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On the $3n+1$ Problem: Something Old, Something New

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RIASSUNTO – *Una ben nota congettura in teoria dei numeri tratta dell'andamento delle iterazioni della funzione che trasforma gl'interi n in $(3n+1)/2$ (se n è dispari) o in $n/2$ (se n è pari). Tale congettura afferma che la cardinalità della successione generata da un qualsiasi numero naturale n è finita e l'ultimo elemento della successione vale* 1. *In questa nota vengono trattati alcuni aspetti di tali successioni con particolare riferimento alla loro cardinalità.*

ABSTRACT – *A notorious number-theoretic conjecture concerns the behavior of the iterates of the function which takes integers n to $(3n+1)/2$ (if n is odd) or to $n/2$ (if n is even). This conjecture asserts that the cardinality of the sequence generated by any natural number n is finite and the last element of the sequence equals* 1. *This note deals with some aspects of such sequences, most of which concern their cardinality.*

KEY WORDS – *Number theory - Open problems in number theory.*

A.M.S. CLASSIFICATION: 11A99 - 11B99

1 – Introduction and generalities

The so-called $3n+1$ *problem* is a notorious number-theoretic conjecture which is also referred to as *the Collatz problem, Hasse's algorithm, Syracuse algorithm, Ulam's problem,* etc. The exact origin of this problem is obscure (see [2] for a complete reference list): let us recall it briefly.

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On the basis of a given natural number n, let us generate the sequence $S(n) = \{s_k(n)\}(k = 1, 2, \ldots, l)$ obeying the following rule

(i) $$s_0(n) = n \notin S(n)$$

(ii) $$s_k(n) = \begin{cases} \dfrac{s_{k-1}(n)}{2} & \text{if } s_{k-1}(n) \text{ is even} \\ \dfrac{3s_{k-1}(n)+1}{2} & \text{if } s_{k-1}(n) \text{ is odd} \end{cases}$$

(iii) when $s_k(n) = 1$, the generation of $S(n)$ stops.

From (i)-(iii) it follows that the last element of $S(n)$ is $s_l(n) = 1$ and $n = 1$ generates an empty sequence. As an example, we show in detail the generation of $S(3)$:

$$(1.1)\quad \begin{aligned} s_1(3) &= (9+1)/2 = 5 \\ s_2(3) &= (15+1)/2 = 8 \\ s_3(3) &= 8/2 = 4 \\ s_4(3) &= 4/2 = 2 \\ s_5(3) &= 2/2 = 1 . \end{aligned}$$

Let us define the cardinality l of $S(n)$ as the *length* of the natural number n and denote it by $\ell(n)$. The example (1.1) shows that $\ell(3) = 5$. By definition we have

$$(1.2)\quad \ell(1) = 0 .$$

The following conjecture has been offered (e.g., see [2])

CONJECTURE 1: $\ell(n) < \infty$ for all n.

This apparently intractably hard to solve conjecture has been checked up numerically to $n = 2^{40}$ by Nabuo Yoneda at the University of Tokyo [2].

The truth of Conj. 1 would imply that all elements of $S(n)$ are distinct. The values of $\ell(n)$ for the first few values of n are:

$$\begin{array}{ll} \ell(1)=0 & \ell(6)=6 \\ \ell(2)=1 & \ell(7)=11 \\ \ell(3)=5 & \ell(8)=3 \\ \ell(4)=2 & \ell(9)=13 \\ \ell(5)=4 & \ell(10)=5. \end{array} \tag{1.3}$$

As soon as we became aware of the $3n+1$ problem, we were attracted by the rule of construction of $S(n)$ in view of a possible application to modern cryptography. Nevertheless, the aim of this note is neither to discuss practical applications of these sequences nor to try to prove Conj. 1, but rather to present some properties of $S(n)$ emerged during our study; most of them concern the length of positive integers (secs. 2 and 3). Some simple considerations on the possible smallest counterexample to the above conjecture are offered in sec. 4. In sec. 5 further properties of $S(n)$ and the results of some brief computer experiments are discussed.

Let us conclude this section by stating the following theorems the proofs of which are omitted because of their triviality.

THEOREM 1. $\ell(n)=k+\ell(s_k(n)) \quad (0\leq k\leq l)$

THEOREM 2. $\ell(2^k d)=k+\ell(d)$.

COROLLARY 1. $\ell(2^k)=k$.

2 – On the numbers having the same length

A glance taken at table (1.3) leads us to realize that there exist distinct numbers having the same length. This phenomenon has a simple explanation; it is caused by the coalescence of the $S(n)$'s, generated by distinct n, after a certain number of steps. We can clearly see it, if we refer to the so-called *Collatz tree* [2, p. 5] whose *root* is any positive m; by Theor. 1, the length of all leaves at the h^{th} level is $h+\ell(m)$. We recall [3, p. 20] that, if $m\equiv 0 \pmod 3$ the number of leaves is 1 for all h, if

$m \not\equiv 0 \pmod 3$ the expected number $T_h(m)$ of leaves at the h^{th} level is $3(4/3)^h/2$, while, for arbitrary m we have

$$T_h(m) = (4/3)^h . \tag{2.1}$$

We shall make use of (2.1) in sec. 5.

The aim of this section is to find out sets of integers $\{a_k\}$ and $\{b_k\}$, depending on a positive parameter k, such that

$$\ell(a_k) = \ell(b_k) \text{ for } k = (0), 1, 2, \ldots . \tag{2.2}$$

It can be sometimes hard to find out such sets but, once they have been discovered, (2.2) is in general easy to check by proving that $s_j(a_k) = s_j(b_k)$ for some j.

THEOREM 3. *If k and d are arbitrary natural numbers, then*

$$\ell(2^k d - 1) = \ell(6^k d - 2^k).$$

PROOF. On the basis of (ii) it is readily seen that $s_k(2^k d-1) = 3^k d-1$. Therefore, from Theor. 1 we have $\ell(2^k d - 1) = k + \ell(3^k d - 1)$ and, from Cor. 1, $\ell(2^k d - 1) = \ell(2^k(3^k d - 1)) = \ell(6^k d - 2^k)$. We point out that, for $n = 2^k d - 1$, the quantities $s_1(n) < s_2(n) < \ldots < s_{k-1}(n)$ are odd. □

The study of the behavior of $\ell(3^k - 1)$ as k varies led us to discover the following cute property

THEOREM 4. *If k is an arbitrary natural number, then*

$$\ell(3^{2k+1} - 1) = \ell(3^{2k+2} - 1).$$

PROOF. It can be readily proved that

$$(2.3)\qquad 3^h \equiv \begin{cases} 1 \pmod 4 \text{ if } h \text{ is even} \\ -1 \pmod 4 \text{ if } h \text{ is odd}. \end{cases}$$

Therefore, we can write

$$(2.4)\qquad \begin{cases} 3^{2k+1} - 1 \equiv 2 \pmod 4 \\ 3^{2k+2} - 1 \equiv 0 \pmod 4. \end{cases}$$

(2.4) and (ii) allow to prove that $s_2(3^{2k+1} - 1) = s_2(3^{2k+2} - 1) = (3^{2k+2} - 1)/4$. □

COROLLARY 2. *If k is an arbitrary natural number, then*

$$\ell(2^{2k+1} - 1) = \ell(2^{2k+2} - 1) - 1.$$

PROOF. Let us write (cf. the proof of Theor. 3) $\ell(2^{2k+1} - 1) = 2k + 1 + \ell(3^{2k+1} - 1)$ and $\ell(2^{2k+2} - 1) = 2k + 2 + \ell(3^{2k+2} - 1)$. Theor. 4 proves the statement. □

THEOREM 5. *If k and d are arbitrary nonnegative integers, then*

$$\ell(2^k d + 1) = \begin{cases} \ell(2^k(3^{k/2}d + 1)) & \text{if } k \text{ is even} \\ \ell(2^k(3^{(k+1)/2}d + 2)) & \text{if } k \text{ is odd}. \end{cases}$$

PROOF. From (ii) it is seen that, for $1 \le j \le k$,

$$(2.5)\qquad s_j(2^k d + 1) = \begin{cases} 3^{(j+1)/2}\, 2^{k-j} d + 2 & \text{if } j \text{ is odd} \\ 3^{j/2}\, 2^{k-j} d + 1 & \text{if } j \text{ is even}. \end{cases}$$

If k is even, from (2.5) we have $s_{k-1}(2^k d+1) = 3^{k/2}2d+2 = 2(3^{k/2}d+1)$. On the other hand, it is clear that $s_{k-1}(2^k(3^{k/2}d + 1)) = 2(3^{k/2}d + 1)$. If k is odd, from (2.5) we have $s_k(2^k d + 1) = 3^{(k+1)/2}d + 2$. On the other hand, it is clear that $s_k(2^k(3^{(k+1)/2}d + 2)) = 3^{(k+1)/2}d + 2$. □

Theorem 6. *If k is an arbitrary nonnegative integer, then*

$$\ell(32k+23) = \ell(96k+69).$$

Proof. It can be readily checked that $s_5(32k+23) = s_5(96k+69) = 27k+20$. □

Theorem 7. *If k is an arbitrary nonnegative integer, then*

$$\ell(64k+15) = \ell(192k+45).$$

Proof. It can be readily checked that $s_6(64k+15) = s_6(192k+45) = 81k+20$. □

2.1 – On the k-tuples of consecutive numbers having the same length

A slight extension of (1.3) suffices to show that there exist several k-tuples of consecutive integers $\{n, n+1, \ldots, n+k-1\}$ such that

$$\ell(n) = \ell(n+1) = \ldots = \ell(n+k-1). \tag{2.6}$$

For given k, let us denote as $n_0(k)$ the smallest n for which (2.6) holds. It can be readily checked that $n_0(2) = 12$, $n_0(3) = 28$ and $n_0(5) = 98$ (this means obviously that also $n_0(4)$ equals 98). By means of a simple computer experiment carried out for $1 \leq n \leq 10^6$ we worked out the results shown in Table 1.

Table 1 - Behavior of $n_0(k)$ vs k.

k	$n_0(k)$	k	$n_0(k)$
2	12	14	2987
3	28	17	7083
5	98	25	57346
6	386	27	252548
7	943	29	331778
8	1494	30	524289
9	1680	40	596310

Obviously, the values of $n_0(k)$ pertaining to the values of k missing in Table 1 equal the value of $n_0(k)$ corresponding to the smallest k exceeding these missing values. The comparatively slow growth of $n_0(k)$ as k increases suggests the following

CONJECTURE 2: There exists a finite $n_0(k)$ for arbitrarily large k.

The principal aim of this subsection is to find out classes of k-tuples ($k = 2, 3, 4$) of consecutive integers $\{n_h, n_h + 1, \ldots, n_h + k - 1\}$ for which (2.6) is satisfied. *Mutatis mutandis*, the statement just after (2.2) is still valid.

THEOREM 8 (see [2, p.12]). *If h is an arbitrary natural number, then*

$$\ell(8h+4) = \ell(8h+5).$$

PROOF. It can be readily checked that $s_3(8h+4) = s_3(8h+5) = 3h+2$. □

Analogously, it can be easily proved that

$$\ell(32h+2) = \ell(32h+3) \tag{2.7}$$

$$\ell(32h+22) = \ell(32h+23). \tag{2.8}$$

THEOREM 9. *If h is an arbitrary natural number, then*

$$\ell(32h+4) = \ell(32h+5) = \ell(32h+6).$$

PROOF. It can be readily checked that $s_5(32h+4) = s_5(32h+5) = s_5(32h+6) = 9h+2$ □

THEOREM 10. *If h is an arbitrary nonnegative integer, then*

$$\ell(256h+98) = \ell(256h+99) = \ell(256h+100) = \ell(256h+101).$$

PROOF. It can be readily checked that $s_8(256h+98) = \ldots = s_8(256h+101) = 27h + 11$. □

3 – Closed form expressions for the length of certain integers

In this section we show how it is possible to establish closed form expressions for the length of certain classes of integers. From Theor. 2, it is apparent that we may confine ourselves to investigate odd integers only.

First, let us consider the class of integers $\{Z_k\}(k = 1,2,\ldots)$ mentioned in [1, p. 264]. The numbers Z_k are defined by the first order recurrence relation

$$Z_k = 4Z_{k-1} + 1 \qquad (Z_1 = 1). \tag{3.1}$$

THEOREM 11. *If k is a positive integer greater than 2, then $\ell(Z_k) = 2k$.*

PROOF. First observe that $\ell(Z_1) = \ell(1) = 0$. By induction on k it can be readily proved that

$$Z_k = (2^{2k} - 1)/3. \tag{3.2}$$

Since Z_k is odd, from (3.2) we can write

$$s_1(Z_k) = (3Z_k + 1)/2 = 2^{2k-1} \qquad (k \geq 2) \tag{3.3}$$

whence, by Theor. 1 and Cor. 1, we obtain $\ell(Z_k) = 1 + \ell(2^{2k-1}) = 1 + 2k - 1 = 2k$. □

The use of the *predecessors* of a positive integer allows to find out several classes of integers whose length can be given by means of a closed form expression. The predecessors of a positive integer are defined in the next subsection where some simple properties of them are also pointed out.

3.1 – The predecessors of a positive integer

The *predecessors* of a positive integer n are the numbers x such that $s_1(x) = n$. Each number n has the *even* predecessor $p(n)$

$$p(n) = 2n\,. \tag{3.4}$$

The number n has also the *odd* predecessor $q(n)$

$$q(n) = (2n-1)/3 \tag{3.5}$$

if and only if

$$n \equiv 2 (\text{mod}\, 3) \quad (n \geq 5)\,. \tag{3.6}$$

Let us define also the quantities $p_j(n)$ and $q_j(n)$ as

$$\begin{cases} p_j(n) = p(\dots p(p(n))\dots) \\ q_j(n) = q(\dots q(q(n))\dots) \end{cases} \quad (j \text{ pairs of brackets}) \tag{3.7}$$

whence $p(n) = p_1(n)$ and $q(n) = q_1(n)$. It is evident that

$$p_j(n) = 2^j n \tag{3.8}$$

exists for arbitrarily large n and j. As to the quantities $q_j(n)$, let us state the following theorem which generalizes (3.5) and (3.6)

THEOREM 12. *$q_j(n)$ exists iff $n+1 \equiv 0$ (mod 3^j).*

PROOF. Rewriting (3.5) as $q_1(n) = 2(n+1)/3 - 1$, by (3.7) it is seen that

$$q_j(n) = \frac{2^j(n+1)}{3^j} - 1\,. \tag{3.9}$$

The quantity on the right-hand side of (3.9) is integral (i.e., $q_j(n)$ exists) iff $n+1 \equiv 0$ (mod 3^j). □

3.2 – Some examples

The technique for obtaining classes of odd integers whose length can be expressed in closed form is rather simple. Starting from a class of integers $\{n_k\}$ whose length is known (powers of 2, in the simplest case), we determine $q_j(n_k)$ (for some j) with the aid of (3.9) and obtain the quantity $\ell(q_j(n_k))$ by Theor. 1.

EXAMPLE 1. The numbers $q_1(2^k)$ exist iff $k \geq 3$ is odd. They are the numbers $Z_{(k+1)/2}$ (cf. (3.1) and (3.2)). These numbers satisfy the congruences

$$(3.10) \qquad q_1(2^k) \equiv \begin{cases} 0 \pmod 3 \text{ if } k \equiv 5 \pmod 6 \\ 1 \pmod 3 \text{ if } k \equiv 1 \pmod 6 \\ 2 \pmod 3 \text{ if } k \equiv 3 \pmod 6 . \end{cases}$$

It can be readily checked that

$$(3.11) \qquad P_{j,k} = p_j(q_1(2^k)) \equiv 2 \pmod 3 \begin{cases} \text{for even } j \text{ if } k \equiv 3 \pmod 6 \\ \text{for odd } j \text{ if } k \equiv 1 \pmod 6 . \end{cases}$$

The condition (3.6) and (3.11) ensure the existence of the integers

$$(3.12) \qquad W_k^{(j)} = q_1(P_{j,k}) = \frac{2^{j+1}(2^{k+1}-1)-3}{9} \begin{cases} k \equiv 3 \pmod 6 \text{ if } j \text{ is even} \\ k \equiv 1 \pmod 6 \text{ if } j \text{ is odd} . \end{cases}$$

By Theor. 1 and Cor. 1, we get

$$(3.13) \qquad \begin{aligned} \ell(W_k^{(j)}) &= 1 + \ell(P_{j,k}) = 1 + \ell(2^j) + \ell(q_1(2^k)) = \\ &= 1 + \ell(2^j) + 1 + \ell(2^k) = j + k + 2 . \end{aligned}$$

As a numerical example, let us put $k = 7$ ($\equiv 1 \pmod 6$) and $j = 3$ (odd) in (3.12) thus getting $W_7^{(3)} = 453$ and $\ell(453) = 3 + 7 + 2 = 12$; with the aid of a pocket calculator, it can be immediately checked that $s_{12}(453) = 1$.

In the next two examples, we shall proceed in a different way. More precisely, we firstly shall give the class of odd integers and the closed form expression for their length, then we shall prove the validity of this expression.

EXAMPLE 2. The numbers Z_k (see (3.2)) can be generalized by considering the numbers

$$Z_k^{(n)} = \left(\frac{2}{3}\right)^n \epsilon_{n,k} - 1\,, \tag{3.14}$$

where

$$\epsilon_{n,k} = 2^{3^{n-1}(2k-1)} + 1\,. \tag{3.15}$$

These numbers include, as a particular case, the numbers Z_k. In fact, from (3.2), (3.14) and (3.15), it can be easily seen that $Z_k^{(1)} = Z_k$. The following lemma allows us to state that the numbers $Z_k^{(n)}$ are odd integers.

LEMMA 1. $\epsilon_{n,k} \equiv 0 \pmod{3^n}$.

PROOF. Denoting by $\phi(\cdot)$ the Euler totient function, from Euler's theorem we have

$$2^{\phi(3^n)} \equiv 2^{2\cdot 3^{n-1}} \equiv \left(2^{3^{n-1}}\right)^2 \equiv 1 \pmod{3^n}$$

whence

$$2^{3^{n-1}} \equiv \pm 1 \pmod{3^n}\,. \tag{3.16}$$

Since $2^{2h-1} \equiv -1 \pmod 3$ for all h, the positive solution to the congruence (3.16) must be clearly disregarded. Therefore we can write

$$2^{3^{n-1}} \equiv \left(2^{3^{n-1}}\right)^{2k-1} = \epsilon_{n,k} - 1 \equiv -1 \pmod{3^n}\,. \qquad \square$$

Now we are in a position to state the following

THEOREM 13.

$$\ell(Z_k^{(n)}) = \begin{cases} 0 \;\; \textit{if}\; n = k = 1 \\ n + (2k-1)3^{n-1} \;\; \textit{otherwise}\,. \end{cases}$$

PROOF. The proof of the particular case $n = k = 1$ is trivial. Since $(Z_k^{(n)})$ is odd, from (ii) we can write

$$s_j(Z_k^{(n)}) = \left(\frac{2}{3}\right)^{n-j} \epsilon_{n,k} - 1 \quad (\text{odd, for } 1 \le j \le n-1). \tag{3.17}$$

Observe that

$$s_n(Z_k^{(n)}) = \epsilon_{n,k} - 1 = 2^{3^{n-1}(2k-1)}. \tag{3.18}$$

Therefore, from Theor. 1 and Cor. 1 we have

$$\ell(Z_k^{(n)}) = n + \ell\left(2^{3^{n-1}(2k-1)}\right) = n + (2k-1)3^{n-1}. \qquad \square$$

EXAMPLE 3. Finally, let us consider the numbers

$$Y_k^{(n)} = \left(\frac{4}{3}\right)^n \delta_{n,k} + 1 \tag{3.19}$$

where

$$\delta_{n,k} = 4^{3^{n-1}k} - 1. \tag{3.20}$$

Observe that $Y_k^{(1)} = Z_{k+1}^{(1)}$. The proof that the numbers $Y_k^{(n)}$ are odd integers is analogous to that of Lemma 1 and is omitted for brevity.

THEOREM 14. $\ell(Y_k^{(n)}) = 2(n + k3^{n-1})$.

PROOF. Since $Y_k^{(n)}$ is odd, from (ii) it is readily seen that, for $1 \le j \le 2n-1$,

$$s_j(Y_k^{(n)}) = \begin{cases} \dfrac{2^{2n-j}\delta_{n,k}}{3^{n-(j+1)/2}} + 2 & (j \text{ odd}) \\[2ex] \dfrac{2^{2n-j}\delta_{n,k}}{3^{n-j/2}} + 1 & (j \text{ even}). \end{cases} \tag{3.21}$$

Observe that

$$s_{2n-1}(Y_k^{(n)}) = 2\delta_{n,k} + 2 = 2^{2k3^{n-1}+1}. \tag{3.22}$$

Therefore, from Theor. 1 and Cor. 1 we have

$$\ell(Y_k^{(n)}) = 2n - 1 + 2k3^{n-1} + 1 = 2(n + k3^{n-1}). \qquad \square$$

4 – On the possible smallest n such that $\ell(n) = \infty$

Let n_0 be the possible *smallest* counterexample to Conj. 1. In this section we show that n_0 must necessarily belong to particular classes of integers: this result is very useful in view of further computer search after its existence. The proofs are carried out by *reductio ad absurdum*, that is we shall prove that, if n_0 belongs to certain classes of integers, then there exists either one of its predecessors or one of its successors $s_k(n_0)$ (both having, obviously, infinite length) smaller than n_0: a contradiction!

PROPOSITION 1. *n_0 is odd.*

PROOF. If n_0 is even, then $s_1(n_0) = n_0/2 < n_0$ □

PROPOSITION 2. *$n_0 = 4k+3$ with $k \not\equiv 0 \pmod 4$ and $k \not\equiv 5 \pmod 8$.*

PROOF. If $n_0 = 4k+1$, it is immediately seen that $s_2(n_0) = 3k+1 < n_0$.

If $n_0 = 4k+3$ with $k \equiv 0 \pmod 4$ $(k = 4h)$, we can write $n_0 = 16h+3$ whence $s_4(n_0) = 9h+2 < n_0$.

If $n_0 = 4k+3$ with $k \equiv 5 \pmod 8$ $(k = 8h+5)$, we can write $n_0 = 32h+23$ whence (cf. the proof of Theor. 6) $s_5(n_0) = 27h+20 < n_0$. □

PROPOSITION 3.

$$n_0 = \begin{cases} 12k+3 & (k \not\equiv 0 \pmod 4) \\ 12k+7 & (k \not\equiv 1 \pmod 4)\,. \end{cases}$$

PROOF. Let us consider odd integers as expressed by the forms $6h+5$, $6h+3$ and $6h+1$.

Case 1: $n_0 = 6h+5$.

$$q_1(n_0) = (2n_0 - 1)/3 = 4h+3 < n_0\,.$$

Case 2: $n_0 = 6h+3$.
(a) $h = 2k+1$, odd

$n_0 = 12k + 9$ whence $s_2(n_0) = 9k + 7 < n_0$.

(b) $h = 2k$, even
$n_0 = 12k + 3$. If $k \equiv 0 \pmod 4$ $(k = 4m)$, we can write $n_0 = 48m + 3$ whence $s_4(n_0) = 27m + 2 < n_0$.

Case 3: $n_0 = 6h + 1$.

(a) $h = 2k$, even
$n_0 = 12k + 1$ whence $s_2(n_0) = 9k + 1 < n_0$.

(b) $h = 2k + 1$, odd
$n_0 = 12k + 7$. If $k \equiv 1 \pmod 4$ $(k = 4m + 1)$, we can write $n_0 = 48m + 19$ whence $s_4(n_0) = 27m + 11 < n_0$.

□

Combining Prop. 2 and Prop. 3, after some simple manipulations leads to the following admissible forms for n_0

$$n_0 = \begin{cases} 12k + 3 & (k \not\equiv 0, 4, 7 \pmod 8) \\ 12k + 7 & (k \not\equiv 1, 4, 5 \pmod 8). \end{cases} \tag{4.1}$$

5 – Some numerical results

In this section some properties of the sequences $S(n)$ are pointed out and the results of three computer experiments are shown.

1^{ST} Experiment

The length $\ell(n)$ has been found for $1 \leq n \leq 10^7$ and the quantity

$$\bar{\ell}(n) = \frac{1}{n} \sum_{i=1}^{n} \ell(i) \tag{5.1}$$

has been calculated for all integers n lying within the above interval. The numerical evidence that turns out from this experiment leads us to believe that

- $\ell(n) = n$ only for $n = 6, 73$
- the largest n such that $\ell(n) > n$ is $n = 63$.

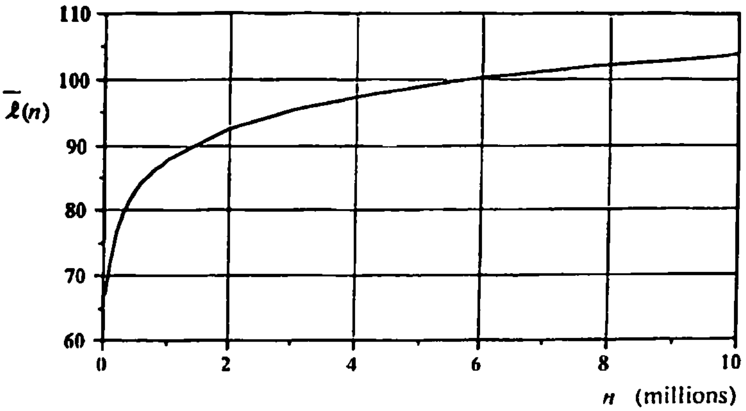

Fig. 1 Behavior of $\bar{\ell}(n)$ vs n.

The behavior of $\bar{\ell}(n)$ vs n is shown in fig. 1.
This behavior suggests (see [2, p.6]) that there exist a positive constant c such that

$$\bar{\ell}(n) \sim c \ln n . \tag{5.2}$$

2^{ND} Experiment

It is know [3, p. 5] that, if n is a random integer, the residues modulo 2 of the quantities $s_k(n)$ are uniformly distributed. The aim of this experiment has been to evaluate the distribution of the residues modulo 3 of $s_k(n)$.

We generated 10^6 sequences $S(n)$ (n randomly chosen, with replacement, within $\{1, 2, \ldots, 10^7\}$) thus obtaining 103,719,207 elements $s_k(n)$ (cf. fig. 1 for $n = 10^7$): 68,737,144 of them (66.27%) turned out to be congruent to 2 (mod 3), 34,648,305 (33.41%) congruent to 1 (mod 3) and 333,758 (0.32%) congruent to 0 (mod 3). These results are strictly close to those presented in [4]. The non-uniformity of this distribution can be justified by a heuristic argument based on the proof of the following

Theorem 15. *If n is any natural number, then $S(n)$ contains at most one odd integer divisible by 3.*

PROOF.

Case 1: $n \equiv 1 \pmod 3$
We have either $n = 6k+4$ (even) or $n = 6k+1$ (odd).

$$(5.3) \qquad s_1(n) = \begin{cases} 3k+2 & (n \text{ even}) \\ 9k+2 & (n \text{ odd}) \end{cases} \equiv 2 \pmod 3.$$

Case 2: $n \equiv 2 \pmod 3$
We have either $n = 6k+2$ (even) or $n = 6k+5$ (odd).

$$(5.4) \qquad s_1(n) = \begin{cases} 3k+1 & (n \text{ even}) \\ 9k+8 & (n \text{ odd}) \end{cases} \equiv \begin{cases} 1 \pmod 3 \\ 2 \pmod 3 \end{cases}.$$

Congruences (5.3) and (5.4) show that, if $n \not\equiv 0 \pmod 3$, $s_k(n) \equiv 1$ or $2 \pmod 3$ $\forall\, k$, that is $S(n)$ does not contain any element divisible by 3.

Case 3: $n \equiv 0 \pmod 3$

Subcase 3a: n odd
We can write $n = 3d$ with d an arbitrary odd integer and

$$(5.5) \qquad s_1(n) = (9d+1)/2 \equiv 2 \pmod 3.$$

Again, (5.3) and (5.4) show that $s_k(n) \equiv 1$ or $2 \pmod 3$ $\forall\, k$.

Subcase 3b: n even
We can write $n = 2^h 3d$ $(h \geq 1)$ and

$$(5.6) \qquad s_1(n) = \begin{cases} 2^{h-k}3d \text{ (even) for } 1 \leq k \leq h-1 \\ 3d \text{ (odd) for } k = h. \end{cases}$$

From (5.6) and (5.5) it is apparent that $S(n)$ contains $h-1$ even elements and exactly one odd element (namely, $s_h(n)$) which are divisible by 3. $\square$

3^{RD} EXPERIMENT

It is clear that, if Conj. 1 is true, $s_k(n)$ must eventually equal a power of 2 (say, 2^h) for some $k = t$ and

$$s_{t+i}(n) = 2^{h-i} \qquad (i = 0, 1, \ldots, h). \tag{5.7}$$

Let us define this final portion ($h+1$ elements) of $S(n)$ as the *tail* of $S(n)$ and denote its length by $\lambda(n)$. From (3.4)-(3.6) it can be readily proved that

$$\lambda(n) \geq 4 \text{ for } n \notin \{1, 2, 4, 8\}, \tag{5.8}$$

and there is a unique n having a prefixed odd tail length, i.e.

$$\lambda(n) = 2m + 1 \implies n = 2^{2m+1}. \tag{5.9}$$

The quantity

$$\overline{\lambda}(n) = \frac{1}{n} \sum_{i=1}^{n} \lambda(i). \tag{5.10}$$

has been found for $1 \leq n \leq 10^7$ and its behavior vs n is shown in fig. 2

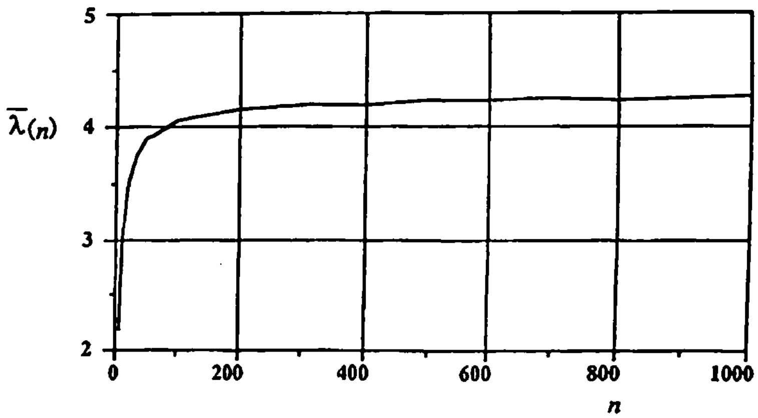

Fig. 2 Behavior of $\overline{\lambda}(n)$ vs n.

for $1 \leq n \leq 10^3$. The maximum value of $\lambda(n)$ within the above interval equals 24 and pertains, obviously, to $n = (2 \cdot 2^{23} - 1)/3 = 5,592,405$.

For $10^3 \leq n \leq 10^7$, the value of $\overline{\lambda}(n)$ oscillates about $d \approx 4.32$. The naturals less than n having tail length equal to 4 are 933 in number for $n = 10^3$ and 9,379,078 for $n = 10^7$. Mainly due to (5.9), we suspected that $\bar{\lambda}(n)$ should have a rather slow growth as n increases. The numerical evidence turning out from this experiment led us to believe that there exists a finite limit $d \approx 4.32$ of $\overline{\lambda}(n)$ as n tends to infinity. In fact, the following theorem can be stated

THEOREM 16. *The limit* $\lim_{n\to\infty} \overline{\lambda}(n) = d$ *exists and* $4 < d < 4.75$.

PROOF. Let us consider the $k^{th}(k \geq 4)$ level of the Collatz tree [2, p.5] whose root is 1. From (5.8), (5.9) and (2.1) we can state that at most one leave has odd tail length and that the expected number $T_k^{(2i)}(1)$ of leaves having tail length $2i$ ($i = 2,3,\ldots,h = \lfloor k/2 \rfloor$) is

$$T_k^{(2i)}(1) = (4/3)^{k-2i}. \tag{5.11}$$

It follows that the expected value λ_k of the tail length of the leaves at the k^{th} level can be expressed by

$$\lambda_k = \frac{1}{(4/3)^k} \sum_{i=2}^{h} 2i(4/3)^{k-2i}. \tag{5.12}$$

After some manipulations omitted for brevity, we can write

$$\lambda_k = 2\,\frac{hr^{h+2} - (h+1)r^{h+1} - r^3 + 2r^2}{(1-r)^2}, \tag{5.13}$$

where $r = 9/16$. Taking the limit of both sides of (5.13) as k tends to infinity yields

$$\lim_{k\to\infty} \lambda_k = \lim_{h\to\infty} \lambda_k = \lambda_\infty = 2\,\frac{2r^2 - r^3}{(1-r)^2} \approx 4.75. \tag{5.14}$$

Since $\lambda_i < \lambda_j$ for $i < j$ and from (5.8), the theorem is proved. □

As a consequence of Theor. 16, we can assert that the expected value of $\lambda(n)$ is a negligible fraction of $\ell(n)$. This fact agrees with the known fact [3, p. 5] that the residues modulo 2 of $s_k(n)$ are uniformly distributed.

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