Rendiconti di Matematica, Serie VII
Volume 11 , Roma (1991), 123-131

The Ruscheweyh's Derivative and some Criteria for Univalence in the Unit Disc

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RIASSUNTO – *Per una funzione $f(z) = z+a_2z^2+\ldots$, analitica in $|z| < 1$, vengono dati alcuni criteri di univalenza mediante derivate di Ruscheweyh:*
$D^n f(z) = \frac{1}{(1-z)^{n+1}} \star f(z)$ *("$\star$" indica il prodotto di Hadamard).*

ABSTRACT – *For a function $f(z) = z + a_2z^2 + \ldots$, analytic in $|z| < 1$, some criteria for univalence in terms on the Ruscheweyh's derivative:*
$D^n f(z) = \frac{1}{(1-z)^{n+1}} \star f(z)$ *("$\star$" means the Hadamard product) are given.*

KEY WORDS – *Ruscheweyh's derivative - Univalence.*

A.M.S. CLASSIFICATION: 30C45

1 – Introduction and preliminaries

Let A denote the class of functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the unit disc $U = \{z: |z| < 1\}$.

As usual, by $S^\star(\alpha)$, $0 \le \alpha < 1$, we denote the class of starlike functions of order α in U, i.e.

$$S^\star(\alpha) = \left\{ f \in A:\ \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha\,,\ z \in U \right\}.$$

The class $S^\star(0) = S^\star$ we call the class of starlike functions.

In [5] RUSCHEWEYH introduced the classes $K_n \subset A$, $n \in \mathbb{N}_0 = \{0,1,2,\dots\}$ under the condition

$$\text{(1)} \qquad \operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \frac{1}{2}, \qquad z \in U,$$

where

$$\text{(2)} \qquad D^n f(z) = \frac{z}{(1-z)^{n+1}} \star f(z)$$

and "$\star$" means the Hadamard product of two analytic functions. He showed that $K_{n+1} \subset K_n \subset K_0 \equiv S^\star(1/2)$ holds for $n \in \mathbb{N}_0$. This implies that K_n, $n \in \mathbb{N}_0$, are the subclasses of univalent functions in U.

Let f and g be analytic functions in U. We say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if g is univalent in U, $f(0) = g(0)$ and $f(U) \subset g(U)$.

In the present paper by using the Ruscheweyh's derivative (2) we give some criteria for univalence in U. The similar method was given in [2].

For our results in the second part of this paper we need the next lemmas.

LEMMA A. *Let g be a convex univalent in U, $g(0) = 1$. Let f be analytic in U, $F(0) = 1$ and let $f \prec g$ in U. Then for all $n \in \mathbb{N}_0$,*

$$(n+1)z^{-n-1}\int_0^z t^n f(t)dt \prec (n+1)z^{-n-1}\int_0^z t^n g(t)dt.$$

This lemma in more general form is due to HALLENBECK and RUSCHEWEYH [1].

LEMMA B. [7] *Let f and g be analytic functions in U, with $f(0) = g(0)$. If the function $h(z) = zg'(z)$ is starlike and*

$$zf'(z) \prec zg'(z),$$

then

$$f(z) \prec g(z) = g(0) + \int_0^z \frac{h(t)}{t} dt .$$

LEMMA C. [7] *Let μ be a positive measure on [0,1] and let $q(z,t)$ be a complex function on $U \times [0,1]$ such that $q(z,\cdot)$ is μ-integrable on $[0,1]$ for all $z \in U$. Suppose that* $\operatorname{Re} q(z,t) > 0$ *for $z \in U$, $t \in [0,1]$, $q(-r,t)$ is real and*

$$\operatorname{Re}\left\{\frac{1}{q(z,t)}\right\} \geq \frac{1}{q(-r,t)}, \quad for\ |z| \leq r < 1, \quad t \in [0,1].$$

If

$$q(z) = \int_0^1 q(z,t) d\mu(t),$$

then

$$\operatorname{Re}\left\{\frac{1}{q(z)}\right\} \geq \frac{1}{q(-r)} \quad for\ |z| \leq r .$$

2 – On some criteria for univalence

First we give the following

LEMMA 1. *Let $p(z)$ be analytic in U, $p(0) = 1$, and let $n \in \mathbb{N}_0$, $0 < k \leq 1$. If*

$$\frac{1}{n+2}(1 + (n+1)p(z) + zp'(z)) \prec 1 - kz, \tag{3}$$

then $p(z) \prec 1 - kz$.

PROOF. Since the function $1 - kz (0 < k \leq 1)$ is a convex function in U and since

$$(n+1)p(z) + zp'(z) = \frac{(z^{n+1}p(z))'}{z^n}, \quad n \in \mathbb{N}_0,$$

then by applying Lemma A we get

$$(n+1)z^{-n-1}\int_0^z t^n \frac{1}{n+2}\left(1+\frac{(t^{n+1}p(t))'}{t^n}\right)dt \prec (n+1)z^{-n-1}\int_0^z t^n(1-kt)dt\,.$$

From there we easily obtain

$$\frac{1}{n+2}+\frac{n+1}{n+2}p(z) \prec 1-k\frac{n+1}{n+2}z\,,$$

i.e.

$$p(z) \prec 1-kz\,.$$

THEOREM 1. *Let $f(z) \in A$ and $D^n f(z) \neq 0$ for $0 < |z| < 1$, and $n \in \mathbb{N}_0$. If there exists a real number k, $0 < k \leq 1$, such that*

$$\left|\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-1\right| < k\left|\frac{D^{n+1}f(z)}{D^n f(z)}\right|, \quad z \in U \tag{4}$$

then $f(z)$ is univalent in U and $\dfrac{D^{n+1}f(z)}{D^n f(z)} \prec \dfrac{1}{1-kz}$.

PROOF. The condition (4) and $D^n f(z) \neq 0$ for $0 < |z| < 1$ implies that $D^{n+1}f(z) \neq 0$ for $0 < |z| < 1$. If $p(z) \neq 0$, $z \in U$ in Lemma 1, then the condition (3) can be written in the form

$$\frac{p(z)}{n+2}\left(\frac{n+1}{p(z)}-(n+1)-\frac{zp'(z)}{p(z)}\right) \prec kz\,. \tag{5}$$

If we put $p(z) = \dfrac{D^n f(z)}{D^{n+1}f(z)}$, then we have $p(0) = 1$ and $p(z) \neq 0$ for $0 < |z| < 1$. After taking the logarithmic differentiation and by using identity

$$z(D^m f(z))' = (m+1)D^{m+1}f(z) - mD^m f(z), \quad m \in \mathbb{N}_0\,, \tag{6}$$

we get

$$\frac{p(z)}{n+2}\left(\frac{n+1}{p(z)}-(n+1)-\frac{zp'(z)}{p(z)}\right) = \frac{D^n f(z)}{D^{n+1}f(z)}\left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-1\right)\,. \tag{7}$$

From the condition (4) and the relation (7) we conclude that the condition (5), i.e. (3) is satisfied. Now, by Lemma 1 we have that $p(z) \prec 1 - kz$, which implies $\frac{1}{p(z)} \prec \frac{1}{1-kz}$, i.e.

$$\frac{D^{n+1}f(z)}{D^n f(z)} \prec \frac{1}{1-kz}.$$

Since $\operatorname{Re}\left\{\frac{1}{1-kz}\right\} > \frac{1}{1+k} \geq \frac{1}{2}$ for $0 < k \leq 1$, we obtain $\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \frac{1}{2}$, $z \in U$. By the Ruscheweyh's result this proves that f is univalent in U.

For $n = 0$ in the previous theorem we have.

COROLLARY 1. *Let $f \in A$ and $f(z) \neq 0$ for $0 < |z| < 1$. If there exists a real number $0 < k \leq 1$ such that*

$$\left|\frac{zf''(z)}{f'(z)}\right| < 2k\left|\frac{zf'(z)}{f(z)}\right|, \quad f \in U,$$

then $f \in S^*\left(\frac{1}{1+k}\right)$ and $\frac{zf'(z)}{f(z)} \prec \frac{1}{1-kz}$.

This is the earlier result due to ROBERTSON [4].

THEOREM 2. *Let $f \in A$ and let $D^n f(z) \neq 0$ for $0 < |z| < 1$. If there exists a real number $k > n+1$, $n \in \mathbb{N}_0$, such that*

$$\left|(n+2)\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right| < k\left|\frac{D^{n+1}f(z)}{D^n f(z)}\right|, \quad z \in U, \tag{8}$$

then

$$\frac{D^n f(z)}{D^{n+1}f(z)} \prec 1 + \frac{(n+1)^2 - k^2}{n+1}\ln\left(1 + \frac{n+1}{k}z\right) \tag{9}$$

(where we take $\ln 1 = 0$).

If, in addition, $n+1 < k \leq \frac{1+\sqrt{1+4(n+1)^2}}{2}$, then

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \alpha(n,k),$$

where

$$\alpha(n,k) = \left(1 + \frac{(n+1)^2 - k^2}{n+1} \ln\left(1 - \frac{n+1}{k}\right)\right)^{-1}. \tag{10}$$

PROOF. From the condition (8) we have that $D^{n+1}f(z) \neq 0$ for $0 < |z| < 1$, hence $p(z) = \dfrac{D^n f(z)}{D^{n+1}f(z)}$ is analytic in U and $p(z) \neq 0$. Hence, the condition (8) is equivalent to

$$\left|\frac{D^n f(z)}{D^{n+1}f(z)}\left((n+2)\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1\right)\right| < k,$$

or if we use the identity (6),

$$\left|(n+1) - z\left(\frac{D^n f(z)}{D^{n+1}f(z)}\right)'\right| < k,$$

i.e.

$$|n+1 - zp'(z)| < k. \tag{11}$$

The previous relation (11) we can write in the form

$$zp'(z) \prec \frac{((n+1)^2 - k^2)z}{k + (n+1)z} \equiv zq'(z),$$

and since zq' is starlike, by Lemma B we have that

$$p(z) \prec q(z) = 1 + \int_0^z \frac{(n+1)^2 - k^2}{k + (n+1)z} dz =$$

$$= 1 + \frac{(n+1)^2 - k^2}{n+1} \ln\left(1 + \frac{n+1}{k} z\right), \tag{12}$$

which was to be prove.

Since the function q, given by (12), is convex and $q(U)$ is symmetric with respect to the real axis, we conclude

$$\operatorname{Re}\left\{\frac{D^n f(z)}{D^{n+1}f(z)}\right\} > 1 + \frac{(n+1)^2 - l^2}{n+1} \ln\left(1 + \frac{n+1}{k}\right), \quad z \in U. \tag{13}$$

For the proof of the second part of this theorem we will use Lemma C. Namely the function q, given by (12), we can write in the form

$$q = \int_0^1 \frac{1 + \frac{(n+1)^2 - k^2 + (n+1)t}{k} z}{1 + \frac{n+1}{k} tz} dt = $$

$$= \int_0^1 q(z,t)dt ,$$

where

$$q(z,t) = \frac{1 + \frac{(n+1)^2 - k^2 + (n+1)t}{k} z}{1 + \frac{n+1}{k} tz} \tag{14}$$

Since $-1 \leq \dfrac{(n+1)^2 - k^2 + (n+1)t}{k} \leq 1$ for $t \in [0,1]$ and $n + 1 < k \leq \dfrac{1 + \sqrt{1 + 4(n+1)^2}}{2} = k_0(n)$ we get $\text{Re}\{q(z,t)\} > 0$ for $z \in U$ and $t \in [0,1]$. Because $\text{Re}\{1/q(z,t)\} > 1/q(-1,t)$, by Lemma C we have

$$\text{Re}\left\{\frac{1}{q(z)}\right\} = \text{Re}\left\{\frac{1}{\int_0^1 q(z,t)dt}\right\} \geq \frac{1}{q(-1)} =$$

$$= \left(1 + \frac{(n+1)^2 - k^2}{n+1} \log\left(1 - \frac{n+1}{k}\right)\right)^{-1}$$

Finally combining this result with the relation (9) already proved, we obtain the statement of Theorem.

Taking $n = 0$ we have the following.

COROLLARY 2. *If $f \in A$, $f(z) \neq 0$ for $0 < |z| < 1$ and if there exists $k > 1$ such that*

$$\left|\frac{zf''(z)}{f'(z)} + 1\right| < k \left|\frac{zf'(z)}{f(z)}\right| , \quad z \in U , \tag{15}$$

then

$$\frac{f(z)}{zf'(z)} \prec 1 + (1-k^2)\log\left(1+\frac{z}{k}\right).$$

Later, if $k_0 = 1.8089\ldots$ *is the root of the equation* $1+(1-k^2)\ln\left(1+\frac{1}{k}\right) = 0$, *then* $f(z) \in S^\star$ *for* $1 < k \le k_0$.

Moreover, if $1 < k \le (1+\sqrt{5})/2 = 1.618\ldots$, *then* $f(z) \in S^\star(\alpha)$, *where*

$$\alpha = \alpha(0,k) = \left(1 + (1-k^2)\log\left(1-\frac{1}{k}\right)\right)^{-1}.$$

This is the earlier result given by MOCANU [2].

REMARK 1. Let's put $\beta(k) = \dfrac{1}{\alpha(n,k)}$, where $\alpha(n,k)$ defined by (10), i.e.

$$\beta(k) = 1 + \frac{(n+1)^2 - k^2}{n+1}\ln\left(1-\frac{n+1}{k}\right),$$

and let consider this function for $n+1 < k \le \dfrac{1+\sqrt{1+4(n+1)^2}}{2} = k_0(n) < n+2$, and fixed $n \in \mathbb{N}_0$.

$$\text{Since } \beta'(k) = -\frac{2k}{n+1}\ln\left(1-\frac{n+1}{k}\right) - \frac{n+1}{k} - 1 =$$

$$= 1 + 2\sum_{m=2}^{\infty}\frac{1}{m+1}\left(\frac{n+1}{k}\right)^m > 0 \text{ for } k > n+1$$

we deduce that $\beta(k)$ is an increasing function in that interval. Also we easily get that $\lim_{k\to n+1+0}\beta(k) = 1$, while

$$\beta(k_0(n)) = 1 - \frac{k_0(n)}{n+1}\ln\left(1-\frac{n+1}{k_0(n)}\right) =$$

$$= 2 + \frac{1}{2}\frac{n+1}{k_0(n)} + \ldots >$$

$$> 1 + \frac{1}{2}\frac{n+1}{n+2} \ge \frac{9}{4} = 2.25$$

(because $k_0(n)$ satisfies the equation $k^2-(n+1)^2=k$, and $k_0(n)<n+2$. In that sense, the function $\alpha(n,k)=1/\beta(k)$ is decreasing function on $n+1<k\le k_0(n)$ from 1 to $\alpha(n,k_0(n))=-\dfrac{1}{\beta(k_0(n))}<0.44\ldots$

For $n=1$ in Theorem 2 we have the following criteria for univalence.

COROLLARY 3. *Let $f(z)\in A$ for $0<|z|<1$ and $f'(z)\neq 0$. If there exists a real number k,*

$$2<k\le k_0(1)=\frac{1+\sqrt{17}}{2}=2.5615\ldots \textit{ such that}$$

$$\left|3\frac{D^3f(z)}{D^2f(z)}-1\right|<k\left|\frac{D^2f(z)}{D^1f(z)}\right|,\quad z\in U$$

then

$$\operatorname{Re}\left\{\frac{D^2f(z)}{D^1f(z)}\right\}>\alpha(1,k),$$

and f is univalent in U,
where

$$\alpha(1,k)=\left(1+\frac{4-k^2}{2}\ln\left(1-\frac{2}{k}\right)\right)^{-1}. \tag{16}$$

PROOF. We only need to prove that f is univalent in U. According to Remark 1 we have that $0.339\ldots=\alpha(1,k_0(1))\le\alpha(1,k)<1$. Since $\operatorname{Re}\left\{\dfrac{D^2f(z)}{D^1f(z)}\right\}>\alpha(1,k)\ge 0.339\ldots>\dfrac{1}{4}$, which implies $\operatorname{Re}\left\{1+\dfrac{zf''(z)}{f'(z)}\right\}>-\dfrac{1}{2}$, we conclude that f is univalent [3].

If we use the result of Ruscheweyh for univalence in the unit disc $\left(\operatorname{Re}\left\{\dfrac{D^{n+1}f(z)}{D^nf(z)}\right\}>\dfrac{1}{2}, z\in U\right)$, from Theorem 2 and Remark 1 we obtain for $n\ge 2$:

COROLLARY 4. *Let $f\in A$ and let $D^nf(z)\neq 0$, $0<|z|<1$. If there exists a real number $n+1<k\le k_n$, where k_n is the root of the equation*

$$\left(1+\frac{(n+1)^2-k^2}{n+1}\log\left(1-\frac{n+1}{k}\right)\right)^{-1}=\frac{1}{2},$$

and if

$$\left|(n+2)\frac{D^{n+2}f(z)}{D^{n+1}f(z)}-1\right| < k\left|\frac{D^{n+1}f(z)}{D^{n}f(z)}\right|, \quad z \in U,$$

then

$$\operatorname{Re}\left\{\frac{D^{n+1}f(z)}{D^{n}f(z)}\right\} > \frac{1}{2}, \quad z \in U$$

i.e. f is univalent in U.

REFERENCES

[1] D.J. HALLENBERK - S. RUSCHEWEYH: *Subordination by convex functions*, Proc. Amer. Math. Soc. 52 (1975), 191-195.

[2] P.T. MOCANU: *On a Theorem of M. Robertson*, "Babes-Bolyai" University, Seminar on Geometric Function Theory, Preprint No. 5. (1986), 77-82.

[3] S. OZAKI: *On the theory of multivalent functions*, II, Sci. Rep. Tokyo Bunrika Daigaku, 4(1941), 45-87.

[4] M.S. ROBERTSON: *Certain classes of starlike functions*, Michigan Math. J. 32 (1985), 135-140.

[5] S. RUSCHEWEYH: *New criteria for univalent functions*, Proc. Amer. Math. Soc. 49 (1975), 109-115.

[6] T.J. SUFFRIDGE: *Some remarks on convex maps of the unit disk*, Duke Math. J. 37 (1970), 775-777.

[7] D.R. WILKEN - J. FENG: *A remark on convex and starlike functions*, J. London Math. Soc. (2), 21 (1980), 287-290.

Lavoro pervenuto alla redazione il 2 dicembre 1989
ed accettato per la pubblicazione il 6 giugno 1990
su parere favorevole di A. Ossicini e di P.E. Ricci

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