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On a Certain Class of Meromorphic Univalent Functions with Positive Coefficients

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RIASSUNTO – *Si definisce la classe* $\Sigma_p(\alpha,\beta,A,B,\gamma)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1,\ -1 \le A < B \le 1,\ 0 < B \le 1$ *e* $A/(A-B) \le \gamma \le 1)$ *delle funzioni meromorfe univalenti in* $U^* = \{z\colon 0 < |z| < 1\}$. *Per questa classe si ottengono stime per i coefficienti, e si studiano proprietà di distorsione. Vengono anche determinati i raggi dei campi circolari nei quali le funzioni sono stellate o convesse. Infine viene dimostrato che la classe* $\Sigma_p(\alpha,\beta,A,B,\gamma)$ *è chiusa rispetto alle operazioni di combinazione lineare convessa e di convoluzione.*

ABSTRACT – *In this paper, we introduce a class* $\Sigma_p(\alpha,\beta,A,B,\gamma)$ $(0 \le \alpha < 1,\ 0 < \beta \le 1,\ -1 \le A < B \le 1,\ 0 < B \le 1$ *and* $A/(A-B) \le \gamma \le 1)$ *of meromorphic univalent functions in* $U^* = \{z\colon 0 < |z| < 1\}$ *and investigate coefficient estimates, distortion properties, radii of starlikeness and convexity for this class. Furthermore, it is shown that the class* $\Sigma_p(\alpha,\beta,A,B,\gamma)$ *is closed under convex linear combinations and convolutions.*

KEY WORDS – *Univalent - Meromorphic - Coefficients.*

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1 – Introduction

Let $\sum$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are regular in $U^* = \{z\colon 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there. And let $\sum_s$ denote the subclass of $\sum$ consisting of analytic and univalent functions $f(z)$ in U^*. A function $f(z)$ in $\sum_s$ is said to be meromorphically starlike of order α if

$$\text{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \quad , \quad (z \in U^*) \tag{1.2}$$

for some $\alpha(0 \le \alpha < 1)$. We denote $\sum^*(\alpha)$ the class of all meromorphically starlike funcitons of order α. Similarly, a function $f(z)$ in $\sum_s$ is said to be meromorphically convex of order α if

$$\text{Re}\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \quad , \quad (z \in U^*) \tag{1.3}$$

for some $\alpha(0 \le \alpha < 1)$. And we denote by $\sum_k(\alpha)$ the class of all meromorphically convex functions of order α. The class $\sum^*(\alpha)$ and similar classes have been extensively studied by POMMERENKE[6], CLUNIE [3], MILLER [4] and others.

Let Σ_p dente the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad , \quad (a_n \ge 0) \tag{1.4}$$

that are analytic and univalent in U^*. In [5] Mogra, Reddy and Juneja defined the class $\sum_p^*(\alpha,\beta)$ of meromorphically starlike functions of order $\alpha(0 \le \alpha < 1)$ and type $\beta(0 < \beta \le 1)$ which is a subclass of Σ_p and investigated interesting properties for the class $\Sigma_p^*(\alpha,\beta)$ which is denoted by $\sum_p^*(\alpha,\beta,A,B)(0 \le \alpha < 1, 0 < \beta \le 1, -1 \le A < B \le 1$ and $0 < B \le 1)$. Recently CHO, LEE and OWA [2] defined the class $\Sigma_p(\alpha,\beta\gamma)(0 \le \alpha < 1, 0 < \beta \le 1$ and $\frac{1}{2} \le \gamma \le 1)$, which is a subclass of Σ_p and investigated properties for the class $\Sigma_p^*(\alpha,\beta,\gamma)$.

2 – Coefficient estimates

We begin with the definition of $\sum(\alpha,\beta,A,B,\gamma)$.

DEFINITION. *A function $f(z)$ in $\sum$ is in the class $\sum(\alpha,\beta,A,B,\gamma)$ if it satisfies the condition*

$$\left|\frac{z^2f'(z)+1}{[(B-A)\gamma+A]z^2f'(z)+[(B-A)\gamma\alpha+A]}\right|<\beta \qquad (z\in U^*)$$

for some $\alpha(0\leq\alpha<1)$, $\beta(0<\beta\leq 1)$, $-1\leq A<B\leq 1$, $0<B\leq 1$ and $\gamma(A/(A-B)\leq\gamma\leq 1)$.

THEOREM 1. *Let $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n$ be analytic in U^*. If*

$$\sum_{n=1}^{\infty}(1+(B-A)\beta\gamma+A\beta)|a_n|\leq(B-A)\beta\gamma(1-\alpha), \tag{2.1}$$

for some $\alpha(0\leq\alpha<1)$, $\beta(0<\beta\leq 1)$, $-1\leq A<B\leq 1$, $0<B\leq 1$ and $\gamma(A/(A-B)\leq\gamma\leq 1)$, then $f(z)$ is in the class $\sum(\alpha,\beta,A,B,\gamma)$.

PROOF. Suppose (2.1) holds for alla admissible values of α,β,A,B, and γ. Then we have

$$|z^2f'(z)+1|-\beta\,|z^2f'(z)[(B-A)\gamma+A]+[(B-A)\gamma\alpha+A]|=$$

$$=\left|\sum_{n=1}^{\infty}na_nz^{n+1}\right|-\beta\left|(B-A)\gamma(\alpha-1)+\sum_{n=1}^{\infty}[(B-A)\gamma+A]na_nz^{n+1}\right|\leq$$

$$\leq\sum_{n=1}^{\infty}n(1+(B-A)\beta\gamma+A\beta)|a_n||z|^{n+1}-(B-A)\beta\gamma(1-\alpha).$$

Since the above inequality holds for all $r=|z|$, $0<r<1$, letting $r\to 1$, we have

$$\sum_{n=2}^{\infty}n(1+(B-A)\beta\gamma+A\beta)|a_n|\leq(B-A)\beta\gamma(1-\alpha)$$

by (2.1). Hence it follows that $f(z)$ is in the class $\sum(\alpha,\beta,A,B,\gamma)$.

Let us write $\Sigma_p(\alpha,\beta,A,B,\gamma) = \Sigma_p \wedge \sum(\alpha,\beta,A,B,\gamma)$ where Σ_p is the class of functions of the form (1.4) that are analytic and univalent in U^*.

Theorem 2. *Let the function* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ *($a_n \geq 0$) be analytic in* U^*. *Then* $f(z)$ *is in the class* $\Sigma_p(\alpha,\beta,A,B,\gamma)$ *if and only if* (2.1) *is satisfied.*

Proof. By Theorem 1, it is sufficient to show the "only if" part. Let us assume that $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n (a \geq 0)$ is in the class $\Sigma_p(\alpha,\beta,A,B,\gamma)$. Then

$$\left|\frac{z^2 f'(z)+1}{[(B-A)\gamma+A]z^2 f'(z)+[(B-A)\gamma\alpha+A]}\right| =$$

$$= \left|\frac{\sum_{n=1}^{\infty} n a_n z^{n+1}}{[(B-A)\gamma+A]\left(1-\sum_{n=1}^{\infty} n a_n z^{n+1}\right)-[(B-A)\gamma\alpha+A]}\right| < \beta,$$

$$(z \in U^*).$$

Using the fact $|\operatorname{Re}(z)| \leq |z$ for all z,

(2.2)
$$\operatorname{Re}\left\{\frac{\sum_{n=1}^{\infty} n a_n z^{n+1}}{[(B-A)\gamma+A]\left(1-\sum_{n=1}^{\infty} n a_n z^{n+1}\right)-[(B-A)\gamma\alpha+A]}\right\} < \beta,$$

$$(z \in U^*).$$

Now we choose the values of z on real axis so that $f(z)$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1$ through positive values, we obtain

$$\sum_{n=1}^{\infty} n(1+(B-A)\beta\gamma+A\beta)a_n \leq (B-A)\beta\gamma(1-\alpha).$$

Hence the result follows.

COROLLARY 1. *If the function* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ *is in the class* $\Sigma_p(\alpha, \beta, A, B, \gamma)$, *then*

$$(2.3) \qquad a_n \leq \frac{(B-A)\beta\gamma(1-\alpha)}{n(1+(B-A)\beta\gamma + A\beta)} \quad , \quad (n \geq 1).$$

The result is sharp for the function

$$(2.4) \qquad f_n(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma(1-\alpha)}{n(1+(B-A)\beta\gamma + A\beta)} z^n \quad , \quad (n \geq 1).$$

3 – Distortion properties and radii of starlikeness and convexity

THEOREM 3. *If the function* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ *is in the class* $\Sigma_p(\alpha, \beta, A, B, \gamma)$, *then*

$$\frac{1}{|z|} - \frac{(B-A)\beta\gamma(1-\alpha)}{(1+(B-A)\beta\gamma + A\beta)}|z| \leq |f(z)| \leq$$
$$\leq \frac{1}{|z|} + \frac{(B-A)\beta\gamma(1-\alpha)}{(1+(B-A)\beta\gamma + A\beta)}|z|.$$

The result is sharp.

PROOF. Suppose $f(z)$ is in $\Sigma_p(\alpha, \beta, A, B, \gamma)$. By Theorem 2, we have

$$\sum_{n=1}^{\infty} a_n \leq \frac{(B-A)\beta\gamma(1-\alpha)}{(1+(B-A)\beta\gamma + A\beta)}.$$

Thus

$$|f(z)| \leq \frac{1}{|z|} + |z| \sum_{n=1}^{\infty} a_n$$
$$\leq \frac{1}{|z|} + \frac{(B-A)\beta\gamma(1-\alpha)}{(1+(B-A)\beta\gamma + A\beta)}|z|.$$

Also,

$$|f(z)| \geq \frac{1}{|z|} - |z| \sum_{n=1}^{\infty} a_n$$
$$\geq \frac{1}{|z|} - \frac{(B-A)\beta\gamma(1-\alpha)}{(1+(B-A)\beta\gamma + A\beta)}|z|.$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma(1-\alpha)}{(1+(B-A)\beta\gamma + A\beta)} z.$$

THEOREM 4. *If the function* $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ *is in the class* $\Sigma_p(\alpha, \beta, A, B, \gamma)$, *then* $f(z)$ *is meromorphically starlike of order* $\delta (0 \leq \delta < 1)$ *in* $|z| < r = r(\alpha, \beta, A, B, \gamma, \delta)$, *where*

$$r(\alpha, \beta, A, B, \gamma, \delta) = \inf_{n \geq 1} \left\{ \frac{n(1+(B-A)\beta\gamma + A\beta)(1-\delta)}{(B-A)\beta\gamma(1-\alpha)(n+2-\delta)} \right\}^{1/(n+1)}.$$

The result is sharp.

PROOF. Let $f(z)$ be in the class $\Sigma_p(\alpha, \beta, A, B, \gamma)$. Then, by Theorem 2,

$$\sum_{n=1}^{\infty} \frac{n(1+(B-A)\beta\gamma + A\beta)}{(B-A)\beta\gamma(1-\alpha)} a_n \leq 1. \tag{3.1}$$

It is sufficient to show that

$$\left| 1 + \frac{zf'(z)}{f(z)} \right| \leq 1 - \delta$$

for $|z| < r(\alpha, \beta, A, B, \gamma, \delta)$, where $r(\alpha, \beta, A, B, \gamma, \delta)$ is as specified in the statement of the theorem. Then

$$\left| 1 + \frac{zf'(z)}{f(z)} \right| = \left| \frac{\sum_{n=1}^{\infty} (n+1) a_n z^n}{\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n} \right| \leq \frac{\sum_{n=1}^{\infty} (n+1) a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} a_n |z|^{n+1}}.$$

This will be bounded by $1-\delta$ if

$$\sum_{n=1}^{\infty}\left(\frac{n+2-\delta}{1-\delta}\right)a_n|z|^{n+1}\leq 1\,. \tag{3.2}$$

By (3.1), it follows that (3.2) is true if

$$\left(\frac{n+2-\delta}{1-\delta}\right)|z|^{n+1}\leq\frac{n(1+(B-A)\beta\gamma+A\beta)}{(B-A)\beta\gamma(1-\alpha)}\quad,\quad(n\geq 1)$$

or

$$|z|\leq\left\{\frac{n(1+(B-A)\beta\gamma+A\beta)(1-\delta)}{(B-A)\beta\gamma(1-\alpha)(n+2-\delta)}\right\}\quad,\quad(n\geq 1). \tag{3.3}$$

Setting $|z|=r(\alpha,\beta,A,B,\gamma,\delta)$ in (3.3), the result follows. The result is sharp for the functions

$$f_n(z)=\frac{1}{z}+\frac{(B-A)\beta\gamma(1-\alpha)}{n(1+(B-A)\beta\gamma+A\beta)}z^n\quad,\quad(n\geq 1). \tag{3.4}$$

Using the method of Theorem 4, we obtain the following.

THEOREM 5. *If the function* $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n$ *is in the class* $\Sigma_p(\alpha,\beta,A,B,\gamma)$ *then* $f(z)$ *is meromorphically convex of order* $\delta(0\leq\delta<1)$ *in* $|z|<r=r(\alpha,\beta,A,B,\gamma,\delta)$, *where*

$$r(\alpha,\beta,A,B,\gamma,\delta)=\inf_{n\geq 1}\left\{\frac{(1+(B-A)\beta\gamma+A\beta)(1-\delta)}{(B-A)\beta\gamma(1-\alpha)(n+2-\delta)}\right\}^{\frac{1}{(n+1)}}.$$

The result is sharp for the functions defined by (3.4).

4 – Convex linear combinations and convolution properties

THEOREM 6. *Let* $f_0(z) = \frac{1}{z}$ *and*

$$f_n(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma(1-\alpha)}{n(1+(B-A)\beta\gamma + A\beta)} z^n \quad , \quad (n \geq 1).$$

Then the function $f(z)$ *is in the class* $\Sigma_p(\alpha,\beta,A,B,\gamma)$ *if and only if it can be expressed in the form* $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$, *where* $\lambda_n \geq 0$ *and* $\sum_{n=0}^{\infty} \lambda_n = 1$.

PROOF. Let $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$ with $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$. Then

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \lambda_n \frac{(B-A)\beta\gamma(1-\alpha)}{n(1+(B-A)\beta\gamma + A\beta)} z^n .$$

Since

$$\sum_{n=1}^{\infty} \frac{n(1+(B-A)\beta\gamma + A\beta)}{(B-A)\beta\gamma(1-\alpha)} \lambda_n \frac{(B-A)\beta\gamma(1-\alpha)}{n(1+(B-A)\beta\gamma + A\beta)} =$$

$$= \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1 ,$$

by Theorem 2, $f(z)$ is in the class $\Sigma_p(\alpha,\beta,A,B,\gamma)$.

Conversely, suppose the function $f(z)$ is in the class $\Sigma_p(\alpha,\beta,A,B,\gamma)$. Since

$$a_n \leq \frac{(B-A)\beta\gamma(1-\alpha)}{n(1+(B-A)\beta\gamma + A\beta)} \quad , \quad ((n \geq 1),$$

setting

$$\lambda_n = \frac{n(1+(B-A)\beta\gamma + A\beta)}{(B-A)\beta\gamma(1-\alpha)} a_n \quad , \quad (n \geq 1)$$

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n ,$$

it follows that

$$f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z).$$

This completes the proof of the theorem.

THEOREM 7. *The class $\Sigma_p(\alpha,\beta,A,B,\gamma)$ is closed under convex linear combinations.*

PROOF. Suppose $f_i(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,i}z^n,(i=1,2)$ are in the class $\Sigma_p(\alpha,\beta,A,B,\gamma)$. Let

$$f(z)=(1-t)f_1(z)+tf_2(z)\quad,\quad(o\leq t\leq 1).$$

Then

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}((1-t)a_{n,1}+ta_{n,2})z^n.$$

By Theorem 2, we have

$$\sum_{n=1}^{\infty}n(1+(B-A)\beta\gamma+A\beta)((1-t)a_{n,1}+ta_{n,2})\leq$$
$$\leq(1-t)(B-A)\beta\gamma(1-\alpha)+t(B-A)\beta\gamma(1-\alpha)$$
$$=(B-A)\beta\gamma(1-\alpha).$$

Therefore it follows that $f(z)$ is in the class $\Sigma_p(\alpha,\beta,A,B,\gamma)$.

ROBERTSON [7] has shown that if $f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n$, $g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_nz^n$ are in $\sum_s$, then so their convolution $f*g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nb_nz^n$.

THEOREM 8. *If the functions $f_i(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,i}z^n$ are in the class $\Sigma_p(\alpha,\beta,A,B,\gamma)$ for each $i=1,2,\ldots,m$, respectively, then the convolution $f_1*f_2*\ldots*f_m(z)$ is in the class $\Sigma_p(1-\prod_{i=1}^{m}(1-\alpha_i),\beta,A,B,\gamma)$.*

Proof. Since the functions $f_i(z)$ are in the class $\Sigma_p(\alpha,\beta,A,B,\gamma)$ for each $i=1,2,\dots,m$, respectively, by Theorem 2,

$$\sum_{n=1}^{\infty} n(1+(B-A)\beta\gamma+A\beta)a_{n,i}\leq(B-A)\beta\gamma(1-\alpha_i)$$

and

$$a_{n,i}\leq\frac{(B-A)\beta\gamma(1-\alpha_i)}{1+(B-A)\beta\gamma+A\beta}$$

for any $n\geq 1$. Hence we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n(1+(B-A)\beta\gamma+A\beta)\prod_{i=1}^{m}a_{n,i}\leq\\
&\leq\sum_{n=1}^{\infty} n(1+(B-A)\beta\gamma+A\beta)\prod_{i=1}^{m-1}\frac{(B-A)\beta\gamma)(1-\alpha_i)}{(1+(B-A)\beta\gamma+A\beta)}a_{n,m}=\\
&=\frac{((B-A)\beta\gamma)^{m-1}}{(1+(B-A)\beta\gamma+A\beta)^{m-1}}\prod_{i=1}^{m-1}(1-\alpha_i)\cdot\\
&\cdot\sum_{n=1}^{\infty} n(1+(B-A)\beta\gamma+A\beta)a_{n,m}\leq\\
&\leq\frac{((B-A)\beta\gamma)^{m}}{(1+(B-A)\beta\gamma+A\beta)^{m-1}}\prod_{i=1}^{m}(1-\alpha_i)\leq\\
&\leq(B-A)\beta\gamma\left\{1-(1-\prod_{i=1}^{m}(1-\alpha_i))\right\}.
\end{aligned}$$

Hence the result follows.

Corollary 2. *The class $\Sigma_p(\alpha,\beta,A,B,\gamma)$ is closed under the convolution.*

Remark. Putting $A=-1$ and $B=1$ in the above results we get the results of Cho, Lee and Owa [2].

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