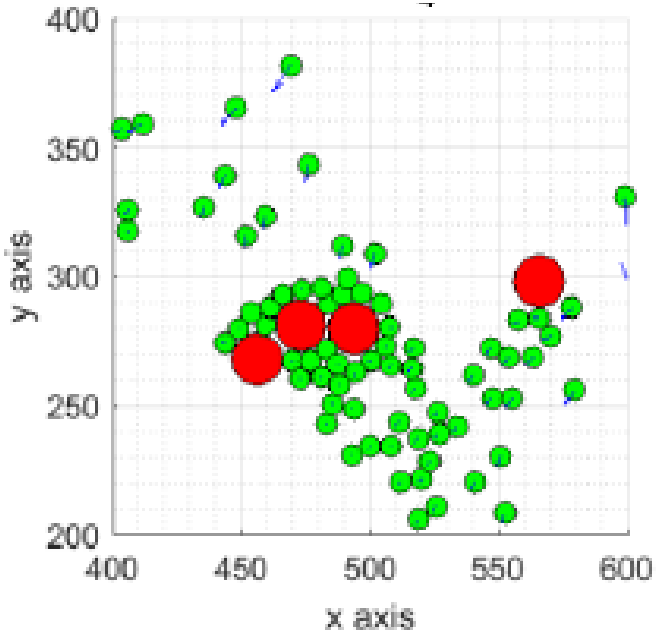
where $\mu^t, u^t \in C([0,t]; H^s) \cap C^1([0,T]; H^{s-1})$, $\psi^t \in C([0,t]; H^s) \cap C^1([0,T]; H^{s-2}) \cap L^2(0,T; H^{s+1})$ [3].

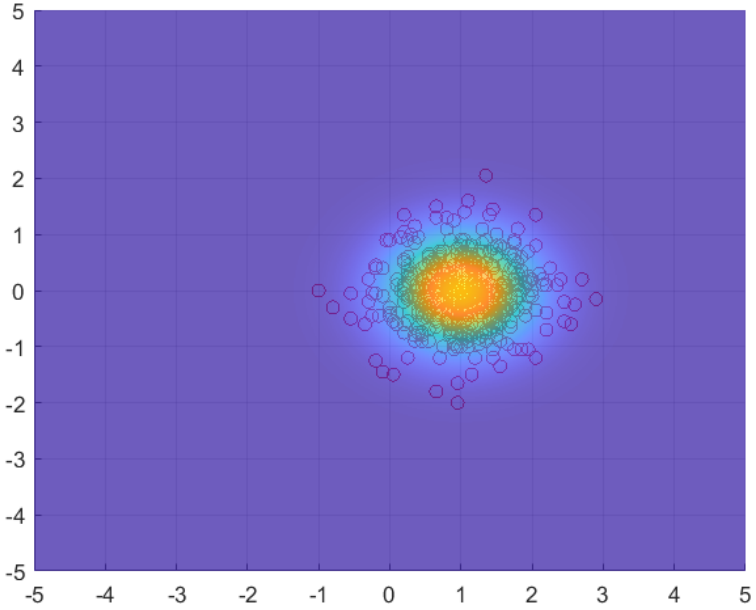
Then $\rho^t(x,v) := \mu^t(x)\delta(v - u^t(x))$ solves the following system

$$\begin{cases} \partial_t \rho^t + v \cdot \nabla_x \rho^t = \nabla_v(\nu(t,x,v)\rho^t), \\ \nu(t,x,v) = \gamma(x,v) * \rho^t + \eta \nabla_x \psi^t(x) + F_{ext}(x), \\ \partial_s \psi^s(z) = D\Delta_z \psi - \kappa\psi + g(z, \rho^s), \ \ \psi^0 = \psi^{in}, \\ \rho^0(x,v) = \mu^{in}(x)\delta(v - u^{in}(x)). \end{cases}$$

Looking for a good macroscopic model

Marta Menci, Roberto Natalini, Thierry Paul, MICROSCOPIC, KINETIC AND HYDRODYNAMIC HYBRID MODELS OF COLLECTIVE MOTIONS WITH CHEMOTAXIS: A NUMERICAL STUDY, preprint 2023





Comparison between Vlasov and the Hydrodynamic model

Nonlocal Valsov equation:

$$\begin{cases} \partial_t \rho^t + v\partial_x \rho^t = \partial_v(\nu(t,x,v)\rho^t), \\ \nu(t,x,v) = \int_2 \dfrac{v-w}{(1+|x-y|^2)^\beta}\rho^t(y,w)dydw + \eta\partial_x\psi^t(x) - \alpha v, \\ \partial_s\psi^s(x) = D\partial_x^2\psi - \kappa\psi + \int_{\mathbb{R}}\int_{x-R}^{x+R}\rho^s(y)dyd\xi, \end{cases}$$

Compare the momenta to the hydrodynamic quantities

$$\nu_0^t(x) = \int \rho^t(x,\xi)d\xi \approx \mu^t(x)$$

$$\nu_1^t(x) = \int \xi\rho^t(x,\xi)d\xi \approx \mu^t(x)u^t(x)$$

Our nonlocal hydrodinamic equation:

$$\begin{cases} \partial_t\mu^t + \partial_x(u^t\mu^t) = 0 \\ \partial_t(\mu^t u^t) + \partial_x(\mu^t(u^t)^2) = \mu^t\int\gamma(\cdot - y, u^t(\cdot) - u^t(y))\mu^t(y)dy + \eta\mu^t\partial_x\psi^t - \alpha\mu^t u^t \\ \partial_s\psi^s = D\partial_x^2\psi - \kappa\psi + \displaystyle\int_{x-R}^{x+R}\mu^s(y)dy, \ \ s\in[0,t]. \end{cases}$$

Comparison between Vlasov and the Hydrodynamic model

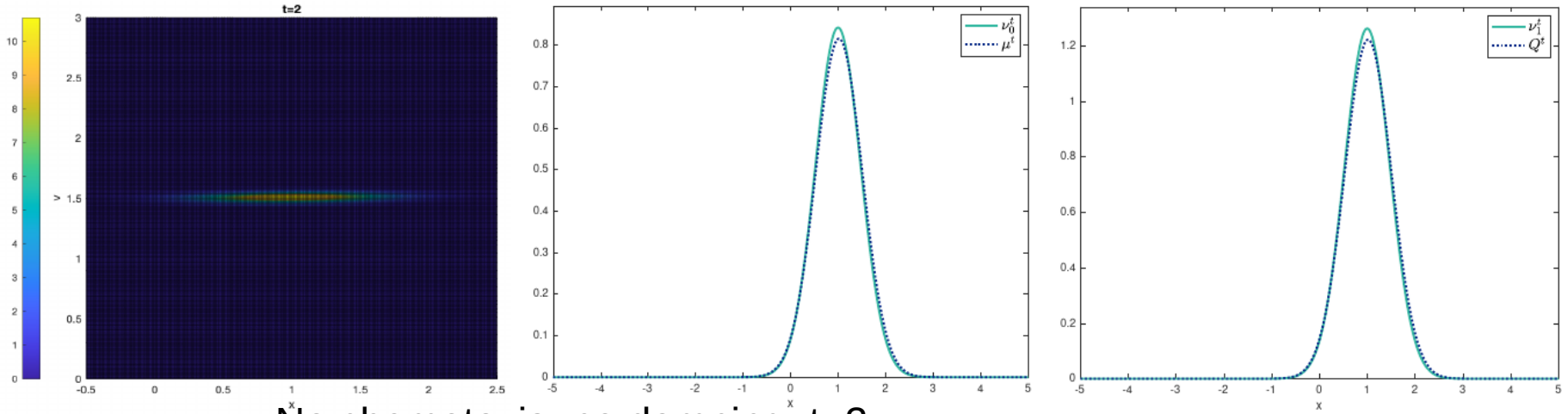
The monokinetic case with initial data:

$$\rho^0(x,v) = \frac{1}{2\pi\sigma_x\sigma_v} e^{\frac{-(x-x_0)^2}{2\sigma_x^2} + \frac{-(v-v_0)^2}{2\sigma_v^2}}$$

Vlasov phase space

ν_0^t and μ^t

ν_1^t and Q^t

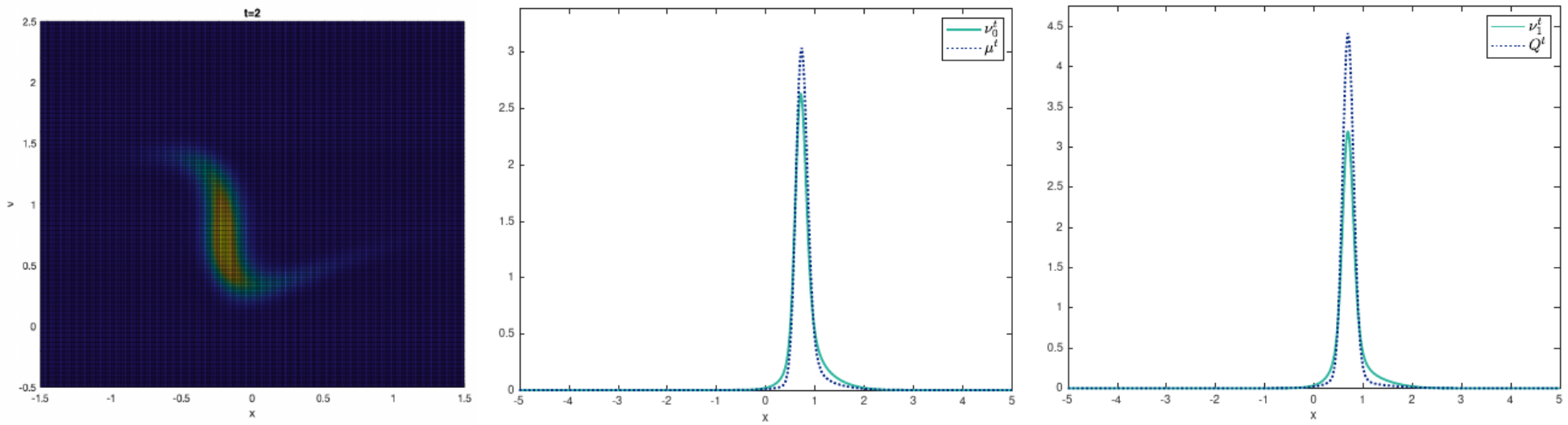


No chemotaxis, no damping, t=2

Comparison between Vlasov and the Hydrodynamic model

The monokinetic case with initial data:

$$\rho^0(x,v) = \frac{1}{2\pi\sigma_x\sigma_v} e^{\frac{-(x-x_0)^2}{2\sigma_x^2} + \frac{-(v-v_0)^2}{2\sigma_v^2}}$$

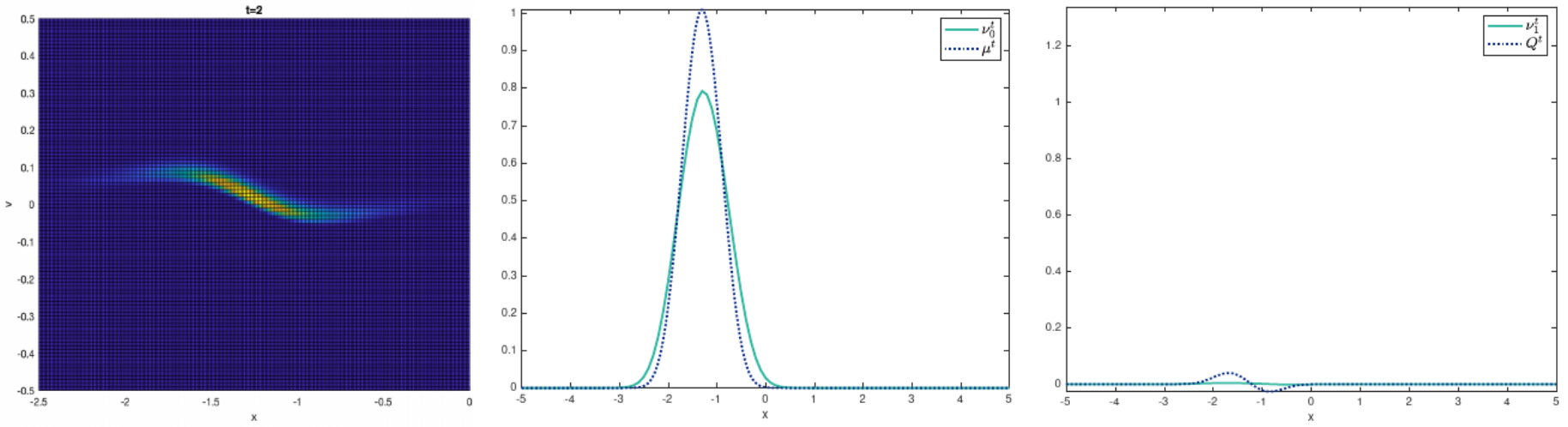


chemotaxis, no damping, t=2: no longer monokinetic

Comparison between Vlasov and the Hydrodynamic model

The monokinetic case with initial data:

$$\rho^0(x,v) = \frac{1}{2\pi\sigma_x\sigma_v} e^{\frac{-(x-x_0)^2}{2\sigma_x^2} + \frac{-(v-v_0)^2}{2\sigma_v^2}}$$



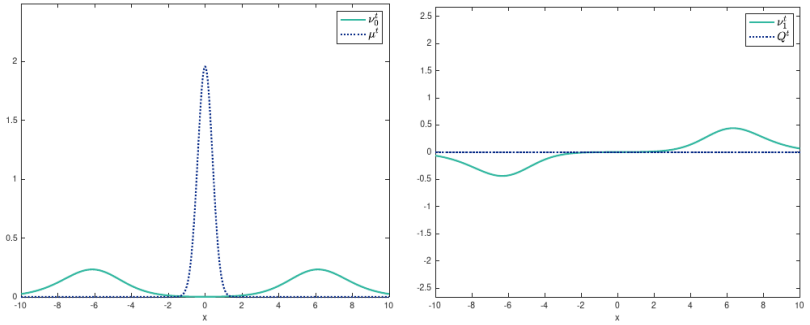
chemotaxis, damping, t=2: dissipation of energy

Comparison between Vlasov and the Hydrodynamic model

The non-monokinetic case with initial data:

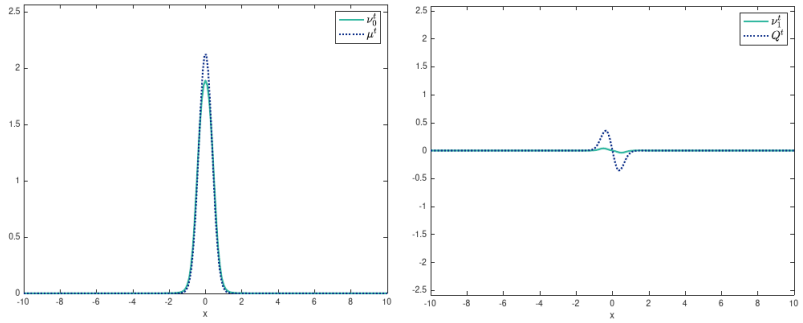
$$\rho^0(x,v) = \frac{1}{2\pi\sigma_x\sigma_v} e^{\frac{-x^2}{2\sigma_x^2}} \left(e^{\frac{-(v+v_0)^2}{2\sigma_v^2}} + e^{\frac{-(v-v_0)^2}{2\sigma_v^2}} \right) \Longrightarrow u^0(x) = 0$$

No chemotaxis
No damping
t=3



No convergence

chemotaxis
No damping
t=7



Convergence for large times (not for all data)

The Hydrodynamic model with a small pressure term

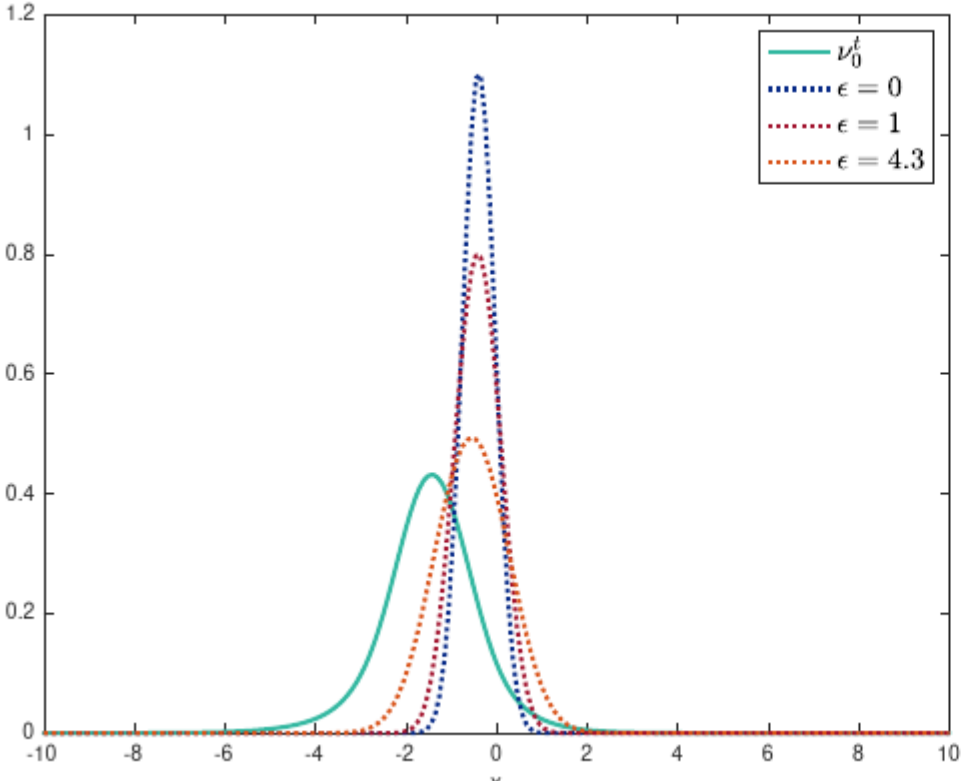
Our nonlocal hydrodinamic equation with a small pressure (to avoid blow-up)

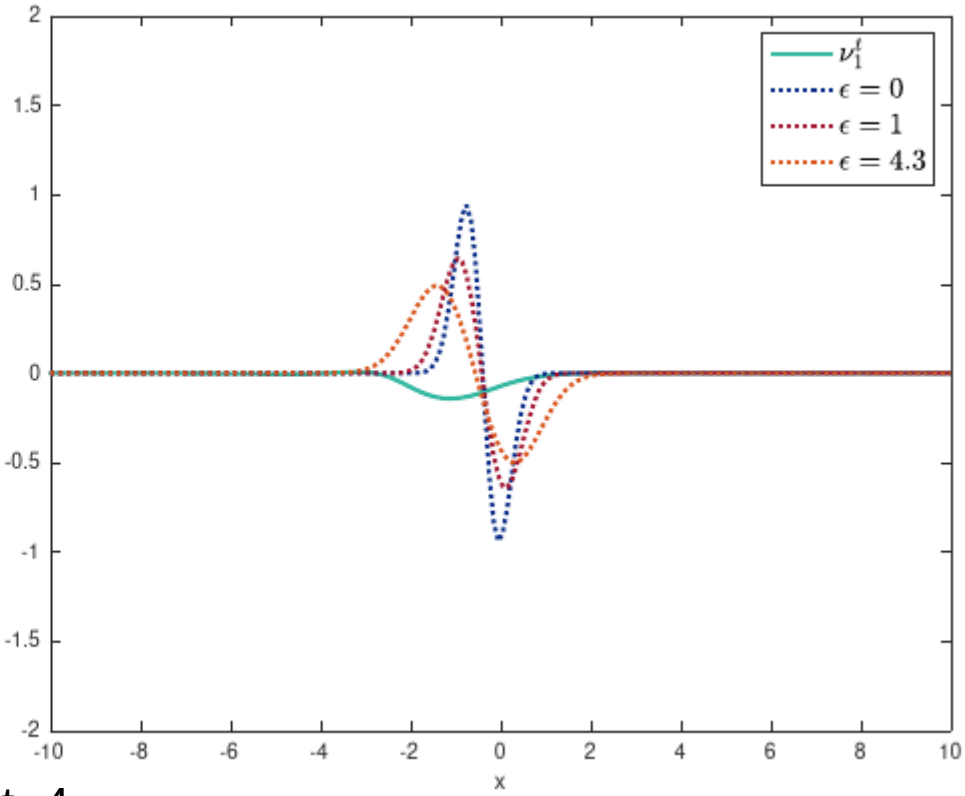
$$\begin{cases} \partial_t \mu^t + \partial_x(u^t \mu^t) = 0 \\ \partial_t(\mu^t u^t) + \partial_x(\mu^t (u^t)^2 + \varepsilon P(\mu^t)) = \mu^t \int \gamma(\cdot - y, u^t(\cdot) - u^t(y))\mu^t(y)dy + \eta \mu^t \partial_x \psi^t - \alpha \mu^t u^t \\ \partial_s \psi^s = D\partial_x^2 \psi - \kappa \psi + \int_{x-R}^{x+R} \mu^s(y)dy, \ \ s \in [0,t]. \end{cases}$$

Compare with the Preziosi-Euler model for chemotaxis in vasculogenesis

$$\begin{cases} \partial_t \mu^t + \partial_x(u^t \mu^t) = 0 \\ \partial_t(\mu^t u^t) + \partial_x(\mu^t (u^t)^2 + P(\mu^t)) = +\eta \mu^t \partial_x \psi^t - \alpha \mu^t u^t \\ \partial_s \psi^s = D\partial_x^2 \psi - \kappa \psi + \mu^t. \end{cases}$$

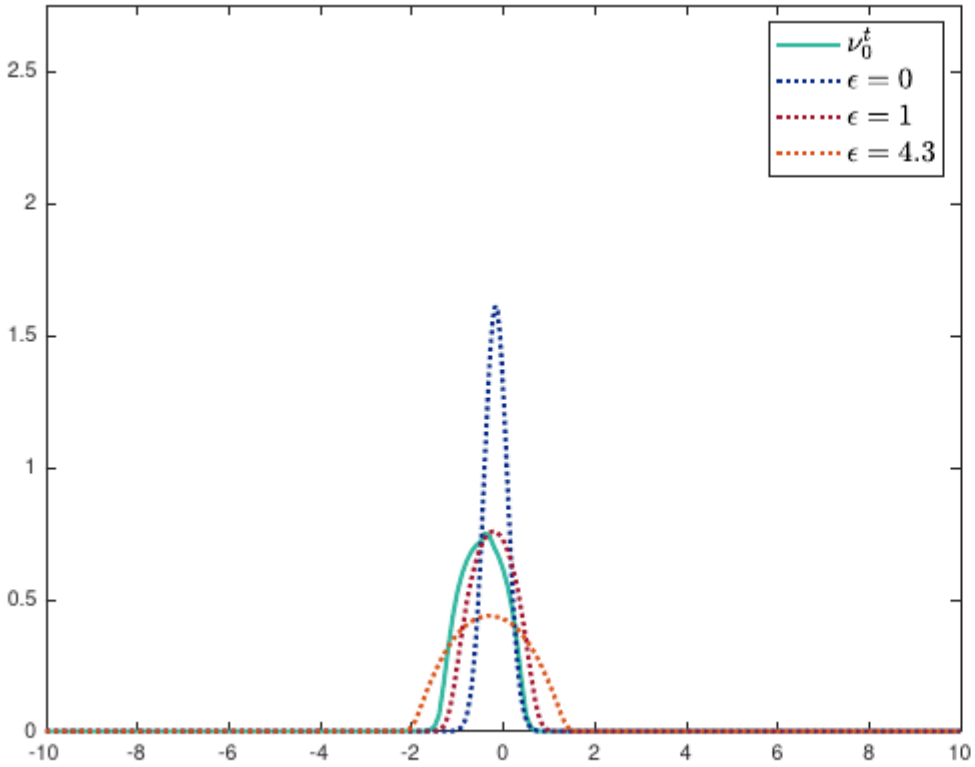
Comparison between Vlasov and the Hydrodynamic model

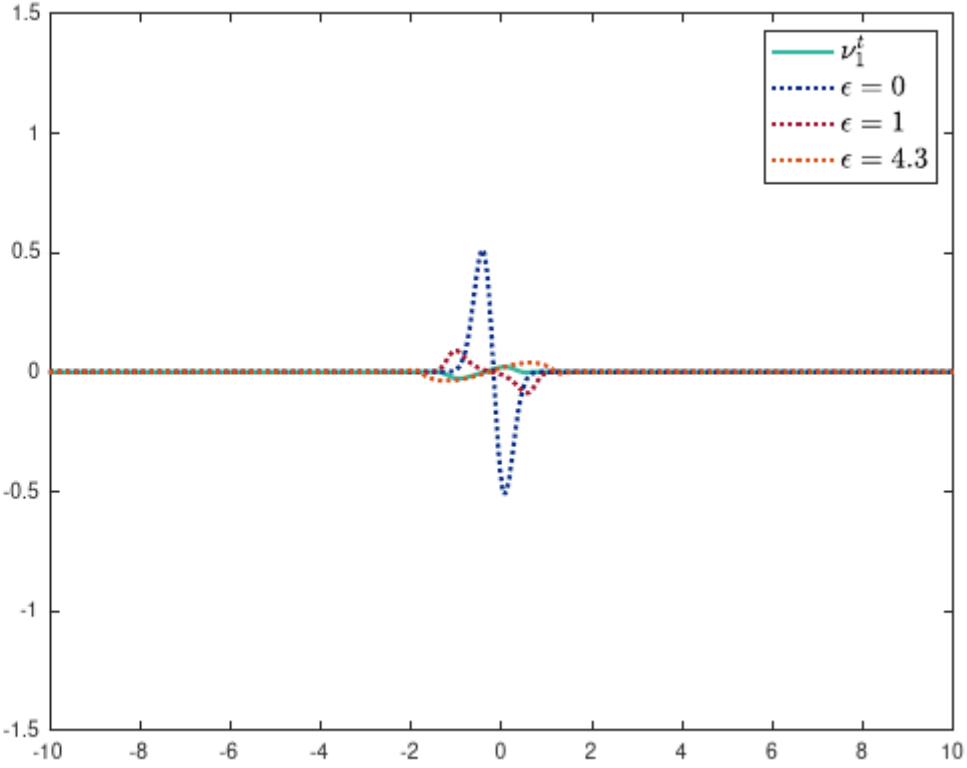




chemotaxis, no damping, t=4

Comparison between Vlasov and the Hydrodynamic model





chemotaxis, damping, t=4

Conclusions

- The Vlasov model gives a reliable approximation of the microscopic case
- The Hydrodynamic non local model is a good approximation in the monokinetic case before blow-up (pressureless gas). And in general the Vlasov remains monokinetic only for short times
- Adding a (small) pressure, in the spirit of Preziosi, takes into account for the neglected momenta, but only for certain values of epsilon.
- Damping, which is a natural ingredient in these models even at the microscopic scale, helps a lot in keeping the hydrodynamics near the kinetic.

Some references

- Di Costanzo E., Natalini R., Preziosi L. A hybrid mathematical model for self-organizing tissue migration in the zebrafish lateral line, Journal of Mathematical Biology, 71(1):171–214, 2015
- Braun E.C., Bretti G., Natalini R. Mass-Preserving Approximation of a Chemotaxis Multi-Domain Transmission Model for Microfluidic Chips. Mathematics, 9(6)688, 1–34, 2021
- Gabriella Bretti, Adele De Ninno, Roberto Natalini, Daniele Peri and Nicole Roselli, Estimation Algorithm for a Hybrid PDE–ODE Model Inspired by Immunocompetent Cancer-on-Chip Experiment, Axioms 2021, 10(4), 243
- Elishan C. Braun, Gabriella Bretti, Roberto Natalini, Parameter estimation techniques for a chemotaxis model inspired by Cancer-on-Chip (COC) experiments, International Journal of Non-Linear Mechanics, Volume 140, April 2022, 103895 https://doi.org/10.1016/j.ijnonlinmec.2021.103895
- Roberto Natalini, Thierry Paul, On The Mean Field limit for Cucker-Smale models, Discrete & Continuous Dynamical Systems – B, 2022, 27(5): 2873-2889. doi: 10.3934/dcdsb.2021164
- Roberto Natalini, Thierry Paul, The Mean-Field limit for hybrid models of collective motions with chemotaxis, SIAM J. Math. Anal. 55, No. 2, 900-928 (2023)
- Marta Menci, Roberto Natalini, Thierry Paul, MICROSCOPIC, KINETIC AND HYDRODYNAMIC
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Ciao Maurizio!