CASTELNUOVO E. (1986). Fractals: an interdisciplinary subject, *Actes CIEAEM* 38, 171-177

EMMA CASTELNUOVO

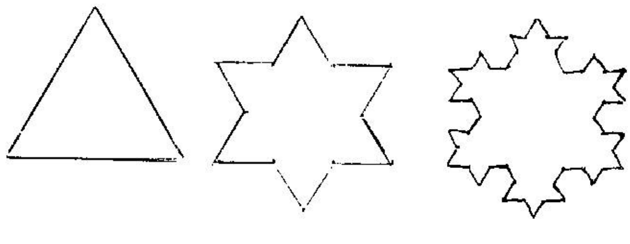
FRACTALS: AN INTERDISCIPLINARY SUBJECT

INTRODUCTION

This subject has been developed by Daniela Gori Giorgi with her 17-year-old students. Given the great interest of all students, we have thought to present it in this Meeting. In class, after having laid a particular emphasis on the difference between figures of "ordinary geometry" (like polygons, circles, cylinders,...) and those, quite complex, found in nature, we drew students' attention to many natural shapes which seem to repeat the same motif on different scales: for instance, a stretch of fragmented coastline observed from an airplane at various altitude, the branches of a tree, a snowflake examinated under a microscope by changing the len's power. It is the regularity of a snowflake that gives help in mathematizing natural shapes. After having considered the snowflake-curve, other curves got by repeating the same construction have been studied; among these, the Peano curve. Then, we have been led to the concept of dimension and the fractal curves. From these abstract considerations we came back to reality: natural phenomena, reproduced by simulation, have given an idea of some up-to-date research on fractal geometry.

SOME CLASSIC PROBLEMS ON THE SNOWFLAKE

The classic construction of a snowflake is described by the following drawings.



One starts with the equilateral triangle of side **3a**.

On the middle third of each side on constructs an equilateral triangle of side **a** and delets the base of each triangle. By continuing this process we get a polygon with encreasingly small sides.

The values for sides and perimeters are the following:

sides: 3a, a, $^a/_3$, $^a/_9$, ... ; perimeters: 9a, 12a, 16a,...

It is clear that the perimeter's value becomes infinite.

Let us, now, calculate the successive areas A_1, A_2, A_3,...We have:

$$A_1=\frac{9}{4}a^2\sqrt{3}\approx 3.9a^2\ ,\quad A_2=\frac{3}{4}a^2\sqrt{3}\approx 1.3a^2\ ,\quad A_3=\frac{1}{3}a^2\sqrt{3}\approx 0.6a^2\ ,\ \ldots$$

The sum of these areas results in:

$$A=\frac{9}{4}a^2\sqrt{3}+\frac{3}{4}a^2\sqrt{3}+\frac{1}{3}a^2\sqrt{3}+\ldots$$

This is, apart from the first term, an infinite geometric progression with common ratio 4/9.

Then total area is given by:

$$A=\frac{9}{4}a^2\sqrt{3}+\frac{\frac{3}{4}a^2\sqrt{3}}{1-\frac{4}{9}}=\frac{18}{5}a^2\sqrt{3}\approx 6a^2$$

We discover that the area is not infinite, its limit value being a value only a little higher than one and half times the area A_1 of the triangle!

These classic results about infinity always fascinate students.

FROM THE SNOWFLAKE TO THE PEANO CURVE

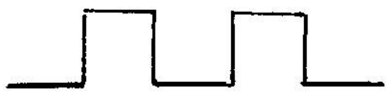
Now, let us come to some more modern reserach. Let us examine step by step the construction concerning one side: we divide one side into 3 equal segments and, by deleting the central part, we construct a polygonal with 4 segments.

Then, starting from each segment, we repeat the same operation. By continuing the same process we get polygonals with a bigger and bigger number of sides; at the limit we get a curve: the snowflake curve.

A spontaneous question arises: is it possible to express this construction by means of a formula? Can we tie **3**, the number of parts into which the initial segment is divided, with **4**, the number of segments we get every time?

Let us present another example: the construction of Greek fret (in its simpler shape) starting from a segment.

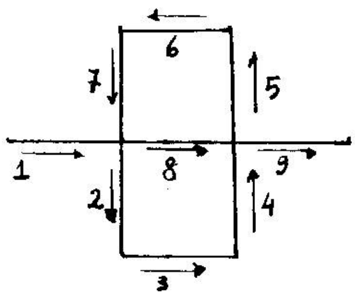
We divide the segment into 5 equal parts and delete the medial ones; then, we construct a polygonal composed of 9 segments.



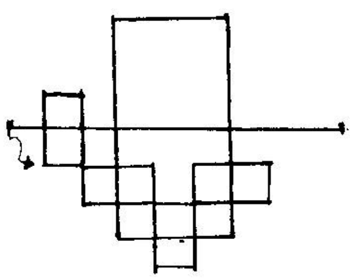
How can we find a formula tying **5** and **9**?

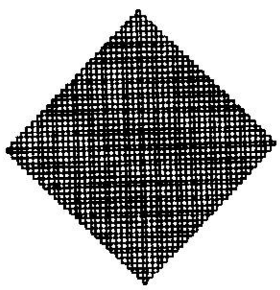
And still another example. By this one we will grasp some ideas, and we will be led to a formula.

Let us divide a segment into 3 parts, and construct a figure made up by utilizing 9 segments. We number these segments so that we can run along our polygonal by a continuous path.



By repeating the same operation on each segment we get the following figures.

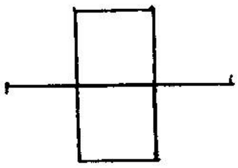


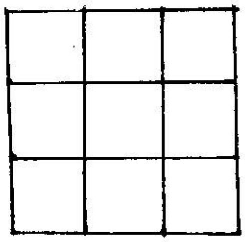


By continuing the same process the polygonal tends to a curve: the famous Peano curve. It is a plane-filling curve, that is a curve running through all the points of a square at least once. The discovery of such a curve shocked the mathematicians of the end of last century, leading to a crisis about the concept of dimension and of curve: in fact, if the dimension of a curve is 1, and the dimension of a plane region, like a square, is 2. How are we to explain the behaviour of the Peano curve? We will try to clarify this contradiction by grasping intuitively that the Peano curve runs through all the points of a square.

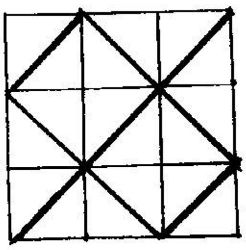
Let us come back to our construction: after having divided a segment into 3 parts, we construct a polygonal made up by 9 segments.

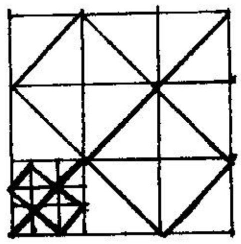
And now let us consider the following figures got, both, by dividing a segment into 3 equal parts.





The first is made by means of 9 segments, the second by means of 9 small squares. Then, it is possible to establish a one to one correspondence between the segments and the squares. Such correspondence becomes very clear if our polygonal is drawn, as in Peano's work, on the diagonal of the square, each segment-diagonal corresponding to "its" square.





It is clear that this one to one correspondence between the sides of the polygonal and the small squares keeps its validity when, by repeating the process, the segments become smaller and smaller. Passing to the limit, that is when the polygonal tends to a curve, both diagonal and square tend to one point. We grasp, this way, that the curve will pass through all the pints of the great square at least once, that is it will fill the square. For this reason one is led to say that the dimension of the Peano curve is 2, like that of the square.

If we indicate by s the number of parts into which the initial segment is divided, and by **n** the number of segments of the polygonal, we have:

$$s = 3\ ,\quad n = 9 \text{ and therefore } n = s^2.$$

We can say that the exponent **2** corresponds to the fact that the dimension of this curve equals - as we discover - **2**.

THE NOTION OF FRACTAL

Coming back to the construction of the snowflake, the situation is the following: the segment is divided into 3 parts and the polygonal is made up by 4 segments, that is

$$s = 3\ ,\quad s = 4,$$

and therefore we cannot write $4 = 3^2$, but we can write

$$4 = 3^d\ ,$$

where **d** is the dimension of the snowflake curve.

It is clear that $d < 2$, and this value corresponds to the fact that is is impossible to establish a one to one correspondence between 4 segments and 9 squares: there are more squares. Then, by continuing the same process, the snowflake-curve doesn't fill the plane.

Calculating the dimension would mean having to find the "compactness" of the curve in comparison with the square. We have:

$$d = \log_3 4 \text{ or } d = \frac{\log 4}{\log 3} \approx 1.3.$$

The dimension **d** is a real number between 1 and 2.
In the case of the Greek fret, being

$$s = 5\ ,\quad n = 9,$$

we have

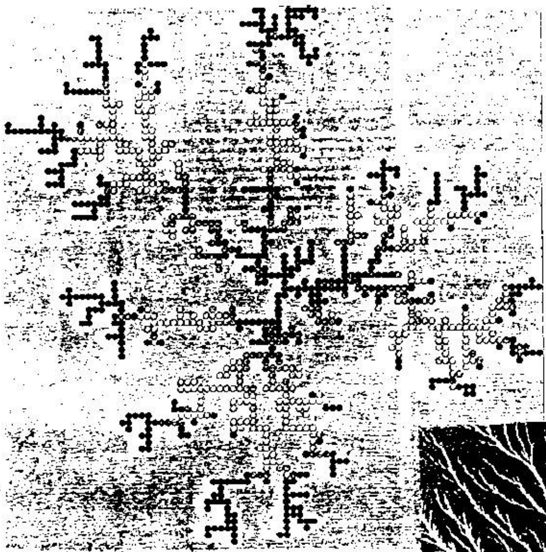
$$9 = 5^d \text{ and then } d = \frac{\log 9}{\log 5} \approx 1.4.$$

Also in this case the dimension **d** is a real number between 1 and 2.

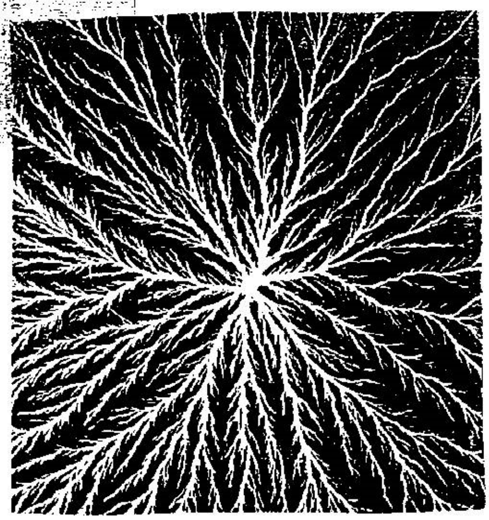
When the dimension d of a curve got by repeating a process "by dilatation" is, as in these cases, included between 1 and 2, the curve does not fill the plane, that is there are some "empty spaces". In these cases we have a *fractal curve*. The word "fractals" was created by Benoit Mandelbrot in 1975.

The importance of such kind of curves is due to its connection with some natural phenomena. In order to study these connections, some models of natural phenomena have been simulated. The photo 1 shows the model of an aerosol aggregate, that is of a suspension of minute solid of liquid particles in the air. The model was built on the basis of the Brownian movement; its fractal structure is quite striking.

The model reminds us the shape of branches and stems of some trees. And, also, it reminds us the physical phenomenon of lightening; it is interesting to notice that the photo 2, looking like a lightening, reproduces, instead, the lines of the cracks in an unbreakable glass. It easy to realize what importance the prediction of these lines could have in the manufacture of glass.



2



Another possible use of the fractal geometry is to model a mass of clouds on the basis of observations through radar and satellites, and to investigate in what way they tend to spread out. The day in which these studies will be advanced enough, weather forecasting could have a real revolution.

It is interesting to realize how live a simulation got by means of the fractal geometry can be: nobody could say that the landscapes reproduced in several movies as "Star Wars" are "artifical"!

As you can see, students have been led to consider some up-to-date research. Moreover, some reflection on the abstract concept of dimension arose in a very stimulating context. It is easy to understand that such a study fascinates even students not very inclined to mathematics. Particularly for this reason we thought of presenting this subject in "Mathematics for those between 14 and 17" Conference.

Prepared by Emma Castelnuovo, with Claudio and Daniela Gori Giorgi.

Emma Castelnuovo
Rome
Italy

REFERENCES

[1] A.K. Dewdney: *A computer microscope zooms in for a look at the most complex object in mathematics*, Scientific American, August 1985.
[2] R. Jullien, R. Botet, M. Kolb: *Les agrégats*, La Recherche, Novembre 1985.
[3] J.P. Kahane: *Measures et dimensions*, Proceedings of ICMI 5°, Adelaide 1984.
[4] M. Kilne: *Mathematical thought from ancient to modern times*, Oxford University Press 1972.
[5] B. Mandelbrot: *The fractal geometry of nature*, W.H. Freeman New York 1983.
[6] R. Noss: *Fractals, turtles and snowflakes*, Micromath, spring 1985.
[7] Y. Sawada, H. Honjo: *D'où vient la forme des dendrites?*, La Recherche, Avril 1986.
[8] D. Tahta: *Frontiers of chaos exhibition*, Mathematics Teaching, December 1985.