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A Remark on G-Convergence for Nonsymmetric Operators

M. CHIPOT - G. VERGARA CAFFARELLI

RIASSUNTO – *Si estende agli operatori non simmetrici un risultato di* A. MARINO *e* S. SPAGNOLO *[1].*

ABSTRACT – *We extend to nonsymmetric operators a result of* A. MARINO *and* S. SPAGNOLO *[1].*

KEY WORDS – *G-Convergence - Nonsymmetric elliptic operator.*

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Let Ω be a bounded open set of $\mathbb{R}^n$, $(n \geq 1)$. For $0 < \lambda < \Lambda$ denote by $M(\lambda, \Lambda, \Omega)$ the set of operators

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) \tag{1}$$

where the $a_{ij}(x)$ are bounded measurable functions satisfying

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n \text{, a.e. } x \in \Omega \,. \tag{2}$$

We would like to prove here the following result:

THEOREM. *Let $A \in M(\lambda, \Lambda, \Omega)$. There exists a sequence $A_k \in M(\lambda', \Lambda', \Omega)$ of the type*

$$A_k = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\beta^k(x)\frac{\partial}{\partial x_i}\right), \quad \beta^k \in C^\infty(\overline{\Omega})$$

and a sequence (γ_j^k) of $C^\infty(\overline{\Omega})$-divergence free vector fields such that when $k \to +\infty$:

$$A_k + \sum_{j=1}^{n} \gamma_j^k \frac{\partial}{\partial x_j} \xrightarrow{G} A$$

in the sense of the G-convergence. (See [1] and the definition below).

REMARK 1. The above result is an extension to the nonsymmetric case of a result of MARINO and SPAGNOLO [1] (see also [2]). Since the set of symmetric operators (i.e. satisfying $a_{ij} = a_{ji} \quad \forall i, j = 1, \ldots, n$) of $M(\lambda, \Lambda, \Omega)$ is compact, so in particular closed for the G-topology (see [2]) there is no hope to be able to suppose that the vector field (γ_j^k) vanishes.

Before to give the proof of the theorem let us recall for the reader convenience the following:

DEFINITION. *Let B_k a sequence of second order elliptic operators. We will say that B_k G-convergence towards B (for the Dirichlet problem) if for every $f \in H^{-1}(\Omega)$ the solutions of the Dirichlet problems*

$$\begin{cases} B_k u_k = f \\ u_k \in H_0^1(\Omega) \end{cases} \qquad \begin{cases} Bu = f \\ u \in H_0^1(\Omega) \end{cases}$$

are such that

$$u_k \to u \quad in \quad L^2(\Omega)$$

when $k \to +\infty$ (see [1]). We then write $B_k \xrightarrow{G} B$.

PROOF OF THE THEOREM. By mollifying the a_{ij}'s it is easy to find a sequence $a^k_{ij} \in C^\infty(\overline{\Omega})$ satisfying (2) and such that

$$a^k_{ij} \to a_{ij} \quad \text{in} \quad L^p_{\mathrm{Loc}}(\Omega) \qquad \forall p \geq 1 .$$

Then one can show that:

$$B_k = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a^k_{ij}(x)\frac{\partial}{\partial x_j}\right) \xrightarrow{G} B = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) . \tag{3}$$

Set

$$s^k_{ij} = \frac{a^k_{ij} + a^k_{ji}}{2} , \; t^k_{ij} = \frac{a^k_{ij} - a_{ji}}{2}$$

the symmetric and skew part of a_{ij}. One has:

$$\begin{aligned} B_k &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(s^k_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(t^k_{ij}\frac{\partial}{\partial x_j}\right) = \\ &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(s^k_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(t^k_{ij})\frac{\partial}{\partial x_j} = \\ &= \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(s^k_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{j=1}^{n} \gamma^k_j \frac{\partial}{\partial x_j} \end{aligned} \tag{4}$$

where $\gamma^k_j = \sum_i^n \frac{\partial}{\partial x_i}(t^k_{ij})$ is a $C^\infty(\overline{\Omega})$-divergence free vector field (recall that the matrix (t^k_{ij}) is skew). Thanks to the result of [1], Theorem 3.1, there exists a sequence of operators $A_{k,p}$ in $M(\lambda', \Lambda', \Omega)$ such that

$$A_{k,p} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\beta^k_p(x)\frac{\partial}{\partial x_i}\right) \xrightarrow{G} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(s^k_{i,j}(x)\frac{\partial}{\partial x_j}\right) . \tag{5}$$

when $p \to +\infty$. The result will be proved if we can show that when $p \to +\infty$:

$$A_{k,p} + \sum_{j=1}^{n} \gamma^k_j \frac{\partial}{\partial x_j} \xrightarrow{G} B_k$$

(Indeed since the G-convergence can be defined by a metric (see [1]) the approximation of B reduces to the approximation of B_k (see (3)).

So let $u_{k,p}$ be the solution of

$$A_{k,p}u_{k,p} + \sum_{j=1}^{n} \gamma_j^k \frac{\partial}{\partial x_j} u_{k,p} = f\,, \quad u_{k,p} \in H_0^1(\Omega) \tag{6}$$

i.e.

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left((\beta_p^k \delta_{ij} + t_{ij}^k)\frac{\partial u_{k,p}}{\partial x_j}\right) = f\,, \quad u_{k,p} \in H_0^1(\Omega)\,.$$

Thanks to the ellipticity of $A_{k,p}$ we get easily:

$$|u_{k,p}|_{H_0^1(\Omega)} \leq C$$

where C does not depend on p. Now, the identity being a completely continuous map from $H_0^1(\Omega)$ into $L^2(\Omega)$ and from $L^2(\Omega)$ into $H^{-1}(\Omega)$ one can extract a sequence (still denoted by p) such that:

$$\begin{aligned}
&u_{k,p} \rightharpoonup u_{k,\infty} && \text{in } H_0^1(\Omega)\\
&u_{k,p} \longrightarrow u_{k,\infty} && \text{in } L^2(\Omega)\\
&\sum_{j=1}^{n} \gamma_j^k \frac{\partial}{\partial x_j} u_{k,p} \longrightarrow \sum_{j=1}^{n} \gamma_j^k \frac{\partial}{\partial x_j} u_{k,\infty} && \text{in } H^{-1}(\Omega)\quad .
\end{aligned} \tag{7}$$

Assume that we have proved the following lemma:

LEMMA. *Let A^ϵ be a sequence of operators in $M(\lambda', \Lambda', \Omega)$ and let u^ϵ, u be the solutions of*

$$A^\epsilon u^\epsilon = f^\epsilon\,,\ u^\epsilon \in H_0^1(\Omega)\,; \quad Au = f\,,\ u \in H_0^1(\Omega)\,.$$

If $A^\epsilon \xrightarrow{G} A$, $f^\epsilon \longrightarrow f$ in $H^{-1}(\Omega)$ then $u^\epsilon \longrightarrow u$ in $L^2(\Omega)$.

From the lemma we have clearly, (combine (5), (6), (7)), $u_{k,p} \longrightarrow u'_{k,\infty}$ in $L^2(\Omega)$ where $u'_{k,\infty}$ satisfies

$$A_k u'_{k,\infty} + \sum_{j=1}^{n} \gamma_j^k \frac{\partial}{\partial x_j} u_{k,\infty} = f\,, \quad u'_{k,\infty} \in H_0^1(\Omega)\,.$$

Of course, by (7), $u'_{k,\infty} = u_{k,\infty}$ and is equal to u_k the solution of

$$B_k u_k = f, \quad u_k \in H_0^1(\Omega).$$

By uniqueness of the accumulation points of $u_{k,p}$ we have $u_{k,p} \longrightarrow u_k$ in $L^2(\Omega)$ which completes the proof.

PROOF OF THE LEMMA. Introduce v^ϵ the solution of

$$A^\epsilon v^\epsilon = f, \quad v^\epsilon \in H_0^1(\Omega).$$

One has:

$$\langle A^\epsilon(u^\epsilon - v^\epsilon), u^\epsilon - v^\epsilon\rangle = \langle f^\epsilon - f, u^\epsilon - v^\epsilon\rangle$$

thus, by the uniform ellipticity:

$$|u^\epsilon - v^\epsilon|_{H_0^1(\Omega)} \leq C|f^\epsilon - f|_{H^{-1}(\Omega)} \longrightarrow 0$$

Now thanks to the G-convergence of A^ϵ, $v^\epsilon \longrightarrow u$ in $L^2(\Omega)$ and so does u^ϵ. The result follows.

REMARK 2. In the case where Ω is simply-connected it is clear - due to the Poincaré Theorem - that any operator of the type

$$A = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\left(\beta(x)\frac{\partial}{\partial x_i}\right) + \gamma_i(x)\frac{\partial}{\partial x_i}\right)$$

where (γ_i) is a $C^\infty(\overline{\Omega})$-divergence free vector fields, can be written as

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left((\beta(x)\delta_{ij} + t_{ij}(x))\frac{\partial}{\partial x_j}\right)$$

where (t_{ij}) is a skew matrix. Thus, in this case, these operators remain in $M(\lambda', \Lambda', \Omega)$.

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INDIRIZZO ATTUALE DEGLI AUTORI:

M. Chipot - Université de Metz - Département de Mathématiques - Ile du Saulcy, 57045 Metz-Cedex - France

G. Vergara-Caffarelli - Università di Roma "La Sapienza" - Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate - Via A. Scarpa, 10 - 00161 Roma - Italia