Rendiconti di Matematica, Serie VII
Volume 11 , Roma (1991), 175-186

Fixed Points and Best Approximations for Convexly Condensing Functions in Topological Vector Spaces

E. DE PASCALE - G. TROMBETTA(*)

RIASSUNTO – *Si introducono i concetti di misura di precompattezza convessa e di funzione convessamente condensante. Si ottengono teoremi di punto fisso e di migliore approssimazione in spazi vettoriali topologici non necessariamente localmente convessi.*

ABSTRACT – *The concepts of convex precompactness measure and of convexly condensing function are introduced. Therefore fixed point theorems and best appproximation theorems are obtained in topological vector spaces which are not necessarily locally convex.*

KEY WORDS – *Topological vector space - Fixed point - Measure of convex precompactness - Convexly condensing function - Best approximation.*

A.M.S. CLASSIFICATION: 47H10 - 54H25 - 41A50

1 – Introduction

One of the main tools in fixed point theory is the use of noncompactness measures, since pionereeng work of DARBO ([4]). We reccommend [1] and [3] for more informations and further references to the literature. The above tools are useful with many difficulties when the framework is a

(*)Work supported by research funds of Ministero Pubblica Istruzione - Italia (60% and 40%).

not necessarily locally convex topological vector space (see [7], [8]). This type of problems is of interest to us in this paper.

In what follows E will denote a real Hausdorff topological vector space.

In [11] IDZIK gave the following definition:

DEFINITION 1.1. (cf. Definition 2.2 of [11]) *A set $B \subset E$ is convexly totally bounded (c.t.b. for short), if, for every neighbourhood V of $0 \in E$, there exist a finite subset $\{x_i, i \in I\} \subset B$ and a finite family of convex sets $\{C_i, i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $B \subset \bigcup\{x_i + C_i, i \in I\}$.*

Using the Definition 1.1, he proved:

THEOREM 1.2. (cf. Theorem 2.4 of [11]). *Let B be a convex subset of E. Assume that $K \subset B$ is a compact set and $f: B \longrightarrow K$ a continuous function. If $\overline{f(B)}$ is a convexly totally bounded set, then f has a fixed point in K.*

The above theorem is closely related with the following, more than a half of a century existing:

SCHAUDER'S CONJECTURE: Every continuous function from a nonempty compact, convex set in a topological vector space into itself has a fixed point.

The relation between Theorem 1.2 and the Schauder's Conjecture is given by.

PROBLEM 1.3. (cf. Problem 4.7 of [12]). Is every compact, convex set in a real Hausdorff topological vector space c.t.b.?

In the same order if ideas introduced by SADOVSKII ([14], [15]), HIMMELBERG, PORTER and VAN VLECK ([9]) defined a very general measure of precompactness in the context of locally convex topological vector space. Subsequently they gave a definition of condensing multifunctions, for which the following theorem holds.

THEOREM 1.4. (cf. Theorem 1 of [9]). *Let B be a nonempty complete, convex subset of a Hausdorff locally convex topological vector space E, and let $f: B \multimap B$ be a condensing multifunction with convex values, closed graph and bounded range. Then f has a fixed point.*

In Paragraph 2 we introduce two measures of convex precompactness, related with Definition 1.1. Further, to study them and to make a comparison with the results in [9] easier, we introduce some other measures of precompactness, similar to that of [9].

In Paragraph 3 we define and develop the concept of convexly condensing multifunction. Therefore we obtain our main fixed point theorems for convexly condensing multifunctions. To this purpose, we are compelled to introduce a control on the convex precompactness measure for the convex hull of a set, because we work in the context of not necessarily locally convex spaces.

In Paragraph 4, using the same tools, we prove a Fan's best approximation theorem for convexly condensing multifunctions. Our result is in the same order of ideas of the following theorem.

Let E be a Hausdorff locally convex topological vector space, equipped with a continuous seminorm p.

THEOREM 1.5. (cf. Theorem 1 of [16]). *Let B be an approximatively p-compact, convex subset of E and $f: B \multimap E$ be a continuous multifunction with closed and convex values. If $f(B) = \bigcup\{f(x), x \in B\}$ is relatively compact then there exists an $x \in B$ with $d_p(x, f(x)) = d_p(f(x), B)$. ($d_p(A, B) = \inf\{p(x-y), x \in A$ and $y \in B\}$).*

In our theorem we remove the compactness condition on the range of f. Further we shall work in a space E which is not necessarily locally convex and equipped with a continuous seminorm p. Some examples of such type of spaces can be found in [2].

Finally a few words about terminology, most of which is standard.

A multifunction $f: X \multimap Y$ between two topological spaces X and Y is a function which assigns to each point $x \in X$ a nonempty subset $f(x)$ of Y. The graph of f is the set $\{(x,y) \in X \times Y, y \in f(x)\}$. f is uppersemicontinuous (u.s.c.) if for every closed subset C of Y $f^{-1}(C) = \{x \in X, f(x) \cap C \neq \emptyset\}$ is closed. f is lowersemicontinuous (l.s.c.) if for every open subset C of Y, $f^{-1}(C)$ is open. f is continuous if it is both u.s.c. and l.s.c.

If f is compact valued (i.e. for each $x \in X$, $f(x)$ is compact) and Y is compact and regular, then f is u.s.c. iff the graph of f is closed in $X \times Y$. If $X = Y$, a fixed point for f is a point $x \in X$ such that

$x \in f(x)$. A subset B of E is precompact (or totally bounded) if for every neighbourhood V of $0 \in E$ there exists a finite set F such that $B \subset V + F$, where $V + F = \{x + y, x \in V \text{ and } y \in F\}$.

2 – Convex Precompactness

For a subset A of E and a neighbourhood V of the origin, we consider the following property (P): "there exist a finite family of closed convex sets $\{C_i, i \in I\}$ and a finite subset of $E\{x_i, i \in I\}$ such that $C_i \subset V$ for each $i \in I$ and $A \subset \cup\{x_i + C_i, i \in I\}$". In the remainder of the paper $\mathcal{B}$ will denote a basis of neighbourhoods of $0 \in E$. For a subset A of E we give the following:

DEFINITION 2.1.

$$R_1(A) = \{V \in \mathcal{B}\,,\ \textit{Property } P \textit{ holds for } V\}\,.$$

$$R_2(A) = \{V \in \mathcal{B}\,, \textit{for each } W \in \mathcal{B} \textit{ Property } P \textit{ holds for } V + W\}\,.$$

We denote indifferently with $R(A)$ either $R_1(A)$ or $R_2(A)$. The above sets $R(A)$ measure the lack of convex precompactness of A.

REMARK 2.2. Property P is slightly different from that of Definition 1.1.

First of all we observe that there is no loss of generality in Definition 1.1 if the sets C_i are assumed to be closed.

Further the assumption "$x_i \in A$" is related with the usual possibility to work with internal ($x_i \in A$) or external (x_i do not necessarily belong to A) measure for A.

For the sake of simplicity we confine ourselves to the case of external measures.

We collect some properties of R in the next

PROPOSITION 2.3.

(a) $R_1(A) \subset R_2(A)$.

(b) $A \subset B$ *implies* $R(B) \subset R(A)$.

(c) A *is c.t.b. iff* $R(A) = \mathcal{B}$.

(d) $R(A \cup F) = R(A)$ *for every* $F \subset E$ *finite.*

(e) $R(A) = R(\overline{A})$, *where* $\overline{A}$ *denotes the closure of* A.

It will be clear from 2.5(c), 2.6 and 2.7(a) that even in the context of locally convex spaces, in general $R_1(A) \not\subset R_2(A)$.

To make a comparison with the results of [9] easier, we introduce the following other measures of precompactness. The second one is just that introduced in [9].

DEFINITION 2.4. $Q_1(A) = \{V \in B,$ *"there exists a finite set* F *such that* $A \subset V + F$"$\}$.

$Q_2(A) = \{V \in B,$ *"there exists a precompact set* S *such that* $A \subset V+S$"$\}$.

$Q_3(A) = \{V \in B,$ *"for every* $W \in B$ *there exists a finite set* F *such that* $A \subset V + W + F$, *and the sets* F *are all contained in a precompact set* S, *not depending on* W"$\}$.

$Q_4(A) = \{V \in B,$ *"for every* $W \in B$, $V + W \in Q_1(A)$"$\}$.

We denote by Q the generic Q_i.

PROPOSITION 2.5.

(a) $Q(A) = B$ *iff* A *is precompact.*

(b) $A \subset B$ *implies* $Q(B) \subset Q(A)$.

(c) $Q_1(A) \subset Q_2(A) \subset Q_3(A) \subset Q_4(A)$.

(d) $R_1(A) \subset Q_1(A)$, $R_2(A) \subset Q_4(A)$.

(e) *If the sets belonging to* B *are closed* $Q_2(A) = Q_3(A)$.

PROOF. The properties listed from (a) to (d) are obvious.

We prove (e): if $V \in Q_3(A)$ then there exists S precompact such that $A \subset V + W + S$ for every $W \in B$. Consequently $A \subset \cap\{V + W + S, W \in B\} = \overline{V + S} = V + \overline{S}$ and so $V \in Q_2(A)$.

REMARK 2.6. We think that the inclusions listed in Proposition 2.5(c) are in general strict. We give a counterexample in the case of Q_1 and Q_2. Let $E = l_2$, $\{e_n, n \in N\}$ be the canonical base of l_2, $B = \{B_r, r > 0\}$ where $B_r = \{x \in l_2, \|x\|_2 \leq r\}$. If $A = B_1 + [0,1]e_1 = \{x + \lambda e_1, x \in B_1$ and $\lambda \in [0,1]\}$, we have $B_1 \in Q_2(A) \setminus Q_1(A)$. Since $A = B_1 + [0,1]e_1$ and $[0,1]e_1$ is precompact, $B_1 \in Q_2(A)$. Suppose that $A \subset B_1 + F$ with $F = \{x^1, \ldots, x^s\}$. For a fixed $k \in \mathbb{N}$ and for every $n \geq 2$, there exist

$x(n) \in B_1$ and $x^{i(n)} \in F$ such that $e_n + e_1/k = x(n) + x^{i(n)}$. We consider in the last equality the first and n-th component: $1/k = x_1(n) + x_1^{i(n)}$, $1 = x_n(n) + x_n^{i(n)}$. The inequality $1 \geq x_1(n)^2 + x_n(n)^2 = (1/k - x_1^{i(n)})^2 + (1 - x_n^{i(n)})^2$ and the $\lim_n x_n^{i(n)} = 0$ (because F is finite) imply that $x_1^{i(n)} = 1/k$ eventually. Since $x_1^{i(n)} \in \{x_1^1, \ldots, x_1^s\}$ and k is arbitrary, we have a contradiction.

PROPOSITION 2.7. *If E is a locally convex space and if $\mathcal{B}$ is composed of convex neighbourhoods we have:*

(a) $R_1(A) = Q_1(A)$, $R_2(A) = Q_4(A)$

(b) $Q(\operatorname{co} A) = Q(A)$, *where* $\operatorname{co} A$ *denotes the convex hull of* A.

(c) *Theorem 1.4 holds not only for the measure Q_2, but also for the others Q_i.*

REMARK 2.8. We do not know if the equality in the Proposition 2.7 (a) holds in not locally convex spaces. This is a very difficult question. In fact a positive answer, even in the limit case of $Q_1(A) = Q_4(A) = \mathcal{B}$, implies a positive solution to Shauder's conjecture.

3 – Fixed Points

It is well known that in the fixed point theory in locally convex spaces the relation $Q(\operatorname{co} A) = Q(A)$ (see Proposition 2.7(b)) is of great importance. But we cannot prove, in general, that the inclusion $R(\operatorname{co} A) \subset R(A)$ is an equality. To bypass the difficulties arising from this fact we introduce a control for $R(\operatorname{co} A)$ in a way similar to that of [8] and [17].

DEFINITION 3.1. *Let B be a subset of the space E. A multifunction $f\colon B \multimap E$ is convexly condensing (with respect to R) if there exists $\varphi\colon \mathcal{B} \longrightarrow \mathcal{B}$, such that*

$$\text{(C)} \qquad \varphi(R(Z)) \subset R(\operatorname{co} Z) \text{ for every } Z \subset \overline{\operatorname{co}} f(B)$$

and one of the following implications holds for every bounded subset Z of B:

(a) $\varphi(R(f(Z))) \subset R(Z) \implies f(Z)$ *c.t.b.*

(b) $\varphi(R(f(Z))) \subset R(Z) \implies Z$ *c.t.b.*

Roughly speaking, the sense of the above definition is that convexly condensing multifunctions take bounded sets not c.t.b. to sets which are more nearly c.t.b.

REMARK 3.2. If E is locally convex, $\mathcal{B}$ is composed of closed convex sets, $R = R_1$, and φ is the identity of $\mathcal{B}$, then Definition 3.1 reduces to that of [9].

REMARK 3.3. The multifunctions for which $f(B)$ is c.t.b. verify Definition 3.1 (a), but in general do not verify 3.1 (b). The conditions (a) e (b) in Definition 3.1 are related by the following conjecture: "f u.s.c. and Z c.t.b. implies $f(Z)$ c.t.b.".

Moreover conditions (a) and (b) are particular cases of the more general condition: $\varphi(R(f(Z))) \subset R(Z)$ and $f(Z) \subset Z$ implies $f(Z)$ c.t.b.

But the last condition requires a preliminary knowledge of the subsets which are invariant for f and it is not useful to work with in the proofs (see for example Lemma 3.6 in what follows).

PROPOSITION 3.4. *Let B be a nonempty convex subset of E and $f\colon B \multimap B$ a convexly condensing multifunction, such that $\overline{\mathrm{co}} f(B) \subset B$ is bounded. Then for every $u \in f(B)$ there exists a subset L_0 of B, such that $L_0 = \overline{\mathrm{co}}\{f(L_0) \cup \{u\}\}$ and $f(L_0)$ is c.t.b.*

PROOF. Let $\mathcal{L} = \{L \subseteq \overline{\mathrm{co}} f(B) / L = \overline{\mathrm{co}} L,\ u \in L,\ f(L) \subseteq L\}$. Since $\overline{\mathrm{co}} f(B) \in \mathcal{L}$, $\mathcal{L}$ is not empty. Let $L_0 = \bigcap\{L, L \in \mathcal{L}\}$ and $L_1 = \overline{\mathrm{co}}\{f(L_0) \cup \{u\}\}$. It is easy to see that $L_1 \in \mathcal{L}$ and so $L_0 = L_1 = \overline{\mathrm{co}}\{f(L_0) \cup \{u\}\}$. If Property (a) of Definition 3.1 holds for f we have $R(L_0) = R(\mathrm{co}\{f(L_0) \cup \{u\}) \supseteq \varphi(R(f(L_0)))$. Since $\overline{\mathrm{co}} f(B)$ is bounded, $f(L_0)$ is c.t.b.. Analogously, if Property (b) of Definition 3.1 holds, we have L_0 c.t.b.. But $f(L_0) \subset L_0$ implies $f(L_0)$ c.t.b..

In the next theorem we shall use a multivalued version of Theorem 1.2 by Idzik. Such versione was obtained by IDZIK himself in [12].

He worked in the more general context of multifunctions f defined on almost convex sets (for the definition see [10], [12])

THEOREM 3.5. *Let B be a nonempty convex subset of E and $f\colon B \multimap B$ be a convexly condensing multifunction with convex values. If f has closed graph, $\overline{\mathrm{co}} f(B) \subset B$ is bounded and complete, then f has a fixed point.*

PROOF. For a fixed $u \in f(B)$, let L_0 be the set whose existence is guaranteed by Proposition 3.4. Let $g: L_0 \multimap L_0$ be the multifunction whose graph is defined by graph $g = \text{graph} f \cap (L_0 \times L_0)$. Since graph g is closed and $g(L_0)$ is contained in the compact set $\overline{f(L_0)}$, g is u.s.c.

By Theorem 4.3 of [12] g has a fixed point x, which is fixed for f too, because g is a submultifunction of f (for each $x \in L_0$ $g(x) \subset f(x)$).

We prove a property of multifunctions convexly condensing with respect to the measure R_2. We do not know if the same result holds for the measure R_1.

In the next lemma and theorem following it we need that the fixed basis of neighbourhoods $\mathcal{B}$ is composed of circled sets.

LEMMA 3.6. *Let $f: B \multimap B$ be a multifunction convexly condensing with respect to the measure R_2 and $\lambda: E \longrightarrow [0,1]$ a real function. Then the multifunction $\lambda f: x \in B \multimap \lambda(x)f(x) = \{\lambda(x)y, y \in f(x)\}$ is convexly condensing with respect to the measure R_2.*

PROOF. Let Z be a subset of B and $V \in R_2(f(Z))$. Given $U \in \mathcal{B}$, let $W \in \mathcal{B}$ be such that $W + W \subset U$. By the definition of R_2 there exist a finite family of convex sets $\{C_i, i \in I\}$ and a finite set $\{x_i, i \in I\}$ such that $C_i \subset V + W$ for every $i \in I$ and $f(Z) \subset U\{x_i + C_i, i \in I\}$. Since $V + W$ is circled, we can suppose, without loss of generality that $O \in C_i$. So we have $\lambda f(Z) \subset \cup\{\lambda(x)x_i + \lambda(x)C_i, i \in I \text{ and } x \in Z\} \subset \cup\{[0,1]x_i + C_i, i \in I\}$.

For every $i \in I$ there exist convex subsets $D_{ij} \subset W$ and a finite family $\{y_{ij}, j = 1, \ldots, n_i\}$ such that $[0,1]x_i \subset \cup\{D_{ij} + y_{ij}, i = 1, \ldots, n_i\}$.

Consequently $\lambda f(Z) \subset \cup\{y_i + D_{ij} + C_i, i \in I \text{ and } j = 1, \ldots, n_i\}$.

Since $D_{ij} + C_i \subset V + W + W \subset V + U$ for every i and j, we have $V \in R_2(\lambda f(Z))$. If condition 3.1 (a) holds for f we have $\varphi(R_2(\lambda f(Z)) \subset R_2(Z) \implies \varphi(R_2(f(Z))) \subset R_2(Z) \implies f(Z)$ c.t.b. $\implies \lambda f(Z)$ c.t.b. If 3.1 (b) holds for f, 3.1 (b) holds also for λf.

THEOREM 3.7. *Let $B \subset E$ be a complete convex set and W be a closed neighbourhood of $u \in B$. Let $f: B \cap W \multimap B$ be a multifunction convexly condensing with respect to R_2 and in accordance with Definition 3.1 (a). If f has closed graph, convex values, $\text{co} f(B \cap W)$ is bounded and the following boundary condition holds: " $x \in \partial W \cap B$ and $x \in$*

$tf(x)+(1-t)u \implies$ there exists $s>1$ such that $x \in sf(x)+(1-s)u$", then f has a fixed point.

PROOF. Let $X = \{x \in W \cap B$, there exists $t \in [0,1]$ so that $x \in tf(x)+(1-t)u\}$. The set X is closed. If $x \in X \cap (\partial W \cap B)$ there are $t \in [0,1]$ and $s > 1$ such that $x \in tf(x)+(1-t)u \cap sf(x)+(1-s)u$. Let $\alpha \in [0,1]$ be such that $\alpha s + (1-\alpha)t = 1$. Since $f(x)$ is convex, we have $x \in \alpha[tf(x)+(1-t)u]+(1-\alpha)[sf(x)+(1-s)u] \subset f(x)$, and so we are done. So we can suppose $X \cap (\partial W \cap B) = \emptyset$. Since the space E is completely regular there exists a continuous function $\lambda\colon E \longrightarrow [0,1]$ such that: $\lambda(x)=0$ for every $x \in X$ and $\lambda(x)=1$ for every $x \in \partial W \cap B$.

Let

$$g(x)=\begin{cases}(1-\lambda(x))f(x)+\lambda(x)u & \text{if } x \in W \cap B\\ u & \text{if } x \in B \setminus W.\end{cases}$$

Then $g\colon B \multimap B$ has closed graph and $\operatorname{co} g(B)$ is bounded. We claim that g is convexly condensing with respect to R_2. Let $Z \subset B$ be such that $\varphi(R_2(g(Z))) \subset R_2(Z)$. By Lemma 3.6 the restriction of g to $W \cap B$ is convexly condensing and since $g(Z) = g(Z \cap W \cap B) \cup \{u\}$ we have $\varphi(R_2(g(Z))) = \varphi(R_2(g(Z \cap W \cap B))) \subset R_2(Z) \subset (R_2(Z \cap W \cap B) \implies g(Z \cap W \cap B)$ c.t.b. $\implies g(Z)$ c.t.b. By Theorem 3.5 there exists $x \in B$ such that $x \in g(x)$. If $x \in B \setminus W$ then $x = u \in W \cap B$, which is possible. So $x \in W \cap B$ and consequently $x \in X$. So we have $\lambda(x)=0$ and $x \in f(x)$.

REMARK 3.8. The boundary condition used in Theorem 3.7, more general then the usual one: "$x \in \partial W \cap k$ and $x \in tf(x)+(1-t)u \implies t > 1$", was used first in [5].

4 – Best Approximations

In this paragraph we suppose that the space E is equipped with a continuous seminorm p.

DEFINITION 4.1. *A subset B of E is approximatively p-compact iff for each $y \in E$ and a net $\{x_\alpha\}$ in B satisfying $p(x_\alpha - y) \to d_p(y,B) = \inf\{p(y-z), z \in B\}$ there is a subnet $\{x_\beta\}$ and $x \in S$ such that $x_\beta \to x$.*

For more details and informations about approximatively p-compactness see [13], [16], and references therein.

PROPOSITION 4.2. *If B is an approximatively p-compact subset of E, then for each $y \in E$, $P(y) = \{x \in B, p(y-x) = d_p(y, B)\}$ is nonempty and the multifunction $P: E \multimap B$ is u.s.c.*

For a proof see REICH [13].

DEFINITION 4.3. *Let B be a nonempty subset of E. We say that the metric projection $P: E \multimap B$ is convexly nonexpansive on a subset X of E, when $R(Z) \subset R(P(Z))$ for every $Z \subset X$.*

THEOREM 4.4. *Let B a nonempty approximatively p-compact convex subset of E and $f: B \multimap E$ a continuous multifunction with convex compact values. Suppose that the metric projection $P: E \multimap B$ is convexly non expansive on $f(B)$. If either*

(a) *f is convexly condensing, B is bounded and complete, or*

(b) *$\overline{f(B)}$ is c.t.b.,*

then there exists an $x \in B$ such that $d_p(x, f(x)) = d_p(f(x), B)$.

PROOF. (a) Define a mapping $g: B \multimap B$ by $g(x) = \cup\{P(y), y \in f(x)$ and $d_p(f(x), B) = d_p(y, B)\}$. Note that since $f(x)$ is compact, $g(x) \neq \emptyset$. Further since $f(x)$ is convex, it follows that $g(x)$ is also convex. In fact if u and v are in $g(x)$, then there exist $y_1, y_2 \in f(x)$ such that $u \in Py_1$ and $v \in Py_2$ and $p(y_1 - u) = d_p(y_1, B) = d_p(f(x), B) = d_p(y_2, B) = p(y_2 - v)$. For $t \in [0,1]$ we have $w(t) = ty_1 + (1-t)y_2 \in f(x) \cap B$ and $d_p(w(t), B) \leq d_p(w(t), tu + (1-t)v) \leq td_p(y_1, u) + (1-t)d_p(y_2, v) = d_p(f(x), B) \leq d_p(w(t), B)$. Let Z be a subset of B such that $\varphi(R(g(Z))) \subset R(Z)$. Since $g(Z) \subset Pf(Z)$ we have $\varphi(R(g(Z))) \supset \varphi(R(Pf(Z))) \supset \varphi(Rf(Z))$. The last inclusion follows by the fact that P is convexly nonexpansive on $f(B)$. Consequently $\varphi(Rf(Z)) \subset R(Z)$. If property (b) of Definition 3.1 holds for f, we are done. If 3.1 (a) holds for f we have that $f(Z)$ is c.t.b. and since $R(g(Z)) \supset R(Pf(Z)) \supset R(f(Z))$, $g(Z)$ is c.t.b. We show that g has closed graph. Let $\{x_\alpha\}$ be a net converging to x_0 and $\{z_\alpha\}$ be a net converging to z_0 such that $z_0 \in g(x_\alpha)$. By the definition of g, there exist $y_\alpha \in f(x_\alpha)$ such that $z_\alpha \in P(y_\alpha)$ and $d_p(f(x_\alpha), B) = d_p(y_\alpha, B)$. Since f is u.s.c. and compact valued we can suppose $y_\alpha \to y_0 \in f(x_0)$. Since P is

u.s.c. and compact valued we can suppose $z_\alpha \to z_0 \in P(y_0)$. Further d_p and f are continuous and so $d_p(y_\alpha, B) = d_p(f(x_\alpha), B) \to d_p(f(x_0), B) = d_p(y_0, B)$. Consequently $z_0 \in g(x_0)$. At last, since $\overline{co}g(B)$ is bounded and complete, we can apply to g Theorem 3.5, to obtain a fixed point $x \in g(x)$, i.e. there exists an $x \in B$ such that $d_p(x, f(x)) = d_p(f(x), B)$.
(b) In this case the multifunction g is convexly condensing, because $g(B) \subset Pf(B)$. So we can apply again Theorem 3.5 to g to obtain the result.

REMARK 4.5. If E is locally convex, B is composed of convex sets, B is approximatively p-compact then the metric projection P is convexly nonexpansive on every X relatively compact. So we obtain Theorem 1.5 as a corollary of Theorem 4.4.

REFERENCES

[1] R.R. AKHMEROV - M.I. KAMENSKII - A.S. POTAPOV - B.N. SADOVSKII: *Condensing operators*, J. Sov. Math. 18 (1980), 185-250.

[2] A. AVALLONE: *Il duale topologico di L_0*, preprint (1989).

[3] J. BANAŚ - K. GOEBEL: *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Appl. Math., Marcel Dekker, 60 (1980), New York and Basel.

[4] G. DARBO: *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padova 24 (1955), 84-92.

[5] E. DE PASCALE - R. GUZZARDI: *On boundary conditions and fixed point for α-nonexpansive multivalued mappings*, Rend. Acad. Naz. dei Lincei LVIII (1975), 300-305.

[6] O. HADŽIĆ: *Some fixed point and almost fixed point theorems for multivalued mappings in topological vector spaces*, Nonlinear Anal. Theory Methods Appl. 5 (1981), 1009-1019.

[7] O. HADŽIĆ: *Fixed point theorems in not necessarily locally convex topological vector spaces*, Functional Analysis Proc. Dubrovnik 1981, Lecture Notes in Math. 948 (1982), Springer-Verlag, Berlin, Heidelberg, New York, 118-130.

[8] O. HADŽIĆ: *Some properties of measures of noncompactness in paranormed spaces*, Proc. Amer. Math. Soc. 102 (1988), 843-849.

[9] C.J. HIMMELBERG - J.R. PORTER - F.S. VAN VLECK: *Fixed point theorems for condensing multifunctions*, Proc. Am. Math. Soc. 23 (1969), 635-641.

[10] C.J. HIMMELBERG: *Fixed point theorems for condensing multifunctions*, J. Math. Anal. Appl. 38 (1972), 205-207.

[11] A. IDZIK: *On γ-almost fixed point theorems. The single-valued case*, Bull. Polish Acad. Sci. Math. 35 (1987), 461-464.

[12] A. IDZIK: *Almost fixed point theorems*, Proc. Amer. Math. Soc. 104 (1988), 779-784.

[13] S. REICH: *Approximate selections, best approximations, fixed points and invariant sets*, J. Math. Anal. Appl. 62 (1978), 104-113.

[14] B.N. SADOVSKII: *A fixed point principle*, Funkts. Anal. Prilozh 1 (1967), 74-76.

[15] B.N. SADOVSKII: *Limit-Compact and condensing operators*, Russ. Math. Surv. 27 (1972), 85-154.

[16] V.M. SEHGAL - S.P. SINGH: *A generalization to multifunctions of Fan's best approximation theorem*, Proc. Amer. Math. Soc. 102 (1988), 534-537.

[17] K. ZIMA: *On the Schauder's fixed point theorem with respect to paranormed space*, Comm. Math., Prace Matematyczne 19 (1977), 421-423.

Lavoro pervenuto alla redazione il 30 luglio 1990
ed accettato per la pubblicazione il 3 settembre 1990
su parere favorevole di A. Vignoli e di A. Figà Talamanca

INDIRIZZO DEGLI AUTORI:

E. De Pascale - G. Trombetta - Dipartimento di Matematica - Università della Calabria - 87036 Arcavacata di Rende (Cosenza) - Italy