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Central Critical Points in the Spectrum of the Fourier-Stieltjes Algebra of a Locally Compact Group

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RIASSUNTO – *In questo articolo si considera l'algebra $B(G)$ di Fourier-Stieltjes su un gruppo localmente compatto G, e il suo spazio degli ideali massimali, o spettro, ΔB. Si studiano certi idempotenti centrali nello spettro ΔB, detti punti critici centrali. Si dimostra un teorema di decomposizione di $B(G)$ per un'ampia classe di gruppi localmente compatti. Tale decomposizione è associata in modo naturale ai punti critici centrali. Per i gruppi di tale classe che soddisfano un'ulteriore ipotesi tecnica, si ottiene una descrizione di ΔB che generalizza i casi dei gruppi euclidei e dei gruppi di Lie, semisemplici, connessi, e con centro finito.*

ABSTRACT – *In this paper we consider the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G and its spectrum ΔB. We study certain central idempotents in ΔB, called central critical points. We prove a decomposition theorem of $B(G)$ for a fairly large class of locally compact groups. This decomposition is naturally associated with the central critical points. For the groups in such class that satisfy an additional technical hypothesis, we also obtain a description of ΔB that generalizes the case of the Euclidean motion groups and the connected, semisimple Lie group with finite centre.*

KEY WORDS – *Locally compact groups - Fourier and Fourier-Stieltjes algebras - Maximal ideal space or spectrum.*

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– Introduction

In the well-known work by Taylor about commutative convolution measure algebras it appears clear that many important properties of such algebras can be expressed in terms of the critical points. Probably the most typical example of commutative measure algebra is the algebra $M(G)$ of bounded regular Borel measures on a locally compact abelian group. In this case $M(G)$ is isomorphic to the Fourier-Stieltjes algebra of the dual group $\widehat{G}$: $B(\widehat{G})$.

Among other things Taylor characterized the critical points as the idempotents in the spectrum of $B(\widehat{G})$ that give a decomposition of $B(\widehat{G})$ as direct sum of an algebra and an ideal for which the algebra is (canonically) isomorphic the Fourier-Stieltjes algebra for another locally compact group.

In the non-commutative case a study of the subject was began by WALTER in [10]. He introduced the notion of critical point and central critical point in the spectrum of the Fourier-Stieltjes algebra $B(G)$ of a locally compact group G. In particular Walter proved that the almost periodic decomposition of $B(G)$,

$$B(G) = AP(G) \cap B(G) \oplus I\,,$$

is given by a central critical point ζ_F in the sense that

$$AP(G) \cap B(G) = \zeta_F \cdot B(G)$$
$$I = (e - \zeta_F) \cdot B(G)\,.$$

Extending the property holding in the commutative case, Walter proved that for any locally group G

$$\zeta_F \cdot B(G) \cong B(\overline{G})$$

where $\overline{G}$ is the almost periodic compactification of G.

In this note we study a decomposition analogous to the one of Taylor for all central critical points for a fairly large class of locally compact groups; these includes particularly the connected semisimple Lie groups with finite centre and the groups of Euclidean motions. We prove a criterion to establish if a central idempotent of $\Delta B(G)$ is critical. Finally we prove that, in the case in which the critical points are only finitely many and under another technical hypothesis holding in the case cited above, it is possible to give a decomposition of $B(G)$ corresponding to an explicit description of the maximal ideal space of $B(G)$.

1 – Preliminaries

For a locally compact group G, let $B(G)$ be the Fourier-Stieltjes algebra as defined in [5]. $B(G)$ consists of the complex linear combinations of continuous positive definite functions. $B(G)$ has an interesting ideal, $A(G)$. This consists of the functions that can be written as a convolution $f * g$ of two functions f and g in $L^2(G)$. It is well known that if G is commutative then $B(G)$ is the algebra of the Fourier-Stieltjes transforms of bounded regular Borel measures of the dual group $\widehat{G}$ and $A(G)$ is the algebra of the Fourier transforms of the L^1-functions. We refer to [5] for further properties of $B(G)$ and $A(G)$ needed here.

$B(G)$ is a commutative semisimple Banach algebra. The dual space of $B(G)$, $W^*(G)$, is a W^*-algebra. Indeed $B(G)$ is the dual of the C^*-algebra of the group G so that $W^*(G)$ is the bidual of a C^*-algebra, see [2]. $B(G)$ can be identified with the space of weakly continuous linear functionals on $W^*(G)$. $W^*(G)$ it can be realized as an algebra of bounded linear operators on a suitable Hilbert space. In this way it is endowed with a natural (binary) product. We will write xy to indicate the product of x and y in $W^*(G)$. This product defines actions of $W^*(G)$ on $B(G)$

$$u \longmapsto x \cdot u \qquad (u \cdot x)$$

$$B(G) \longrightarrow B(G)$$

by the equalities

$$\langle y, x \cdot u \rangle = \langle yx, u \rangle \qquad (\langle y, u \cdot x \rangle = \langle xy, u \rangle).$$

An element x of $W^*(G)$ is *hermitian (positive)* if it is hermitian (positive) in the canonical representation of $W^*(G)$ as an algebra of operators on a Hilbert space. If x and y are elements of $W^*(G)$ we write $x \geq 0$ if x is positive or 0 and $x \geq y$ if $x - y \geq 0$. We also write $x > 0$ if x is positive and not identically zero. A positive element x of $W^*(G)$ is said to be an *idempotent* o a *projection* if $x^2 = x$. Each $u \in B(G)$ has a unique *polar* *decomposition*

$$u = U \cdot |u|$$

where $|u|$ is positive definite, $\|u\| = \|\, |u| \,\|$ and $U \in W^*(G)$ is such that UU^*, U^*U are projections and $|u| = U^* \cdot u$. The following inequalities hold for $u \in B(G)$, $x \in W^*(G)$:

$$|\langle x, u\rangle|^2 \leq \|u\| \langle xx^*, |u|\rangle$$

$$|\langle x, |u|\rangle| \leq \langle (xx^*)^{1/2}, |u|\rangle .$$

On $W^*(G)$ we will consider two topologies. The *weak topology* is by definition the weal-$*$ topology of duality. Hence a net $\{x_\alpha\}$ of elements in $W^*(G)$ converges weakly to $x \in W^*(G)$ if and only if for all $f \in B(G)$

$$\langle x_\alpha, f\rangle \longrightarrow \langle x, f\rangle .$$

The *strong topology* can be defined as follows. We say that the net $\{x_\alpha\}$ in $W^*(G)$ converges strongly to $x \in W^*(G)$ if

$$\|x_\alpha \cdot f - x \cdot f\| \longrightarrow 0$$

for every $f \in B(G)$; see [10] and references therein.

A subset A of $B(G)$ is said to be *left invariant* if

$$x \cdot A \subseteq A \qquad \text{for all} \quad x \in W^*(G)$$

and *right invariant* if

$$A \cdot x \subseteq A \qquad \text{for all} \quad x \in W^*(G) .$$

A subset is said to be *invariant* if it both is left and right invariant. The group G can identified with the subset of $W^*(G)$ of multiplicative functionals on $B(G)$

$$u \longrightarrow u(g) \quad u \in B(G) , \quad g \in G .$$

The topology of G coincides with the topology induced by the weak topology on $W^*(G)$. The set $\left\{\sum_{i=1}^{n} c_i g_i : c_i \in \mathbf{C}, g_i \in G\right\}$ is weakly dense in $W^*(G)$, [4]. Then a subspace A of $B(G)$ is invariant if and only if

$$g \cdot A, A \cdot g \subseteq A \qquad \text{for all} \quad g \in G.$$

The *maximal ideal space, or spectrum*, $\Delta B(G)$ of $B(G)$ is the subset of $W^*(G)$ of the multiplicative linear functionals not identically zero. ΔB is weakly compact and is a semigroup with the product inheredited by $W^*(G)$. The fact that the product of two elements of ΔB is not identically zero follows from [10, Theorem 1]. Denote by Δ^+ the set positive elements of ΔB. In [10, Theorem 2] the existence of a minimal element ζ_F of Δ^+ is proved. It corresponds to the *almost periodic decomposition* of $B(G)$:

$$B(G) = AP(G) \cap B(G) \oplus I.$$

where $AP(G)$ are the almost periodic functions on G, I is an ideal, and

$$AP(G) \cap B(G) = \zeta_F \cdot B(G)$$

and

$$I = (e - \zeta_F) \cdot B(G).$$

ζ_F is a central idempotent, hence gives rise to an invariant decomposition.

To each idempotent ρ of $\Delta B(G)$ is associated a group G_ρ defined by

$$G_\rho = \{\delta \in \Delta B : \delta\delta^* = \delta^*\delta = \rho\}.$$

Notice that if $\rho = e$, the identity of $W^*(G)$ and G, $G_e = G$, [9, Theorem 1].

Now we introduce some idempotents in the spectrum of $B(G)$ which play a key role in our paper.

DEFINITION. *Let ρ be an idempotent in ΔB. If ρ is weakly isolated in $\rho\Delta^+\rho$ we say that ρ is a critical point in ΔB. Moreover if ρ is a central element we say that ρ is a central point, hereinafter written CCP.*

The following Proposition ([10, Proposition 7]), gives different characterizations of the critical points.

PROPOSITION 1.1. *Let $\rho \in \Delta B$ be an idempotent. The following are equivalent.*

(i) There is no net $\{\phi_\alpha\} \subseteq \Delta^+$ such that $\phi_\alpha < \rho$ and $\phi_\alpha \longrightarrow \rho$ in the weak topology;

(ii) The group $G_\rho \subsetneq \Delta B$ is strongly open in $\rho\Delta B\rho$;

(iii) ρ is the minimal element of a strongly open and closed subset of Δ^+;

(iv) ρ is critical.

REMARKS AND EXAMPLES. We begin by noticing that the identity element e of $W^*(G)$ is a CCP in ΔB. Indeed, let v be any positive definite function in $A(G)$ of norm 1. Let

$$\Gamma(v;1/2;e) \equiv \{\delta \in \Delta B : |\langle \delta, v\rangle - \langle e, v\rangle| < 1/2\}.$$

Consider $\Gamma \cap \Delta^+$. Since $\langle e, v\rangle = v(e) = \|v\| = 1$, we know that $\Gamma \cap \Delta^+ = \{e\}$. Therefore $\{e\}$ is weakly isolated in $\Delta^+(= e\Delta^+)$. Thus e is a CCP.

By condition (ii) of Proposition 1.1 it follows that if ρ is a critical point in ΔB then G_ρ is a locally compact group in the relative weak topoology.

The next definition is fundamental.

DEFINITION. *If A is a bi-invariant subalgebra and I is an ideal and $B(G) = A \oplus I$ we say that $B(G) = A \oplus I$ is a prime decomposition of $B(G)$.*

We also recall, see [10,Proposition 1], that $B(G) = A \oplus I$ is a prime decomposition if and only if there exists a central idempotent ρ in ΔB such that

$$\rho \cdot B(G) = A, \quad (e - \rho) \cdot B(G) = I.$$

Moreover if ρ is a central idempotent in ΔB then the mapping

$$u \longmapsto \rho \cdot u$$

$$B(G) \longmapsto B(G)$$

is an algebra homomorphism.

The aim of this is to study the CCP's in ΔB and the characterization of the prime decomposition defined by them in which the algebra A is (isomorphic to) the Fourier-Stieltjes algebra of another locally compact group, namely G_ρ.

In particular we extend the following theorem due to Taylor, cf. [8], to a non-trivial class of non-commutative locally compact groups. First a definition:

DEFINITION. *Let G, G_1 be locally compact groups. A linear operator*

$$T: B(G_1) \longrightarrow B(G)$$

is said to be order preserving if for each positive definite function $f \in B(G_1)$ it holds that Tf is positive definite.

THEOREM 2.1. *Let G be a locally compact abelian group. Let ρ be an idempotent in ΔB. Consider the prime decomposition of $B(G)$,*

$$B(G) = \rho \cdot B(G) \oplus (e - \rho) \cdot B(G)$$

and the group $G_\rho \subseteq \Delta B$ having ρ as identity. Then there exists a locally compact group G_1 and an order preserving isomorphism onto

$$T: B(G_1) \longrightarrow \rho \cdot B(G)$$

if and only if ρ is critical. In this case $G_1 \simeq G_\rho$, where $\simeq$ is a topological group isomorphism.

REMARKS. In the abelian setting, Taylor worked directly on the algebra of regular counded Borel measures, $M(\widehat{G})$, where $\widehat{G}$ is the dual group of G. His characterization of the critical points was in terms of the locally compact refinements of the topology of $\widehat{G}$. It is known, [7], that the only possible locally compact refinement of the topology of $\mathbb{R}$ is the discrete topology. Therefore the only critical points are the identity and the one given by the almost periodic decomposition

$$B(\mathbb{I}) = AP(\mathbb{R}) \oplus I .$$

The case of the torus T is trivial. Indeed $\Delta B(T)$ reduces to T itself and the only positive element in $\Delta B(T)$ is the identity (which coincides with the projection associated to the almost periodic decomposition).

It is also known that there are uncountably many (locally compact) refinements of the topology of $\mathbb{R}^n$, for $n > 1$; therefore there are uncountably many critical points in the spectrum of $B(\mathbb{R}^n)$, for $n > 1$, [7].

In the non-abelian case a description of the maximal ideal space and a direct sum decomposition of the Fourier-Stieltjes algebra are known for the connected semisimple Lie groups with finite center, [1], and the Euclidean motion groups, [6] and [3]. Namely we have that

$$B(G) = \bigoplus_{j=1}^{N} A_j$$

where the A_j are subalgebras isomorphic to Rad $A(G/H_j)$ for some normal subgroup H_j of G. In particular there exist prime decompositions

$$B(G) = B_j \oplus I_j$$

where $B_j = B(G/H_j)$ (we use here the identification given by [5] (2.26)). As a consequence

$$\Delta B = \bigcup_{j=1}^{N} G/H_j$$

in the sense that if $\delta \in \Delta B$ there exists a central projection $\zeta_j \in \Delta B$ for which

$$\zeta_j \cdot B(G) = \{f \in B(G) : f \quad \text{is constant on the left cosets of} \quad H_j\}$$

and there exists $g \in G$ such that for every $u \in B(G)$

$$\langle \delta, u \rangle = \zeta_j \cdot u(gH) \, .$$

2 – The principal results

DEFINITION. *Let G be a locally compact group G. We say that G satisfies the condition (B) if the followings hold:*
(i) all critical points are central;
(ii) if ζ is a CCP and θ the group homomorphism defined by

$$\theta: G \longrightarrow G_\rho$$
$$g \longmapsto \zeta g$$

then $\theta(G)$ is dense in G_ρ.

REMARKS. The two classes of non-commutative groups discusses in the last section both satisfy the condition (B). Condition (i) in (B) is somewhat technical. Condition (ii), although essential in this paper, is more natural. Of course it holds in all abelian groups by inspection and it holds in the case of maximal and minimal elements in Δ^+ (i.e. e and ζ_F, the critical point given by the almost periodic prime decomposition of $B(G)$). Walter conjectured ,[10], that (ii) holds for all locally compact groups; but a lot of work seems to be needed to establish this more general result.

To our knowledge, no example of a group not satisfying the condition (B) is known.

In this paper we will always assume that G satisfies the condition (B).

The following is a generalization of (1.2).

PROPOSITION 2.1. *Let G satisfy the condition (B). Let ρ be a central idempotent in ΔB. Consider the prime decomposition of $B(G)$*

$$B(G) = \rho \cdot B(G) \oplus (e - \rho) \cdot B(G).$$

Then there exists a locally compact group G_1 and an order preserving isometric isomorphism onto

$$T: B(G_1) \longrightarrow \rho \cdot B(G)$$

if and only if ρ is critical. In this case $G_1 \simeq G_\rho$; here $\simeq$ is a topological group isomorphism.

Proof. Suppose first that ρ is critical. Since G satisfies the condition (B) we have a continuous homomorphism

$$\theta : G \longrightarrow G_\rho$$
$$g \longmapsto \rho g$$

with dense image. By [5] (2.20) we have that

$${}^t\theta : B(G_\rho) \longrightarrow B(G)$$
$$f \longmapsto f \circ \theta$$

is an isometric isomorphism onto a bi-invariant cloded subalgebra E of $B(G)$. We claim that $E = \rho \cdot B(G)$. Let

$${}^{tt}\theta : W^*(G) \longrightarrow W^*(G_\rho)$$

be defined by

$$\langle x, \cdot \rangle_G \longmapsto \langle x, \cdot \circ \theta \rangle_G .$$

We want to show that ${}^{tt}\theta$ is an algebra homomorphism. Notice that for $f \in B(G_\rho)$

$$\begin{aligned} \langle \rho g, f \rangle_{G_\rho} &= {}^t\theta(f)(g) = \\ &= \langle g, {}^t\theta(f) \rangle_G = \\ &= \langle {}^{tt}\theta(g), f \rangle_{G_\rho} \end{aligned}$$

so that ${}^{tt}\theta(g) = \rho g$. Therefore

$$\begin{aligned} \langle {}^{tt}\theta(g_1 g_2), f \rangle_{G_\rho} &= \langle \rho g_1 g_2, {}^t\theta(f) \rangle_G = \\ &= \langle \rho g_1 \rho g_2, {}^t\theta(f) \rangle_G = \\ &= f(\rho g_1 \rho g_2) = \\ &= \langle \rho g_1 \rho g_2, f \rangle_G = \\ &= \langle {}^{tt}\theta(g_1) {}^{tt}\theta(g_2), f \rangle_{G_\rho} . \end{aligned}$$

Thus ${}^t\vartheta$ preserves the product of elements of G and of their complex linear combinations. The weak toplogy on $W^*(G_\rho)$ is given by intersections of the open sets

$$U(y;f;\varepsilon) \equiv \{z \in W^*(G) | \langle y, f\rangle - \langle z, f\rangle| < \varepsilon\}$$

where $f \in B(G)$.

The complex linear combinations $\left\{\sum_{i=1}^{n} c_i g_{\rho i}; c_i \in \mathbb{C}, g_{\rho i} \in G_\rho\right\}$ are weakly dense in $W^*(G_\rho)$. But the weak topology on G_ρ is the restriction of the weak topology of $W^*(G)$ and the set $\{\zeta g \colon g \in G\}$ is weakly dense in G_ρ. Hence the sets

$$\left\{x \in W^*(G_\rho) \colon \left| \langle \sum_{i=1}^{n} c_i \rho g_i, {}^t\vartheta(f)\rangle - \langle \sum_{j=1}^{m} d_j \rho g_j, {}^t\vartheta(f)\rangle \right| < \varepsilon \right\}$$

where $c_i, d_j \in \mathbb{C}, g_i, g_j \in G; f \in B(G_\rho)$, form a neighbourhood sub-basis for the weak topology on $W^*(G_\rho)$. Therefore we can conclude that

$${}^{tt}\vartheta \colon W^*(G) \longrightarrow W^*(G_\rho)$$

is an algebra homomorphism.

Next we want to show that ${}^{tt}\vartheta(\rho) = \rho$. We know that ${}^{tt}\vartheta$ coincides with the multiplication by ρ on the weakly dense set $\{\Sigma c_i g_i \colon c_i \in \mathbb{C}, g_i \in G\}$. By density ${}^{tt}\vartheta$ is the multiplication by ρ on all $W^*(G)$. Since ${}^{tt}\vartheta$ is an algebra homomorphism

$$\rho = \rho^2 = \rho\rho = {}^{tt}\vartheta(\rho).$$

Finally, for $g \in G$,

$$\begin{aligned}\langle g, \rho \cdot {}^t\vartheta(f)\rangle_G &= \langle \rho g, {}^t\vartheta(f)\rangle_G = \\ &= \langle {}^{tt}\vartheta(\rho g), f\rangle_{G_\rho} = \\ &= \langle {}^{tt}\vartheta(\rho)\, {}^{tt}\vartheta(g), f\rangle_{G_\rho} = \\ &= \langle \rho(\rho g), f\rangle_{G_\rho} = \\ &= \langle \rho g, f\rangle_{G_\rho} = \\ &= \langle g, {}^t\vartheta(f)\rangle_G .\end{aligned}$$

Hence $E = \vartheta[B(G_\rho)] \subseteq \rho \cdot B(G)$.

The inverse inclusion is simpler. Let $u = \rho \cdot u \in B(G)$ be positive definite. Define

$$f(\rho g) \equiv u(g) = \langle g, u\rangle_G = \langle g, \rho \cdot u\rangle_G\,.$$

This function is continuous in the topology of G_ρ on a dense subset and it is positive definite. Then it extends uniquely to a function in $B(G_\rho)$ such that $\vartheta(f) = u$. Then

$$\vartheta[B(G_\rho)] \subseteq E\,.$$

The Claim and one implication are established.

Conversely, suppose

$$T\colon B(G_1) \longrightarrow B(G)$$

is an isometric isomorphism preserving the positive definite functions and such that

$$B(G) = T[B(G_1)] \oplus I$$

is a prime decomposition. Let $\rho \in \Delta^+$ be such that

$$T[B(G_1)] = \rho \cdot B(G)$$

and

$$I = (e - \rho) \cdot B(G)\,.$$

Let $v \in A(G_1)$ be positive definite with norm 1. Put

$$\Gamma(Tv; 1/2; \rho) = \{\delta \in \Delta B : |\langle \delta, Tv\rangle - \langle \rho, Tv\rangle| < 1/2\}$$

and consider $\Gamma \cap \rho\Delta^+$. Let $\phi \in \Gamma \cap \rho\Delta^+$. Suppose that

$$|\langle \phi, Tv\rangle - \langle \rho, Tv\rangle| < 1/2\,.$$

Since Tv is positive definite and of norm 1, $\langle \rho, Tv\rangle = 1$ (since $TB(G_1) = \rho \cdot B(G)$). Therefore

$$1/2 < \langle \phi, Tv\rangle \leq 1\,.$$

Also

$$T: W^*(G) \longrightarrow W^*(G_1)$$

is an isomorphism of $\rho W^*(G)$ onto $W^*(G_1)$ sending ϕ to e_1. Then $\phi = \rho$ and

$$\Gamma \cap \rho\Delta^+ = \{\rho\}.$$

Thus ρ is critical. The first part of the Proposition and the Duality Theorem in [11] finish the proof. □

Next we prove a criterion to establish when a central idempotent in ΔB is critical. First a definition:

DEFINITION. *Let ρ be a central idempotent in ΔB, G_ρ be the group defined by ρ. Put*

$$A_\rho = \Big\{u \in B(G) : u = \rho \cdot u \text{ and for } \delta \in \Delta B \text{ it holds that}$$

$$\langle \delta, u\rangle \neq 0 \text{ implies } \rho\delta \in G_\rho\Big\}.$$

Notice that for $\rho = e$, $A_e = \mathrm{Rad}A(G)$.

LEMMA 2.2. *Let ρ be a central idempotent in ΔB. Then A_ρ is a closed bi-invariant subalgebra of $B(G)$ and an ideal in $\rho \cdot B(G)$. Moreover if $v \in A_\rho$ and $v = V \cdot |v|$ is the polar decomposition of v, then $|v| \in A_\rho$.*

PROOF. Notice that $\langle \delta, v_1 + v_2\rangle \neq 0$ implies $\langle \delta, v_i\rangle \neq 0$ for one i at least. In this case $\rho\delta \in G_\rho$. Thus A_ρ is a linear subspace. It is also clear that it is closed. Let $v \in A_\rho$, $u \in \rho \cdot B(G)$ and $\delta \in \Delta B$. Then

$$0 \neq \langle \delta, uv\rangle = \langle \delta, u\rangle\langle \delta, v\rangle$$

implies

$$0 \neq \langle \delta, v\rangle$$

so that $\delta \in G_\rho$. Thus A_ρ is an ideal in $\rho \cdot B(G)$. This also shows that A_ρ is a subalgebra.

In order to show that A_ρ is bi-invariant is suffices to show that

$$g \cdot A_\rho, A_\rho \cdot g \subseteq A_\rho \qquad \text{for all } g \in G.$$

Let $\delta \in \Delta B$, $g \in G$ (again identifying g with the linear functional $u \longmapsto u(g)$). Supposing that $\langle \delta, v \cdot g \rangle \neq 0$, we want to show that $\rho\delta \in G_\rho$. We have $\langle g\delta, v \rangle \neq 0$ so that $\rho(g\delta) \in G_\rho$. But

$$\rho(g\delta) = \rho^2(g\delta) = (\rho g)(\rho\delta).$$

Also $\rho g \in G_\rho$ so that $\rho\delta \in G_\rho$. Therefore $v \cdot g \in A_\rho$. In a similar way we obtain $g \cdot v \in A_\rho$.

Finally we want to show that if $v \in A_\rho$ and $v = V \cdot |v|$ is the polar decomposition of v, then $|v| \in A_\rho$. Since A_ρ is bi-invariant and $|v| = V^* \cdot v$ we have $|v| \in A_\rho$. □

PROPOSITION 2.3. *Let ρ be a central idempotent in ΔB. Then $A_\rho \neq \{0\}$ if and only if ρ is critical.*

PROOF. Suppose $A_\rho \neq \{0\}$. Let $v \in A_\rho \setminus \{0\}$. We can assume that v is positive definite and of norm 1. Consider the open set in ΔB

$$\Gamma \equiv (v; 1/2; \rho) = \{\delta \in \Delta B; |\langle \delta, v \rangle - \langle \rho, v \rangle| < 1/2\}\ .$$

Let $\phi \in \Gamma \cap \rho\Delta^+$. We have

$$|\langle \phi, v \rangle - \langle \rho, v \rangle| > 1/2.$$

Since $\phi \leq \rho$ and $\langle \rho, v \rangle = \|v\| = 1$ it follows that

$$\begin{aligned} 1/2 > |\langle \rho, v \rangle - \langle \phi, v \rangle| &= \\ = \langle \rho, v \rangle - \langle \phi, v \rangle &= \\ = 1 - \langle \phi, v \rangle. \end{aligned}$$

Hence

$$\langle \phi, v \rangle > 1/2.$$

Since $v \in A_\rho$ we have that $\phi = \rho\phi \in G_\rho$. But the only positive element in G_ρ is ρ itself. Then $\phi = \rho$. Therefore

$$\Gamma \cap \rho\Delta^+ = \{\rho\},$$

i.e. ρ is weakly isolated in $\rho\Delta^+$. Thus ρ is critical.

Conversely suppose ρ is critical. Then $\rho \cdot B(G) \simeq B(G_\rho)$ by Proposition 2.1. Let $v \in A(G_\rho)$ be positive definite of norm 1 and identify it with the corresponding element in $\rho \cdot B(G)$. Then $v \in A_\rho$ and $v \neq 0$. Thus $A_\rho \neq \{0\}$. □

Finally we prove a decomposition Theorem which generalizes the case of two non-commutative groups discussed earlier.

THEOREM 2.4. *Let G satisfy the condition (B) and suppose that there are only finitely many CCP's; called them $\left\{\zeta_i\right\}_{i=1}^{N}$. Furthermore suppose that the product of two CCP's is a CCP.*
Define

$$B_i = \{u \in B(G) \colon \zeta_j \cdot u = u \text{ but } \zeta \cdot u = 0 \text{ if } \zeta\zeta_j < \zeta_j\} .$$

Then

$$B(G) = \bigoplus_{i=1}^{N} B_i .$$

Moreover

$$\Delta B(G) = \bigcup_{i=1}^{N} \Delta B(G_{\zeta_i})$$

in the sense that if $\delta \in \Delta B(G)$ then there exists a smallest CCP ζ such that, for all $u \in B(G)$,

$$\langle \delta, u\rangle = \langle \delta, \zeta \cdot u\rangle ;$$

thus δ can be identified with an element of $\Delta B(G_\zeta)$.

PROOF. Let u, $v \in B_i$. Then $\zeta_i \cdot (u+v) = u+v$ and if $\zeta\zeta_i < \zeta_i$ then $\zeta \cdot (u+v) = 0$. Moreover

$$\zeta \cdot (uv) = (\zeta_i \cdot u)(\zeta_i \cdot v) = uv$$

and

$$\zeta \cdot (uv) = (\zeta \cdot u)(\zeta \cdot v) = 0 .$$

Then the B_i's are subalgebras. Let $u \in B_i \cap B_j$, $i \neq j$. We have either $\zeta_i\zeta_j < \zeta_i$ or $\zeta_i\zeta_j < \zeta_j$. Suppose $\zeta_i\zeta_j < \zeta_i$. Then $\zeta_j \cdot u = 0$ since $u \in B_i$; but

$u = \zeta_j \cdot u$ since $u \in B_j$. Thus $u = 0$ and the sum of the B_i's is a direct sum. Next we want to show that $\bigoplus_{i=1}^{N} B_i = B(G)$. This equality is proved easily for $N = 1, 2$. (notice that $N = 1$ if and only if G is compact, see [10]. In this case the identity and the almost periodic projection coincide). We proceed by induction on N. For $\zeta \neq e$ consider G_ζ and $B(G_\zeta)$. Suppose that ω is a CCP in $\Delta B(G_\zeta)$. Then it is easy to see that ω can identified with an element in $\Delta B(G)$ that is a projection and for which $A_\omega \neq \{0\}$. Also if ω_1 and ω_2 are CCP's in $\Delta B(G_\zeta)$ then their product is critical in $\Delta B(G)$ by hypothesis; it follows that this product is critical in $\Delta B(G_\zeta)$ too. Then for every CCP ζ in $\Delta B(G)$, G_ζ satisfies the hypotheses of the theorem and the number of CCP's in $\Delta B(G_\zeta)$ is less of the ones in $\Delta B(G)$. Thus the Theorem is true for each G_ζ, ζ a CCP in $\Delta B(G)$, by induction.

Next, let $\{\zeta_1, \ldots, \zeta_M\}$ be the CCP's satisfying

(i) $e > \zeta_j$

(ii) there exists no CCP ζ such that $e > \zeta > \zeta_j$.

Let $u \in B(G)$. We have

$$u = \prod_{i=1}^{M} (e - \zeta_j) \cdot u + v \equiv u_1 + v.$$

It is easy to see that

$$v \in \{\Sigma \zeta \cdot B(G) : \zeta \,\mathrm{CCP}, \zeta \neq e\}.$$

By the above argument we have $v \in \oplus B_i$. Finally we show that $u_1 \in B_e$. If $u_1 \equiv 0$ then there is nothing to prove. If $u_1 \neq 0$ we have that for any CCP $\zeta \neq e$,

$$\zeta \cdot u_1 = \zeta \cdot \prod_{i=1}^{M} (e - \zeta_i) \cdot u = \prod_{i=1}^{M} (\zeta - \zeta\zeta_i) \cdot u = 0$$

since $\zeta < \zeta_i$ for at least one i because of (ii). Then $u_1 \in B_e$ and $u \in \bigoplus_{i=1}^{N} B_i$.

Finally we consider the statement about the maximal ideal space. Let $\delta \in \Delta B$. Then there exists a minimal CCP ζ such that $\zeta\delta = \delta$ (since the product of two CCP's is a CCP). Let $u \in B(G)$, $u = \Sigma u_i$. Then

$$\langle \delta, u \rangle = \langle \delta, \zeta \cdot u \rangle .$$

This proves the Theorem. □

REMARK. The interesting part of this decomposition is that $B_i \supseteq A_{\zeta_i}$ and this is $\neq \{0\}$ by Proposition 2.3. In the cases of the connected semisimple Lie groups, [1], and of the Euclidean motion groups, [3], the algebras B_i reduce to A_{ζ_i}. Thus the maximal ideal space is extremely simple, as previously noted.

COROLLARY 2.5. *Let G satisfy the hypotheses in Theorem 2.4. Furthermore suppose that if $u \in B(G)$, $u = \Sigma u_i$, then for each u_i there exists a $p > 0$ such that $u_i \in L^p(G_{\zeta_i})$. Then*

$$\Delta B(G) = \prod_{i=1}^{N} G_{\zeta_i}$$

and $B(G)$ is symmetric.

PROOF. This is what actually happens in the cases of [1] and [3]. Indeed if $u_i \in L^p(G_{\zeta_i})$, then $u \in L^q(G_{\zeta_j})$ for any $q > p$. Therefore we can assume p to be an integer. Now $u_i \in L^{2p}(G_{\zeta_i})$ so that $u_i^p \in L^2(G_{\zeta_i})$ i.e. $u_i^p \in A(G_{\zeta_i})$. If $\delta \in \Delta B$ is not zero on u_i then $\zeta_i\delta \in \Delta B(G_{\zeta_i})$ and

$$0 \neq \left[\langle \zeta_i \delta, u_i \rangle_{G_{\zeta_i}} \right]^p = \langle \zeta_i \delta, u_i^p \rangle_{G_{\zeta_i}} .$$

Since $u_i \in A(G_{\zeta_i})$ it follows that $\zeta_i\delta \in G_{\zeta_i}$ and we are done. □

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