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Some Results on the Homogeneous Riemannian Structures of Class $\mathcal{T}_1 \oplus \mathcal{T}_2$

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RIASSUNTO – *Si studiano le strutture Riemanniane omogenee appartenenti alla classe $\mathcal{T}_1 \oplus \mathcal{T}_2$ della classificazione di Tricerri e Vanhecke, ed aventi 1-forma fondamentale chiusa.*

ABSTRACT – *We study homogeneous Riemannian structures of class $\mathcal{T}_1 \oplus \mathcal{T}_2$ whose foundamental 1-form is closed.*

KEY WORDS – *Homogeneous Riemannian structures.*

A.M.S. CLASSIFICATION: 53C20 - 53C30

– Introduction

Let (M, g) be a connected Riemannian manifold of dimension n. A homogeneous Riemannian structure on (M, g) is a tensor field T of type (1,2) satisfying the following equations of Ambrose and Singer:

$$(A-S)\begin{cases} \text{i)}\ g(T_X Y, Z) + g(T_X Z, Y) = 0 \\ \text{ii)}\ (\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_X YZ} - R_{YT_X Z} \\ \text{iii)}\ (\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y} \end{cases}$$

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for any $X, Y, Z \in \mathcal{H}(M)$, [1], [9].
Here, $\mathcal{H}(M)$ denotes the Lie algebra of the tangent vector fields on M, ∇ is the Riemannian connection with the curvature tensor field R defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

It is well-known that, putting $\tilde{\nabla} = \nabla - T$, an equivalent formulation of $(A-S)$ is given by:

$$(A-S)' \begin{cases} \text{i) } \tilde{\nabla} g = 0 \\ \text{ii) } \tilde{\nabla} R = 0 \\ \text{iii) } \tilde{\nabla} T = 0. \end{cases}$$

Furthermore, the curvature tensor fields R and $\tilde{R}$ verify the relation:

$$(1) \qquad R(X,Y) = \tilde{R}(X,Y) + [T_X, T_Y] + T_{\tilde{\Sigma}(X,Y)}$$

where $\tilde{\Sigma}$ denotes the torsion tensor field of $\tilde{\nabla}$.
Now, we recall that among the eight classes founded by F. Tricerri and L. Vanhecke, the class $\mathcal{T}_1 \oplus \mathcal{T}_2$ is characterized by the condition

$$(2) \qquad \underset{X,Y,Z}{\mathfrak{S}} g(T(X,Y), Z) = 0$$

where $T(X,Y) = T_X Y$ and $\underset{X,Y,Z}{\mathfrak{S}}$ denotes the ciclic sum over X, Y, Z.
Furthermore, the class $\mathcal{T}_2$ is characterized by (2) and $c_{12}(T) = 0$, where, by definition, $c_{12}(T)(Z) = \sum_{i=1}^{n} g(T(E_i, E_i), Z)$, with $Z \in \mathcal{H}(M)$ and $(E_1, \ldots, E_n)$ local orthonormal fields.
Finally, putting

$$(3) \qquad \xi = \frac{1}{n-1} \sum_{i=1}^{n} T(E_i, E_i)$$

for a structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$, we have:

$$(4) \qquad T(X,Y) = g(X,Y)\xi - g(Y,\xi)X + \pi(X,Y),$$

where π is a tensor field of type (1,2).
Since, $\xi = 0$ implies $T \in \mathcal{T}_2$, and $\pi = 0$ implies $T \in \mathcal{T}_1$, [9], we say that $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ is a proper structure if $\xi \neq 0$ and $\pi \neq 0$.

For such a structure, we denote by ω the foundamental 1-form i.e. the 1-form which is dual of ξ with respect to the metric g.

There exist examples of proper structures $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ in dimension $n = 3, 4$, [9, p. 85, 93]. In each case, an easy computation shows that $d\omega = 0$.
In dimension $n \geq 4$, many examples can be obtained, taking:

a) the direct product of two structures in $\mathcal{T}_1$ whose 1-form is always closed, as easily follows from Lemma 5.3 in [9]. In the simply-connected case, the underlying Riemannian manifold is a product of two hyperbolic spaces, [9, Th. 5.2].
b) the direct product of two proper structures of class $\mathcal{T}_1 \oplus \mathcal{T}_2$ with closed foundamental 1-forms;
c) the direct product of a structure in $\mathcal{T}_1$ and a structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ having a closed, possibly vanishing, foundamental 1-form.

Namely, consider two manifolds (M_1, g_1) and (M_2, g_2) of dimension $n \geq 2$, $m \geq 2$ and structures T_1, T_2 respectively. Obviously the product structure T is homogeneous, Riemannian on $(M_1 \times M_2, g_1 \times g_2)$ and $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$, when $T_1, T_2 \in \mathcal{T}_1 \oplus \mathcal{T}_2$.

Now, suppose $T_1 \notin \mathcal{T}_2$. Then, we have $\xi = a\xi_1 + b\xi_2$, $\xi_1 \neq 0$, $a = \frac{n-1}{n+m-1} \neq 0$. By a direct computation, we obtain $\pi(X_2, \xi_1) = a\|\xi_1\|^2 X_2$ for any $X_2 \in \mathcal{H}(M_2)$ and then T is proper, since $\xi \neq 0$ and $\pi \neq 0$. Finally, for any Z, orthogonally decomposed as $Z_1 + Z_2$, we have $\omega(Z) = a\omega_1(Z_1) + b\omega_2(Z_2)$ and ω is closed if ω_1 and ω_2 are closed. Now, a), b), c) follow easily.
We do not know any example satisfying $d\omega \neq 0$.

In this paper, we study proper structures of class $\mathcal{T}_1 \oplus \mathcal{T}_2$ with $d\omega = 0$. In the simply-connected case, we prove that the underlying Riemannian manifold (M, g) is foliated by an isoparametric family of $(n-1)$-dimensional submanifolds carrying a homogeneous structure of class $\mathcal{T}_2$. Furthermore, M is the total space of a Riemannian submersion with base $\mathbb{R}$, the mixed sectional curvatures are non-positive and at least one of them is negative.

We discuss the case of warped products, proving that, in this hypothesis, (M,g) has to be isometric to the hyperbolic space $\mathbb{H}^n$, $n \geq 6$, of constant negative curvature $K = -\|\xi\|^2$.

Finally, as we will see in the last section, the existence of such a homogeneous structure on $\mathbb{H}^n$, $n \geq 6$, depends on the existence of a parallel, non-vanishing, homogeneous structure of class $\mathcal{T}_2$ on $\mathbb{R}^{n-1}$.

1 – General properties of homogeneous Riemannian structures of class $\mathcal{T}_1 \oplus \mathcal{T}_2$ with closed foundamental 1-form

Let (M,g) be an n-dimensional connected Riemannian manifold equipped with a proper homogeneous structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$.
Since, in dimension $n = 2$, there exist only structures of class $\mathcal{T}_1$, we have to suppose $n \geq 3$. Observe that there exist examples in dimension $n = 3$, [9]. It is also well-known that M is not compact, since $\xi \neq 0$, [10].

From the conditions (2), (3), (4) and $(A-S)'$ iii) it follows that:

$$\tilde{\nabla}\xi = 0\,,\ \tilde{\nabla}\omega = 0\,,\ \tilde{\nabla}\pi = 0\,,\ c_{12}(\pi) = 0\,,\ \underset{X,Y,Z}{\mathfrak{S}} g(\pi(X,Y),Z) = 0\,.$$

Furthermore, it is easy to see that π is neither symmetric nor alternating, and

$$g(\pi(X,Y),Z) + g(\pi(X,Z),Y) = 0\,, \tag{5}$$

so that we have $g(\pi(X,\xi),\xi) = 0$.

On the other hand, $\tilde{\nabla}\xi = 0$ implies $\|\xi\| = c$, c constant. Hence ξ is bounded and complete, [9]. Moreover, ξ is not conformally Killing, [10, th.4.3].

Finally, the torsion tensor of $\tilde{\nabla}$ is given by:

$$\tilde{\Sigma}(X,Y) = g(Y,\xi)X - g(X,\xi)Y + \pi(Y,X) - \pi(X,Y)\,.$$

LEMMA 1. *The following conditions are equivalent.*

a) ω *is closed,*

b) $\omega \circ \tilde{\Sigma} = 0$,

c) $\pi_\xi = \pi(\xi,\ ) = 0$.

From the relation $(d\omega)(X,Y) = (\tilde{\nabla}\omega)(Y,X) - (\tilde{\nabla}\omega)(X,Y) + (\omega \circ \tilde{\Sigma})(X,Y)$, where $d\omega$ is defined without the factor $\frac{1}{2}$, using (5) and $\tilde{\nabla}\xi = 0$, we have

$$(d\omega)(X,Y) = (\omega \circ \tilde{\Sigma})(X,Y) = g(\pi(Y,X),\xi) - g(\pi(X,Y),\xi) =$$
$$= \mathop{\mathfrak{S}}_{Y,X,\xi} g(\pi(Y,X),\xi) + g(\pi(\xi,X),Y) = g(\pi(\xi,X),Y)$$

since the ciclic sum vanishes.
Note that $d\omega = 0$ implies:

$$g(\pi(X,\xi),Y) = g(\pi(Y,\xi),X). \tag{6}$$

From now on, we suppose $d\omega = 0$ and we denote by $\mathcal{D}$ the distribution orthogonal to ξ. Since π_ξ vanishes, $\tilde{\nabla}\xi = 0$ implies $\nabla_\xi\xi = 0$ so that the integral curves of ξ are geodesics.

PROPOSITION 1.1. *Let (M,g) be a Riemannian manifold equipped with a proper homogeneous structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ such that $d\omega = 0$. Then the distribution $\mathcal{D}$ orthogonal to ξ is integrable and each integral manifold carries a structure $\overline{T} \in \mathcal{T}_2$.*

The integrability of $\mathcal{D}$ follows immediately from the hypothesis $d\omega = 0$. Now, let N be a maximal integral manifold of $\mathcal{D}$. For any $X,Y \in \mathcal{H}(N)$ we have $\tilde{\nabla}_X g(Y,\xi) = 0$ and $\tilde{\nabla}g = 0$, $\tilde{\nabla}\xi = 0$ imply $g(\tilde{\nabla}_X Y,\xi) = 0$ i.e. $\tilde{\nabla}_X Y \in \mathcal{H}(N)$ and so N is autoparallel with respect to $\tilde{\nabla}$.

Applying the theorem 2.1 and 2.8 in [10], we have that the induced structure $\overline{T}$ on N belongs to the class $\mathcal{T}_1 \oplus \mathcal{T}_2$ and $\overline{T}(X,Y) = T(X,Y) - \alpha(X,Y)$. Here α is the second foundamental form of N in M, defined by the Gauss equation $\nabla_X Y = \nabla'_X Y + \alpha(X,Y)$ for each $X,Y \in \mathcal{H}(N)$, where ∇' is the Riemannian connection on N. Using the Gauss equation, for any $X,Y \in \mathcal{H}(N)$, we have:

$$\alpha(X,Y) = g(X,Y)\xi + \frac{1}{c^2} g(\pi(X,Y),\xi)\xi, \quad c^2 = \|\xi\|^2.$$

It follows:

$$\overline{T}(X,Y) = \pi(X,Y) - \frac{1}{c^2} g(\pi(X,Y),\xi)\xi. \tag{7}$$

Finally, if $(E_1,\dots,E_{n-1})$ is a local orthonormal basis of N, putting $\eta = \frac{\xi}{\|\xi\|}$ we have $\pi(\eta,\eta)=0$ and for each $Z \in \mathcal{H}(N)$,

$$c_{12}(\overline{T})(Z) = \sum_{i=1}^{n-1} g(\overline{T}(E_i,E_i),Z) =$$

$$= \sum_{i=1}^{n-1} g(\pi(E_i,E_i),Z) = c_{12}(\pi)(Z) = 0\,.$$

Hence $\overline{T}$ belongs to $\mathcal{T}_2$.

REMARK 1. Obviously, it may happen that the structure $\overline{T} \in \mathcal{T}_2$ induced by T on N is trivial. In this case, N is locally symmetric and for any $X,Y \in \mathcal{H}(N)$ we have $T(X,Y) = \alpha(X,Y)$, and $T(X,Y) = g(X,Y)\xi + \pi(X,Y)$, using (4) and $g(Y,\xi)=0$. As a special case, suppose that $n=3$. Then each integral manifold is locally symmetric, since, in dimension 2, $\overline{T} \in \mathcal{T}_2$ implies $\overline{T}=0$.

REMARK 2. Let N be a maximal integral manifold of $\mathcal{D}$. It is easy to verify that for any $X \in \mathcal{H}(N)$, we have:

$$\nabla_X \xi = -c^2 X + \pi(X,\xi)\,. \tag{8}$$

Hence, the Weingarten operator determined by the unique normal unit vector field η, is given by:

$$A_\eta X = cX - \pi(X,\eta)\,. \tag{9}$$

Let us denote by B the tensor field of type (1,1) on M defined by

$$B(X) = \pi(X,\eta) \qquad X \in \mathcal{H}(M)\,. \tag{10}$$

PROPOSITION 1.2. *B is diagonizable and trace-free. Furthermore, 0 is one of its eigenvalues.*

Using (6), we have $g(B(X),Y) = g(\pi(X,\eta),Y) = g(\pi(Y,\eta),X) = g(X,B(Y))$ for any X, $Y \in \mathcal{H}(M)$, so that B is symmetric with respect to the metric g and it can be diagonalized. Since $B(\eta) = \pi(\eta,\eta) = 0$, η is eigenvector with eigenvalue 0.
Finally

$$\operatorname{tr}(B) = \sum_{i=1}^{n-1} g(B(E_i),E_i) + g(B(\eta),\eta) = -c_{12}(\pi)(\eta) = 0$$

where $(E_1,\ldots,E_{n-1})$ is a local orthonormal basis of $\mathcal{D}$.

REMARK 3. Since $B(\xi) = 0$ and $g(B(X),\xi) = 0$, B induces a tensor field of type (1,1) on every integral manifold of $\mathcal{D}$. Such a tensor will be denoted with the same letter B.

PROPOSITION 1.3. *The eigenvalues of B are constant on M and the integral manifolds are isoparametric hypersurfaces.*

The existence of a homogeneous structure on M, implies that M is locally homogeneous. Now, fixed $p,q \in M$, there exist neighboords U of p and V of q and an isometry ϕ: $U \longrightarrow V$ such that $\phi(p) = q$. ϕ is an affine transformation of $\tilde{\nabla}$ and we have in U: $\phi_*(T(X,Y)) = T(\phi_*(X),\phi_*(Y))$. It follows $\phi_*\xi = \xi$ and $\phi_*(B(X)) = B(\phi_*(X))$.

Now, suppose that X is an eigenvector of B in p with eigenvalue μ. We have:

$$B_q(\phi_{*p}(X)) = \phi_{*p}(B_p(X)) = \phi_{*p}(\mu X) = \mu\phi_{*p}(X)$$

i.e. $\phi_{*p}(X)$ is eigenvector of B in q with eigenvalue μ.
Hence the eigenvalues of B are constant on M.

Let N be an integral manifold of $\mathcal{D}$. From (9) we have $A_\eta X = cX - B(X)$, and the principal curvatures $\lambda_i = c - \mu_i$, where μ_i is eigenvalue of B, are constant on N and they do not depend on the integral manifold. It follows that the mean curvature of each integral manifold is constant and equal to c.

PROPOSITION 1.4. *Let $p \in M$. If Y is an eigenvector of B_p, with eigenvalue μ, and orthogonal to ξ_p, then Y is eigenvector in p with eigenvalue $(\mu - c)^2$ for the curvature operator $R(\eta,\ )\eta$.*

Since $\tilde{\nabla}\eta = 0$ implies $\tilde{R}(\eta, Y)\eta = 0$, from (1) we have:

$$R(\eta, Y)\eta = [T_\eta, T_Y](\eta) + T_{\tilde{\Sigma}(\eta,Y)}\eta .$$

Now, $\tilde{\Sigma}(\eta, Y) = T(Y, \eta) - T(\eta, Y)$, $T_\eta = \pi_\eta = 0$ and (4) give:

$$R(\eta, Y)\eta = T(T(Y, \eta)\eta) = (\mu - c)^2 Y .$$

PROPOSITION 1.5. *The Ricci curvature* Ricc(η, η) *is a constant negative function on* M.

Let N be the maximal integral manifold of $\mathcal{D}$ throught a fixed point $p \in M$ and $(E_1, \dots, E_{n-1})$ an orthonormal basis of $T_p(N)$ given by eigenvectors of A_η.

The proposition 1.4 implies:

$$\mathrm{Ricc}\,(\eta, \eta)(p) = -\sum_{i=1}^{n-1} g(E_i, R(\eta, E_i)\eta) = -\sum_{i=1}^{n-1}(c - \mu_i)^2 \leq 0$$

and the equality does not hold, since B is trace-free and $c \neq 0$. Obviously, that means that for any $p \in M$ there is a 2-plane σ_i spanned by (E_i, η) with sectional curvature $K(\sigma_i) < 0$.
As corollaries, we have:

PROPOSITION 1.6. *The Euclidean space* $\mathbb{R}^n$ *does not admit any proper homogeneous Riemannian structure* $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ *with closed foundamental 1-form.*

PROPOSITION 1.7. *Any manifold* (M, g) *with sectional curvature* $K \geq 0$ *does not admit any proper homogeneous Riemannian structure* $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ *with closed foundamental 1-form.*

Finally, we have the following result:

PROPOSITION 1.8. *The following properties hold:*

1) $\nabla_\eta T = 0$, $\nabla_\eta \pi = 0$, $\nabla_\eta \omega = 0$, $\nabla_\eta R = 0$.

2) $\nabla\omega$ *is a 2-covariant symmetric tensor field. Furthermore, if* M *is simply-connected, then* $\omega = df$ *and* $\nabla\omega$ *is the Hessian of* f.

3) For any $X, Y, Z \in \mathcal{H}(M)$, *we have:*

$$(\nabla_X \pi)(Y,Z) = [\pi_X, \pi_Y](Z) - \pi_{\pi(X,Y)}Z + g(X, \pi(Y,Z))\xi + \\ - g(\pi(Y,Z),\xi)X + g(Y,\xi)\pi(X,Z) - g(X,Z)\pi(Y,\xi) + \\ + g(Z,\xi)\pi(Y,X).$$

4) $\nabla_\eta B = 0$ *and* $L_\eta B = 0$, *where* L *denotes the Lie differentiation.*

Since $\pi_\eta = 0$, we have $T_\eta = 0$ and $\nabla_\eta \eta = 0$. Then, obviously, $\nabla_\eta \omega = 0$ and $(A - S)$ iii) implies $\nabla_\eta T = 0$. Hence $\nabla_\eta \pi = \nabla_\eta T = 0$.

Furthermore, $\nabla_\eta R = 0$ follows from $(A - S)$ ii). A direct computation gives 2) and 3). For 4), $\nabla_\eta B = 0$ follows from 3). Finally, we have $(L_\eta B)(\eta) = 0$ and for each $X \in \mathcal{D}$, $(L_\eta B)(X) = [A_\eta, B](X) = [cK - B, B](X) = 0$, since $A_\eta = cK - B$, where K is the Kronecker tensor field.

The Frobenius theorem implies that M is locally isometric to the product $\mathbb{R} \times N$, where N is a maximal integral manifold of $\mathcal{D}$, with a suitable metric.

For such a metric, we are giving the local expression.

Consider $p \in N$. By the Frobenius theorem, there exists a neighboord U of p in M, with coordinates $(t, x^1, \ldots, x^{n-1})$ centered at p such that $\xi = \dfrac{\partial}{\partial t}$ and $\left\{\dfrac{\partial}{\partial x^i}\right\}_{i=1,\ldots,n-1}$ generate the distribution $\mathcal{D}$ in U.

We have, in U:

$$g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \|\xi\|^2 = c^2;$$

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial t}\right) = 0 \qquad \text{for } i = 1, \ldots, n-1.$$

It follows that

$$g \equiv c^2 dt^2 + \sum_{i,j=1}^{n-1} g_{ij}(t,x) dx^i dx^j$$

where

$$g_{ij}(t,x) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \qquad \text{for } i,j = 1,\dots,n-1\,.$$

Using (8) and $\tilde{\nabla}\eta = 0$, a direct computation (as in [9, p.52-53]) gives:

$$g'_{ij}(t,x) = \frac{\partial}{\partial t} g_{ij}(t,x) = -2c^2 g_{ij}(t,x) - 2g\left(\pi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \xi\right)(t,x)\,.$$

Since

$$g\left(\pi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \xi\right) = g\left(\pi^r_{ij}\frac{\partial}{\partial x^r} + \pi^0_{ij}\xi, \xi\right) = c^2 \pi^0_{ij}\,,$$

we obtain the differential equation $g'_{ij} = -2c^2 g_{ij} - 2c^2\pi^0_{ij}$, and so

$$g_{ij}(t,x) e^{2c^2 t} = -2c^2 \int_0^t \pi^0_{ij}(s,x) e^{2c^2 s} ds + h\,.$$

Now, $t = 0$ implies $h = g_{ij}(0,x)$ where $g_{ij}(0,x)$ is the metric induced on $U \cap N$, hence

$$g_{ij}(t,x) = e^{-2c^2 t} g_{ij}(0,x) - 2c^2 e^{-2c^2 t} \int_0^t \pi^0_{ij}(s,x) e^{2c^2 s} ds\,.$$

2 – Homogeneous Riemannian structures of class $\mathcal{T}_1 \oplus \mathcal{T}_2$ with $d\omega = 0$ on simply-connected manifolds

Let (M,g) be a connected, simply-connected Riemannian manifold equipped with a proper homogeneous Riemannian structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$, with $d\omega = 0$. Obviously, there exists a function $f \colon M \to \mathbb{R}$ such that $\omega = df$ and $\xi = \operatorname{grad} f$. The results of section 1 imply that f is an isoparametric function and the maximal integral manifolds of $\mathcal{D}$ are the level hypersurfaces of f. Since ξ is nowhere zero, the level hypersurfaces are regular so that there are not focal varieties of f, [7]. The existence of f allows us to consider the manifold (M, g) under two points of view: as a manifold foliated by the level sets of f as well as the total space of a Riemannian submersion.

From the first point of view, the foliation is a Riemannian foliation with bundle-like metric g [8]. It is not harmonic, since its leaves are submanifolds with non-zero constant mean curvature $c = \|\xi\|$, by proposition 1.3. Using the theorem 2 in [4], we obtain that the leaves, obviously closed, are simply-connected. Moreover the remark 1 implies that such leaves are symmetric spaces, if M has dimension $n = 3$. On the other hand, the map $f\colon (M,g) \to (\mathbb{R}, g_0)$ with $g_0 = c^{-2}dt^2$, t coordinate in $\mathbb{R}$, is a Riemannian submersion. Namely, since grad $f = \xi$ is nowhere zero, for any $p \in M$ the tangent map f_* is surjective and then f is a submersion. Its fibres are the level sets of f so that the vertical distribution V coincides with $\mathcal{D}$, whereas the horizontal distribution H is spanned by ξ. Furthermore, each fibre is not totally geodesic, since its mean curvature is a non-zero constant. Finally, for any $p \in M$, we have $(g_0)_{f(p)}(f_*\xi_p, f_*\xi_p) = c^{-2}(\omega_p(\xi_p))^2 = g_p(\xi_p, \xi_p)$, so that f_* induces an isometry from H_p to $T_{f(p)}\mathbb{R}$. Thus f is a Riemannian submersion.

Let Q and A be the two tensor invariants of the Riemannian submersion, both immersed in tensor fields on M, [2]. The invariant A vanishes, since the distribution H is integrable.

The other invariant Q is defined by $Q(X,Y) = \mathrm{H}\nabla_{VX}VY + V\nabla_{VX}\mathrm{H}Y$, for any $X, Y \in \mathcal{H}(M)$. Observe that, for $X, Y \in \mathcal{D}$, we have $Q(X,Y) = \mathrm{H}\nabla_X Y$, hence, on each fibre, Q is the second foundamental form.

Since (M, g) is complete, by the theorem of Ehresmann and Hermann, [2, 9.40, 9.42], f is a locally trivial fibration with diffeomorphic fibres. Furthermore, the distribution H is an Ehresmann-connection i.e. Ehresmann-complete with trivial holonomy group Φ_t at $t \in \mathbb{R}$. Namely, Φ_t is the group of all diffeomorphisms of the fibre over t corresponding to closed paths in $\mathbb{R}$ starting at t.

Now, a theorem of J.A. Wolf, [2, 9.48], implies that Φ_t is the structural group of f and, since it reduces to the identity, M is diffeomorphic to the product $\mathbb{R} \times N$ where N is the standard fibre and f is the projection on the first factor. Since A vanishes, M is locally isometric to the product $\mathbb{R} \times N$ with a Riemannian metric $g = g_0 + g_t$ whose value at $(t,p) \in \mathbb{R} \times N$ is given by $g(t,p) = f^*(g_0(t)) + h^*(g_t(p))$, where h is the projection on the second factor and g_t is the metric induced by g on $N_t = f^{-1}(\{t\})$.

Using the O'Neill formulas for the curvature, we obtain the results of proposition 1.5 and the following one.

PROPOSITION 2.1. *The mixed sectional curvatures are non-positive.*

Let be $p \in M$, $X \in \mathcal{D}$, $\|X\| = 1$ and σ the 2-plane generated by η and X at p. Then, we have, [2,9.29b]: $K(\sigma) = g((\nabla_\eta Q)_X X, \eta) - \|Q(X,\eta)\|^2$. Since $\nabla_\eta \pi = 0$ and $g(\pi(X,Y),\eta) = g(\pi(Y,X),\eta)$ (see the proof of Lemma 1), we obtain $K(\sigma) = -\|Q(X,\eta)\|^2 = -\| -cX + \pi(X,\eta)\|^2 \leq 0$.

Observe that $K(\sigma) = 0$ if and only if $\pi(X,\eta) = cX$. Hence the mixed sectional curvatures at p can not vanish simultaneously, since B is trace-free.

As a special case of Riemannian submersion we can consider the warped product of two manifolds (M_1, g_1) and (M_2, g_2) by means of a positive function ϕ: $M_1 \to \mathbb{R}$. Therefore, we can ask is a simply-connected Riemannian manifold (M,g) equipped with a proper homogeneous Riemannian structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ such that $d\omega = 0$, can be a warped product.

From [2, 9.104] we already know that a Riemannian submersion is locally a warped product if and only if the invariant A vanishes, the vector field $(n-1)H$, (H mean curvature vector) is basic and the "trace-free" part Q^0 of the invariant Q, vanishes.

PROPOSITION 2.2. *Let (M,g) be an n-dimensional, connected, simply-connected Riemannian manifold with a proper homogeneous structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ having closed foundamental 1-form. If (M,g) is a warped product, then M is isometric to the hyperbolic space $\mathbb{H}^n$ of constant curvature $-\|\xi\|^2$ and $n \geq 6$.*

We already know that A vanishes. Obviously, $(n-1)H$ is basic. Thus, the hypothesis of warped product reduces to the condition $Q^0 = 0$.

Since, by definition, we have:

$$\begin{cases} Q^0(X,Y) = Q(X,Y) - g(X,Y)H \\ Q^0(X,\eta) = Q(X,\eta) + g(H,\eta)X & X,Y \in \mathcal{D} \\ Q^0(\eta,X) = 0 \end{cases}$$

the condition $Q^0 = 0$ is equivalent to :

$$\begin{cases} \alpha(X,Y) = g(X,Y)H \\ A_\eta X = g(H,\eta)X \end{cases}$$

which can be rewritten as

$$\begin{cases} g(\pi(X,Y),\eta) = 0 \\ \pi(X,\eta) = 0 \end{cases}$$

and these conditions reduce to $B = 0$.
Consequently, the tensor T verifies the relations:

$$\begin{cases} T(X,\xi) = g(X,\xi)\xi - \|\xi\|^2 X \\ T(\xi,X) = 0 \end{cases}$$

for any $X \in \mathcal{H}(M)$. Now, the theorem 6.2 in [10] implies that (M,g) is isometric to the hyperbolic space $\mathbb{H}^n$ with $K = -\|\xi\|^2$. Following the proof of theorem 6.2 in [10] we have that the integral manifolds of $\mathcal{D}$ are flat and isometric to $\mathbb{R}^{n-1}$. Proposition 1.1 implies that $\mathbb{R}^{n-1}$ has to carry a homogeneous induced structure $\overline{T}$ of class $\mathcal{T}_2$. Now, suppose $n < 6$. Since the classification given in [6] excludes the Euclidean spaces $\mathbb{R}^3$ and $\mathbb{R}^4$, it follows that $\overline{T} = 0$ on $\mathbb{R}^{n-1}$ and so we have:

$$\pi(X,Y) = \frac{1}{c^2} g(\pi(X,Y),\xi)\xi = -g(\pi(X,\eta),Y)\eta = 0$$

for any $X,Y \in \mathcal{D}$.
Since $\pi(X,\eta) = \pi(\eta,X) = 0$, we conclude $\pi = 0$ and $T \in \mathcal{T}_1$, a contradiction.

3 – The hyperbolic space $\mathbb{H}^n, n \geq 6$.

In this section, we discuss the existence of a proper homogeneous Riemannian structure of class $\mathcal{T}_1 \oplus \mathcal{T}_2$ with closed foundamental 1-form, on $\mathbb{H}^n$, $n \geq 6$.

PROPOSITION 3.1. *The hyperbolic space $\mathbb{H}^n$, $n \geq 6$, admits a proper structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ with $d\omega = 0$ if and only if the Euclidean space $\mathbb{R}^{n-1}$ admits a parallel, non-vanishing, homogeneous structure $\overline{T} \in \mathcal{T}_2$.*

Suppose that $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ is a proper homogeneous Riemannian structure on $\mathbb{H}^n$, $n \geq 6$, with $d\omega = 0$. Since $\mathbb{H}^n = \mathbb{R} \times_{e^{2t}} \mathbb{R}^{n-1}$ is a warped product, the proposition 2.2 implies $\pi(X, \eta) = 0$ for any $X \in \mathcal{D}$.

Observe that the same result can be achieved using the results obtained by E.CARTAN, [3]. Namely, putting $\omega = df$, the level hypersurfaces of f give a family of isoparametric hypersurfaces. Furthermore, since $\mathbb{H}^n$ has negative constant curvature K, either the principal curvatures coincide, or there exist two principal curvatures $\lambda \neq \overline{\lambda}$. In the first case, the eigenvalues of $B = \pi(\ ,\eta)$ coincide and vanish, since B is trace-free.

In the last case, the relation $d\lambda_i = (K + \lambda_i^2)dt$, $i = 1, \ldots, n-1$, in [3] and the proposition 1.3 imply $K + \lambda^2 = 0$ and $K + \overline{\lambda}^2 = 0$.

But, since $\lambda\overline{\lambda} + K = 0$, we have $\lambda = \overline{\lambda}$ so that we reduce to the first case.

Now, each integral manifold of the distribution $\mathcal{D}$ orthogonal to ξ is isometric to $\mathbb{R}^{n-1}$, [10], and by proposition 1.1 the induced structure $\overline{T} \in \mathcal{T}_2$ on $\mathbb{R}^{n-1}$ is given by $\overline{T}(X,Y) = \pi(X,Y)$ for any $X, Y \in \mathcal{D}$. Obviously $\overline{T} \neq 0$, otherwise we have $\pi = 0$ and $T \in \mathcal{T}_1$.

Since $\mathbb{H}^n$ has constant negative curvature $K = -\|\xi\|^2$, $T(\xi,\ ) = 0$ and $T(X,\xi) = -\|\xi\|^2 X$ for any $X \in \mathcal{D}$, the condition (1) implies:

$$\widetilde{R}(X,\xi)Z = -\|\xi\|^2 \pi(X,Z) \qquad \text{for any} \quad X, Z \in \mathcal{D}.$$

On the other hand $\widetilde{R}(X,\xi)\xi = 0 = -\|\xi\|^2\pi(X,\xi)$ so that we have $\widetilde{R}(X,\xi) = -\|\xi\|^2\pi_X$ for any $X \in \mathcal{D}$.

Now, it is easy to verify that $\widetilde{R}(X,\xi) \cdot g = 0$ and $\widetilde{R}(X,\xi) \cdot \widetilde{\Sigma} = 0$ imply $\widetilde{R}(X,\xi) \cdot \pi = 0$ and $\pi_X \cdot \pi = 0$.

It follows $[\pi_X, \pi_Y](Z) - \pi_{\pi(X,Y)}Z = [\pi_X, \pi_Z](Y) - \pi_{\pi(X,Z)}Y$ for any $X, Y, Z \in \mathcal{D}$. Since π is a homogeneous structure on $\mathbb{R}^{n-1}$, using $(A - S)$

iii) the above relation becomes $(D_X\pi)_Y Z = (D_X\pi)_Z Y$ where D is the Levi-Civita connection on $\mathbb{R}^{n-1}$.

Hence $D_X\pi$ is a symmetric tensor field on $\mathbb{R}^{n-1}$. By covariant derivation of $\underset{X,Y,Z}{\mathfrak{S}} g(\pi(X,Y),Z) = 0$ with respect to an arbitrary $W \in \mathcal{D}$, we have $\underset{X,Y,Z}{\mathfrak{S}} g((D_W\pi)(X,Y),Z) = 0$ and the symmetry of $D_W\pi$ together with its skew-symmetry with respect to g, give $D_W\pi = 0$.

Conversely, given a parallel, non-vanishing homogeneous Riemannian structure $\overline{T} \in \mathcal{T}_2$ on $\mathbb{R}^{n-1}$, $n \geq 6$, we can construct a proper homogeneous Riemannian structure $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ on $\mathbb{H}^n$, having $d\omega = 0$.

Let us consider the Poincaré half-space $\mathbb{H}^n = \mathbb{R}^*_+ \times \mathbb{R}^{n-1}$, $n \geq 6$, with the metric g given by $ds^2 = r^2(y^1)^{-2}\sum_{j=1}^{n}(dy^j)^2$, $r > 0$, and the global orthogonal fields $\xi, E_2, \ldots, E_n$ defined by $\xi = \dfrac{y^1}{r^2}\dfrac{\partial}{\partial y^1}$ and $E_i = \dfrac{y^1}{r^2}\dfrac{\partial}{\partial y^i}$, $i = 2, \ldots, n$.

Now, since the dual form of ξ is $\omega = (y^1)^{-1}dy^1$, we have $\xi = \operatorname{grad} f$, with $f \colon \mathbb{H}^n \to \mathbb{R}$, given by $f(y^1, \ldots, y^n) = \log y^1$. Obviously, the distribution $\mathcal{D}$ orthogonal to ξ is integrable, with maximal integral manifolds isometric to $\mathbb{R}^{n-1}$ and given by $\{a\} \times \mathbb{R}^{n-1}$, $a \in \mathbb{R}^*_+$ Namely, they are the level sets of f with induced metric $g_a = r^2a^{-2}\sum_{j=2}^{n}(dy^j)^2$. Let $N = \{a\} \times \mathbb{R}^{n-1}$, $a > 0$ be a fixed maximal integral manifold of $\mathcal{D}$ and denote by $\overline{T}$ a parallel, non vanishing homogeneous structure of class $\mathcal{T}_2$ on $\mathbb{R}^{n-1}$ and hence on N. Consider the extension of $\overline{T}$ to a tensor field π on $\mathbb{H}^n$ defined as follows. Fix $p = (a, y) = (a, y^2, \ldots, y^n) \in \mathbb{R}^*_+ \times \mathbb{R}^{n-1}$. Since any vector $X_p \in T_p(\mathbb{R}^*_+ \times \mathbb{R}^{n-1})$ can be written as $X_p = \rho\xi_p + \overline{X}_p$ with $\rho \in \mathbb{R}$ and $\overline{X}_p \in T_p(N)$, we define $\pi'_p(X_p, Y_p) = \overline{T}_p(\overline{X}_p, \overline{Y}_p)$ and we denote by π' the tensor field uniquely determined on $\mathbb{R}^*_+ \times \mathbb{R}^{n-1}$ by parallel transport of π'_p with respect to the Levi-Civita connection on the flat space $\mathbb{R}^*_+ \times \mathbb{R}^{n-1}$. Obviously, we have $\pi'(\xi, X) = 0$, $\pi'(X, \xi) = 0$ for any vector field X and $\pi'_{|N} = \overline{T}$.

Now, put $\pi(X,Y) = \dfrac{a}{y^1}\pi'(X,Y)$ for any $X, Y \in \mathcal{H}(\mathbb{H}^n)$.

It is easy to verify that $\pi_{|N} = \overline{T}$ and

$$\text{(11)} \qquad \pi(\xi, \ ) = 0, \quad \pi(\ , \xi) = 0, \quad g(\pi(X,Y),\xi) = 0$$

for any $X, Y \in \mathcal{H}(\mathbb{H}^n)$.

Finally, for any $X, Y \in \mathcal{H}(\mathbb{H}^n)$ we define

$$T(X,Y) = g(X,Y)\xi - g(Y,\xi)X + \pi(X,Y)$$

and we consider the connection $\tilde{\nabla} = \nabla - T$, where ∇ is the Riemannian connection on $\mathbb{H}^n$.
Using the construction of π and the condition (2) for $\overline{T}$, we have

$$\text{(12)} \qquad \underset{X,Y,Z}{\mathfrak{S}} g(T(X,Y),Z) = \underset{X,Y,Z}{\mathfrak{S}} g(\pi(X,Y),Z) = 0\,.$$

Now, the condition $(A-S)$ i) for T, i.e. $\tilde{\nabla} g = 0$ follows from (11) and the analogous condition for $\overline{T}$.
Obviously, we have $\tilde{\nabla} R = 0$, where R is the Riemannian curvature of $\mathbb{H}^n$.
Finally, to obtain $T \in \mathcal{T}_1 \oplus \mathcal{T}_2$ we have to prove that $\tilde{\nabla} T = 0$. Since $T_1(X,Y) = g(X,Y)\xi - g(Y,\xi)X$ determines a structure of class $\mathcal{T}_1$ on $\mathbb{H}^n$, [9], we have $\nabla_1 \xi = 0$, where $\nabla_1 = \nabla - T_1$.
On the other hand, for any $X \in \mathcal{H}(\mathbb{H}^n)$, $\pi(X,\xi) = 0$ implies

$$\tilde{\nabla}_X \xi = \nabla_X \xi - T_1(X,\xi) = (\nabla_1)_X \xi = 0\,.$$

Then, from $\tilde{\nabla}\xi = 0$, $\tilde{\nabla} g = 0$ it follows: $\tilde{\nabla} T = 0 \iff \tilde{\nabla}\pi = 0$.
Now, it is easy to see that for any $X, Y \in \mathcal{H}(N)$ we have $\tilde{\nabla}_X Y = \nabla'_X Y - \overline{T}(X,Y)$ where ∇' is the flat Riemannian connection on N.
It follows that $\tilde{\nabla}_{|N}$ is the canonical connection determined by $\overline{T}$, and, consequently, $\tilde{\nabla}\pi = 0$ on N and on any integral manifold of $\mathcal{D}$, taking account of the definition of π.
A direct computation shows that $\tilde{\nabla}_\xi E_i = 0$ for $i = 2, \dots, n$.

Furthermore, since $\tilde{\nabla}\xi = 0$, $\pi(\xi, X) = \pi(X,\xi) = 0$ and

$$\pi_{(y^1,y)}(E_i, E_j) = \sum_{h=2}^{n} \frac{a}{r} \overline{T}^h_{ij}(a,y) E_h$$

where $\overline{T}^h_{ij}(a,y)$ are constant, we have:

$$(\tilde{\nabla}_\xi \pi)(E_i,\xi) = 0\,, \quad (\tilde{\nabla}_\xi \pi)(E_i,E_j) = 0\,, \quad (\tilde{\nabla}_{E_i}\pi)(\xi,E_j) = 0\,,$$

$$(\tilde{\nabla}_\xi \pi)(\xi,E_i) = 0\,, \quad (\tilde{\nabla}_{E_i}\pi)(E_j,\xi) = 0\,, \text{ so that } \tilde{\nabla}\pi = 0 \text{ on } \mathbb{H}^n\,.$$

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