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On the Analytic Solution of the 3-Point Weber Problem

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RIASSUNTO – *Viene calcolata la soluzione analitica del problema di Weber per 3 punti in $\mathbb{R}^2$. Tale soluzione è espressa per mezzo dei pesi e delle coordinate cartesiane dei 3 punti assegnati.*

ABSTRACT – *The analytic solution for the 3-point Weber problem in $\mathbb{R}^2$ is expressed in terms of the weights and of the cartesian coordinates of the given points.*

KEY WORDS – *Weber problem - Location problems.*

A.M.S. CLASSIFICATION: 49B21 - 90C30

1 – Introduction

The 3-points Weber problem is a special case of the well known Weber problem for n points (e.g. see [1], [2]), and can be stated as follows.
Three points $A_k(x_k, y_k)$ on the Euclidean plane and three positive constants $w_k (k = 1, 2, 3)$ are given (see Fig.1).
It is required to determine a new point $X = (x, y)$ such that it minimizes the cost function

$$F(x,y) = \sum_{k=1}^{3} w_k d_k \tag{1}$$

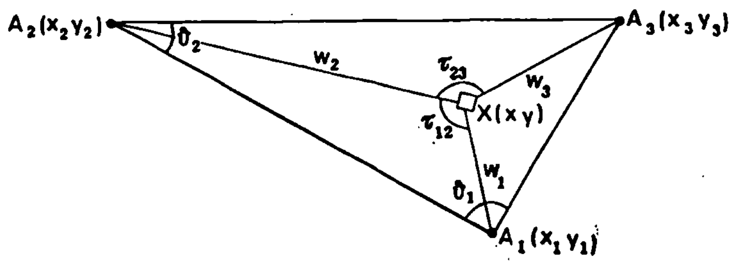

Fig. 1

where $d_k = \|X - A_k\|$ is the euclidean distance between X and A_k. No simple analytic solution for this problem seems to have existed until 1983, when the author of this paper gave, in an unpublished report [3], the solution in terms of cartesian coordinates. Successively the results (without proof and together with other topics) were presented in two scientific italian meetings of 1984 and 1985 (see [4], [5]).

In a recent paper E.E. JONES presented again the analytic solution of the 3-point Weber problem [6]. Nevertheless, his solution is not correct and his proof is not complete when, for some ciclic interchange of the subscripts 1, 2, 3 the inequality $w_i^2 > w_j^2 + w_k^2$ holds.

In this paper the correct solution and its graphic construction (as they were given by the author in the report of 1983) are presented.

2 – The analitical solution of the problem

In the following we will suppose that the points A_1, A_2, A_3 are non collinear, since the solution of the problem in this particular case is trivial. Moreover, we will indicate by Δ the non-degenerate triangle with vertices A_1, A_2, A_3 disposed in counterclockwise order (see Fig. 1).

It was proved by KUHN [1] that the 3-point Weber problem has an unique solution, which is located in A_k if and only if the inequality

$$w_k^2 \geq w_i^2 + w_j^2 + 2w_i w_j \cos\vartheta_k \tag{2}$$

holds, where ϑ_k is the vertex angle of Δ at A_k (see Fig. 1). It follows from (2) that a simple sufficient condition for the vertex A_k solves the problem is $w_k \geq w_i + w_j$.

If the inequality (2) does not hold for any ciclic interchange of the subscripts 1, 2, 3, then the minimum point $X^*(x^*, y^*)$ lies inside Δ, and hence $F(x,y)$ is differentiable in X^*. By equating to zero the gradient of F, we obtain

$$(3) \qquad \begin{cases} \dfrac{\partial F}{\partial x} = \dfrac{w_1(x-x_1)}{d_1} + \dfrac{w_2(x-x_2)}{d_2} + \dfrac{w_3(x-x_3)}{d_3} = 0 \\ \dfrac{\partial F}{\partial y} = \dfrac{w_1(y-y_1)}{d_1} + \dfrac{w_2(y-y_2)}{d_2} + \dfrac{w_3(y-y_3)}{d_3} = 0 \end{cases}$$

A new equation, consequence of system (3), can be found by moving (in both equations (3)) any of the three terms to the right hand side, by squaring the left and right hand sides, and summing the two new equations in this way obtained. If, as an example, the third terms are moved to the right, we obtain:

$$w_1^2 + w_2^2 + 2w_1w_2 \frac{(x-x_1)(x-x_2) + (y-y_1)(y-y_2)}{d_1d_2} = w_3^2$$

which, after setting

$$S_{12}(x,y) = (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = \langle X - A_1, X - A_2 \rangle$$

$$(4) \qquad b_3 = \frac{w_3^2 - w_1^2 - w_2^2}{2w_1w_2} = \frac{S_{12}(x,y)}{d_1d_2} = \cos\tau_{12}$$

can be written as

$$(5) \qquad S_{12}(x,y) = b_3 d_1 d_2 .$$

It follows from (4) that τ_{12} is the angle $A_1\widehat{X}A_2$ of figure 1. By squaring (5), we obtain:

$$S_{12}^2(x,y) = b_3^2\left[(x-x_1)^2 + (y-y_1)^2\right]\left[(x-x_2)^2 + (y-y_2)^2\right]$$

$$(6) \quad S_{12}^2(x,y) = b_3^2\left\{S_{12}^2(x,y) + \left[(x-x_1)(y-y_2) - (x-x_2)(y-y_1)\right]^2\right\} .$$

After introducing the notations

$$m_3 = \frac{b_3}{\sqrt{1-b_3^2}} = \operatorname{cotg} \tau_{12}$$

$$T_{12}(x,y) = (x-x_1)(y-y_2) - (x-x_2)(y-y_1)$$

the equality (6) can be rewritten as

$$C_{12}(x,y) \cdot C_{12}^*(x,y) = 0$$

with $C_{12}(x,y)$ and $C_{12}^*(x,y)$ defined by

$$C_{12}(x,y) = S_{12}(x,y) + m_3 \cdot T_{12}(x,y)\,,$$

$$C_{12}^*(x,y) = S_{12}(x,y) - m_3 \cdot T_{12}(x,y)\,.$$

It can be easily verified that $C_{12}(x,y) = 0$ and $C_{12}^*(x,y) = 0$ are the two circles through A_1, A_2 with radius

$$r_{12} = \frac{\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}}{2\sqrt{1-b_3^2}}$$

and with centres located on the axis of A_1A_2 (see Fig. 2).

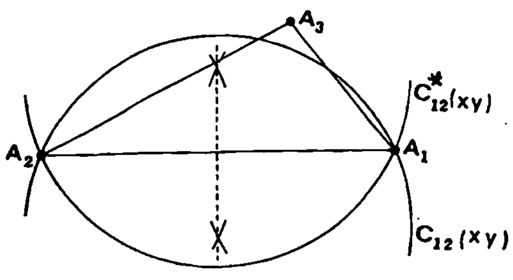

Fig. 2 Case $b_3 < 0$.

More precisely, if $b_3 < 0$ then the centre of $C_{12} = 0$ lies on the other side of the segment A_1A_2 with respect to A_3, and the centre of $C_{12}^* = 0$ on the same side. If, on the contrary, $b_3 > 0$, then the centre of $C_{12} = 0$ lies on the same side of A_3, and the centre of $C_{12}^* = 0$ on the other side. If $b_3 = 0$, then $C_{12} = 0$ and $C_{12}^* = 0$ coincide. Moreover, it must be observed that only the equation

$$S_{12}(x,y) = -m_3 \cdot T_{12}(x,y) \tag{7}$$

is compatible with (5), while the equation

$$S_{12}(x,y) = m_3 \cdot T_{12}(x,y) \tag{8}$$

is extraneous to the problem and arises from the squaring.

This property can be easily verified by studing the signs of the left and right sides of (5), (7), (8) in a system of Cartesian coordinates whith origin in the mid-point of the segment A_1A_2.

As already pointed out before, by reordering system (3) in the three possible ways, we can write three different equations of type (5). As a consequence, the minimum point of the considered problem can be found by solving the system $C_{12}(x,y) = 0$, $C_{23}(x,y) = 0$, $C_{31}(x,y) = 0$, which can be also written as

$$C_{ij}(x,y) = x^2 + y^2 - 2\alpha_k x - 2\beta_k y + \gamma_k = 0 \tag{9}$$

$$(ijk = 123, 231, 312)$$

where:

$$\begin{cases} \alpha_k = [(x_i + x_j) - (y_i - y_j)m_k]/2 \\ \beta_k = [(y_i + y_j) - (x_i - x_j)m_k]/2 \\ \gamma_k = x_i x_j + y_i y_j + m_k(x_i y_j - y_i x_j) \\ m_k = \dfrac{b_k}{\sqrt{1 - b_k^2}}, \quad b_k = \dfrac{w_k^2 - w_i^2 - w_j^2}{2w_i w_j}. \end{cases} \tag{10}$$

The cartesian coordinates of the minimum point $X^*(x^*, y^*)$ can be computed by selecting anyhow two pairs of circles (9) (as an example, $(C_{12}C_{23})$

and $(C_{12}C_{31})$), by finding the two corresponding radical axis, and lastly by intersecting such straight lines. The result is as follows:

$$
(11)\qquad \begin{cases} x^* = [(\beta_3 - \beta_1)(\gamma_3 - \gamma_2) - (\beta_3 - \beta_2)(\gamma_3 - \gamma_1)]\,/2D \\ y^* = [(\alpha_1 - \alpha_3)(\gamma_3 - \gamma_2) + (\alpha_3 - \alpha_2)(\gamma_3 - \gamma_1)]\,/2D \\ D = (\alpha_1 - \alpha_3)(\beta_3 - \beta_2) - (\alpha_2 - \alpha_3)(\beta_3 - \beta_1) \end{cases}
$$

with $\alpha_k, \beta_k, \gamma_k$ given by (10). The point $X^*(x^*, y^*)$ satisfies (3). We omit the computations required for deducing (11), since the algebra involved is complex and tedious.

What said above suggest a graphic solution of the 3-point Weber problem as illustrated in Fig. 3

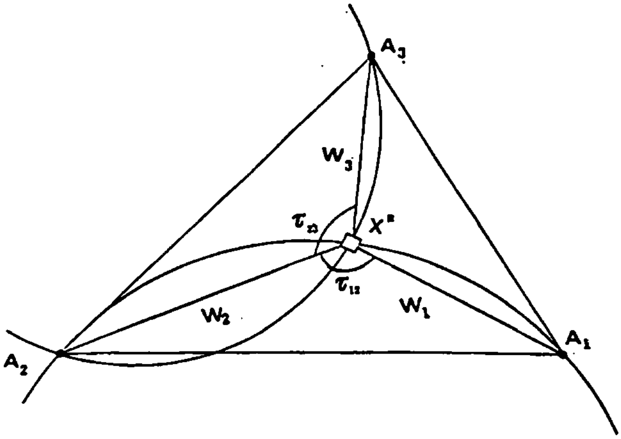

Fig. 3

Provided that for no vertex A_i the inequality (2) is satisfied, the minimum point $X^*(x^*, y^*)$ is located at the intersection of the three circles C_{12}, C_{23}, C_{31}, where C_{ij} is the circle through A_iA_j, whith radius

$$
r_{ij} = \frac{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}}{2\sqrt{1 - b_k^2}}
$$

and whose centre is located on the axis of A_iA_j, externally with respect to Δ if $b_k < 0$, and on the same side of A_k if $b_k > 0$. Moreover, the three numbers $\tau_{ij} = \arccos b_k$ are the angles between the rays projecting A_i, A_j from X^*.

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