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Existence, Uniqueness and Stability for a Semi-Linear Equation of the Viscoelasticity

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RIASSUNTO – *Si considera un'equazione iperbolica del terzo ordine, con un termine forzante non lineare, che regge i moti unidimensionali di un corpo con un comportamento viscoelastico di tipo rilassamento. Si discutono alcuni problemi di valori al contorno per i quali si dimostrano anche dei principi di massimo. Inoltre, si studia un problema di evoluzione e si prova la limitatezza e la stabilità alla Liapunov rispetto a due metriche.*

ABSTRACT – *We deal with a hyperbolic third order partial differential equation with a non-linear forcing term that rules the one-dimensional motions of a body with a viscoelastic behaviour of relaxation type. We discuss some boundary value problems for which maximum theorems are also stated. Moreover, we consider an evolution problem and show the boundedness and the Liapunov stability with respect to two metrics.*

KEY WORDS – *Hyperbolic third order PDE - Boundary value problems - Maximum principles - Liapunov stability and boundedness.*

A.M.S. CLASSIFICATION: 35L35 - 35B50 - 35B35

– Introduction

Let B be an isotropic, intrinsically homogeneous body with a viscoelastic behaviour of relaxation type. Denote by u the only non-vanishing displacement component from a reference configuration. It is known [10,

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13] that the basic motion equations lead to the following third order hyperbolic partial differential equation

$$(*)\qquad \varepsilon(u_{tt} - u_{xx})_t + u_{tt} - ku_{xx} = f\,.$$

Here, ε and k are positive constants and denote, respectively, a relaxation time and the ratio (< 1) of the initial and the regime values of the relaxation function typical of B. Moreover, in this work, the forcing term f is supposed to be a non-linear function of u, u_x, u_t, that is

$$(**)\qquad f = f(x,t,u,u_x,u_t)\,.$$

The evolution equation (*) follows from the relaxation representation; however, the same type of equation is obtained via the creep representation [13].

When $f = 0$ or $f = f(x,t)$ equation (*) has been widely analyzed by P. RENNO, see e.g. [4, 6, 8, 9]. Then, some other problems concerning the computation of the solutions and their qualitative behaviour have been considered [5, 7]. Furthermore, the unilateral phenomenon arising when the free motions of B are impeded by a rigid obstacle has been treated [10, 12, 13, 14]; in this last case the scheme proposed is in complete agreement with the analysis developed by an energetic approach [11].

In this paper, with reference to (*), (**), we discuss several questions, which are a first step to solve physically meaningful problems - e.g., the piston problem, the impact or support problems - and the properties of the solutions of an evolution problem. We start with the analysis of the motion with prescribed tension at an end and boundary conditions on a characteristic of the operator (*). In the linear case we get the explicit solution by applying suitably the classical Riemann method to the third order hyperbolic operator defined by (*) (§1). This is also useful to determine in the non-linear case the integral equation equivalent to the differential problem (§2). The existence and the uniqueness are shown by considering a map F of a metric space into itself and proving that F has a unique fixed point. Afterwards, we study a boundary value problem when, instead of the tension, the displacement of the end is given; we write the (known) solution of the linear case in a new form useful for our aims and treat the non-linear situation with the above mentioned tecnique. Successively, we consider (*), (**) with the only initial conditions and examine an evolution problem by means of successive extensions of (some) previous results.

Since, in special applications, such as unilateral and free boundary problems, maximum principles are essential, we devote to them Section 3, where some maximum theorems are stated.

Finally, in §4 we examine the qualitative behaviour of the solutions of (*), (**) assuming the ends of B are fixed and the forcing term (**) depends linearly on u_t. This situation occurs when the body is subjected to a non-linear positional force. In this case, under suitable hypotheses on f and f_t and referring to the initial perturbations, we show the Liapunov stability for the equilibrium and, besides, the boundedness of the solutions with respect to two metrics, using the energy functional associated to (*).

1 – Explicit solution of a problem with given tension

We consider the operator L defined by

$$L = \varepsilon\partial_t(\partial_t^2 - \partial_x^2) + \partial_t^2 - k\partial_x^2 \tag{1.1}$$

and discuss on $\Omega_T = \{(x,t) \colon x \in I_T,\, x < t < T - x\}$, $I_T =]0, T/2[$, $T > 0$, the following problem

$$Lu = f(x,t)\,, \quad (x,t) \in \Omega_T\,, \tag{1.2}$$

$$u_x(0,t) = b(t)\,, \quad t \in J_T =]0,T[\,, \tag{1.3}$$

$$u(x,x) = u_0(x)\,, \quad Gu(x,x) = g(x)\,, \quad x \in I_T\,. \tag{1.4}$$

Here, G is the operator

$$G = \varepsilon(\partial_t^2 - \partial_x^2) + (1-k)\partial_t\,. \tag{1.5}$$

An explicit solution of the problem (1.2) - (1.4) is achieved by the following

THEOREM 1.1. *Under the hypotheses that* $b \in C^2(J_T)$, $u_0 \in C^2(I_T)$, $g \in C^0(I_T)$, $f \in C^0(\Omega_T)$, *there exists a unique solution* u *of* (1.2) - (1.4) *with* $u \in C^2(\Omega_T)$, $(u_{tt} - u_{xx})_t \in C^0(\Omega_T)$. *Moreover, its explicit expression is given by* (1.16).

PROOF. Let $(x_0, t_0) \in \Omega_T$ and

$$Z_0 = \{(x,t) \colon |x - x_0| < t_0 - t\}, \quad \Omega_0 = \Omega_T \cap Z_0 .$$

We indicate by $v(|x_0 - x|, t_0 - t)$, $(x,t) \in Z_0$, the Riemann function [10, 8, 9] defined by

$$v(|x_0 - x|, t_0 - t) = \int_{|x_0 - x|}^{t_0 - t} F_3(z, t_0 - t) dz$$

(1.6)
$$F_3(r,t) = \varepsilon^{-1} e^{\delta r - \gamma t} \left\{ I_0(\omega) + \int_0^1 [4\eta y I_0(\xi y) + \xi I_1(\xi y)] \cdot \right.$$
$$\left. \cdot I_0(\omega(1 - y^2)^{1/2}) e^{\eta y^2} dy \right\},$$

where I_n is the modified Bessel function of order n and

$$0 \le r \le t, \quad b = (1-k)/2\varepsilon, \quad \gamma = (1+k)/2\varepsilon, \quad \delta = k/\varepsilon,$$
$$\eta = b(t-r)/2, \quad \xi = 2[b\delta r(t-r)]^{1/2}, \quad \omega = b(t^2 - r^2)^{1/2}.$$

Furthermore, we put

$$K^{\pm} = 2\varepsilon(\partial_t^2 \pm \partial_{xt}^2) - (1+k)\partial_t \mp 2k\partial_x .$$

We first solve problem (1.2) - (1.4) when $b(t) = 0$ by applying the Riemann method to the hyperbolic operator (1.1). Setting

$$x_i = (t_0 + (-1)^i x_0)/2 , \quad t_i = x_i , \quad i = 1,2 ,$$
$$\Omega_0' = \{(x,t) \in \Omega_T , 0 < x < x_1 , x < t < t_0 - x_0 - x\} ,$$

and recalling the properties of the function v [10], we obtain the following explicit expression for the solution z

(1.7)

$$\begin{aligned} z(x_0,t_0) &= -\varepsilon u_0(x_1)v_t(x_0+x_1,t_0-t_1) - \varepsilon u_0(x_2)v_t(x_2-x_0,t_0-t_2)+ \\ &+ \frac{1}{2}\int_0^{x_1}\Big\{u_0(x)K^+v(x_0+x,t_0-x) + g(x)v(x_0+x,t_0-x)\Big\}dx+ \\ &+ \frac{1}{2}\int_0^{x_2}\Big\{u_0(x)K^+v(|x_0-x|,t_0-x) + g(x)v(|x_0-x|,t_0-x)\Big\}dx+ \\ &+ \frac{1}{2}\int_{\Omega_0'} f(x,t)v(x_0+x,t_0-t)dxdt + \frac{1}{2}\int_{\Omega_0} f(x,t)v(|x_0-x|,t_0-t)dxdt+ \\ &+ u(0)(\varepsilon\partial_t - k)v(x_0,t_0) . \end{aligned}$$

Moreover, it can be verified that $z \in C^2(\Omega_T)$, $(z_{tt} - z_{xx})_t \in C^0(\Omega_T)$.

Assume, now, $b(t) \neq 0$ and note that, defining

$$P = xb(kv_x - \varepsilon v_{xt}) + b(\varepsilon v_t - kv) ,$$
$$Q = xb(\varepsilon v_{tt} - v_t) + xb'(v - \varepsilon v_t) + \varepsilon v b'' x ,$$

one has

$$-\int_{\Omega_0}(Lxb(t))v(|x_0-x|,t_0-t)dxdt = \int_{\partial\Omega_0}(Qdx - Pdt) . \tag{1.8}$$

Let us compute the right-hand side of (1.8):

$$\tag{1.9}
\begin{aligned}
\int_{(x_2,t_2)}^{(x_0,t_0)} (Q dx - P dt) = \int_{x_2}^{x_0} \Big\{ & xb(\varepsilon v_{tt} - \varepsilon v_{xt} - v_t + k v_x) + \\
& + v(\varepsilon x b'' + x b' - kb) + \varepsilon v_t (b - x b') \Big\} dx = \\
= \int_{x_2}^{x_0} \Big\{ & xb(2\varepsilon v_{tt} - 2\varepsilon v_{xt} - v_t + k v_x) + v(\varepsilon x b'' + x b' - \\
& - kb) \Big\} dx + \varepsilon v_t(0,0) x_0 b(t_0) - \varepsilon v_t(x_2 - x_0, t_0 - t_2) x_2 b(t_2) ;
\end{aligned}$$

$$\tag{1.10}
\begin{aligned}
\int_{(x_0,t_0)}^{(0,t_0-x_0)} (Q dx - P dt) = - \int_{x_0}^{0} \Big\{ & xb(-\varepsilon v_{tt} - \varepsilon v_{xt} + k v_x + v_t) + \\
& + v(-\varepsilon x b'' - kb - x b') + \varepsilon v_t (b + x b') \Big\} dx = \\
= - \int_{x_0}^{0} \Big\{ & xb(-2\varepsilon v_{tt} - 2\varepsilon v_{xt} + k v_x + v_t) + \\
& + v(-\varepsilon x b'' - kb - x b') \Big\} dx + \varepsilon v_t(0,0) x_0 b(t_0) .
\end{aligned}$$

From (1.9), (1.10) and recalling [10]

$$v = 0 , \quad K^{\pm} v = 0 \text{ on } \partial Z_0 , \quad \varepsilon v_t(0,0) = -1 ,$$

we deduce

$$\tag{1.11}
\begin{aligned}
& \int_{(x_2,t_2)}^{(x_0,t_0)} (Q dx - P dt) + \int_{(x_0,t_0)}^{(0,t_0-x_0)} (Q dx - P dt) = \\
& = -2 x_0 b(t_0) - \varepsilon v_t (x_2 - x_0, t_0 - t_2) x_2 b(t_2) .
\end{aligned}$$

For the other two integrals we have

$$(1.12)\qquad \int_{(0,t_0-x_0)}^{(0,0)} (Qdx - Pdt) = \int_0^{t_0-x_0} b(t)(\varepsilon\partial_t - k)v(x_0, t_0 - t)dt\,,$$

and, moreover, proceeding as for (1.10), we obtain

$$(1.13)\qquad \int_{(0,0)}^{(x_2,t_2)} (Qdx - Pdt) = \int_0^{x_2} \Big\{xbK^+v + v[\varepsilon xb'' + (1-k)xb']\Big\}dx - \\ - \varepsilon v_t(x_2 - x_0, t_0 - t_2)x_2 b(t_2)\,.$$

Therefore, from (1.11) - (1.13) we achieve

$$(1.14)\qquad -\int_{\Omega_0} (Lxb(t))v(|x_0 - x|, t_0 - t)dxdt = \int_0^{t_0-x_0} b(t)(\varepsilon\partial_t - k)v(x_0, t_0 - t)dt - \\ -2x_0 b(t_0) + \int_0^{x_2} \Big\{xb(x)K^+v(|x_0 - x|, t_0 - x) + \\ + [\varepsilon xb''(x) + (1-k)xb'(x)]v(|x_0 - x|, t_0 - x)\Big\}dx - \\ - 2\varepsilon v_t(x_2 - x_0, t_0 - t_2)x_2 b(t_2)\,.$$

Then, we replace Ω_0' with Ω_0 in the identity (1.8); using analogous arguments we find

$$(1.15)\qquad - \int_{\Omega_0'} (Lxb(t))v(x_0 + x, t_0 - t)dxdt = \int_0^{t_0-x_0} b(t)(\varepsilon\partial_t - k)v(x_0, t_0 - t)dt - \\ - 2\varepsilon v_t(x_0 + x_1, t_0 - t_1)x_1 b(t_1) + \int_0^{x_1} \Big\{xb(x)K^+v(x_0 + x, t_0 - x) + \\ + [\varepsilon xb''(x) + (1-k)xb'(x)]v(x_0 + x, t_0 - x)\Big\}dx\,.$$

We note, now, that the wanted solution is obtained from the one of the problem (1.2) - (1.4) previously solved, by replacing in (1.7) $z(x,t)$ with $u(x,t) - xb(t)$, $u_0(x)$ and $g(x)$ respectively with $u_0(x) - xb(x)$ and $g(x) - \varepsilon x b''(x) - (1-k)xb'(x)$ and, lastly, $f(x,t)$ with $f(x,t) - Lxb(t)$. Finally, we take into account (1.4), (1.5) obtaining

(1.16)
$$u(x_0,t_0) = -\varepsilon u_0(x_1)v_t(x_0+x_1,t_0-t_1) - \varepsilon u_0(x_2)v_t(x_2-x_0,t_0-t_2) +$$
$$+\frac{1}{2}\int_0^{x_1}\Big\{u_0(x)K^+v(x_0+x,t_0-x) + g(x)v(x_0+x,t_0-x)\Big\}dx +$$
$$+\frac{1}{2}\int_0^{x_2}\Big\{u_0(x)K^+v(|x_0-x|,t_0-x) + g(x)v(|x_0-x|,t_0-x)\Big\}dx +$$
$$+\frac{1}{2}\iint_{\Omega_0'} f(x,t)v(x_0+x,t_0-t)dxdt + \frac{1}{2}\iint_{\Omega_0} f(x,t)v(|x_0-x|,t_0-t)dxdt +$$
$$+u_0(0)(\varepsilon\partial_t - k)v(x_0,t_0) + \int_0^{t_0-x_0} b(t)(\varepsilon\partial_t - k)v(x_0,t_0-t)dt\,.$$

Under the hypotheses on b, the solution u has the same qualitative properties of the function z; thus, the theorem is completely proved.

2 – Initial and boundary value problems for the non-linear equation

Consider the equation

$$Lu = f(x,t,u(x,t),u_x(x,t),u_t(x,t))\,, \tag{2.1}$$

with L defined by (1.1). In Subsection 2.1 we study the following problems

Problem 𝔄

$$Lu = f \quad \text{on} \quad \Omega_T\,,$$

$$u = u_0(x), \quad Gu = g(x) \text{ for } t = x \text{ and } x \in I_T\,; \tag{2.2}$$

$$u = a(t) \text{ for } x = 0\,, \quad t \in J_T\,; \tag{2.3}$$

Problem 𝔅

This problem is obtained from 𝔄 changing (2.3) into

$$u_x = b(t) \text{ for } x = 0\,, \quad t \in J_T\,. \tag{2.4}$$

Moreover, in Subsections 2.2, 2.3 we discuss an initial value question and an evolution problem.

2.1 – When f is a function of x, t, then, as we have shown in [11], **Problem** 𝔄 has an explicit solution $u(x_0, t_0)$. This, setting

$$\begin{aligned} u(u_0, g; x_0, t_0) &= \varepsilon v_t(x_0 + x_1, t_0 - t_1)u_0(x_1) - \\ &-\varepsilon v_t(x_2 - x_0, t_0 - t_2)u_0(x_2) - \frac{1}{2}\int_0^{x_1}\Big\{u_0(x)K^+ v(x_0 + x, t_0 - x) + \\ &+ g(x)v(x_0 + x, t_0 - x)\Big\}dx + \frac{1}{2}\int_0^{x_2}\Big\{u_0(x)K^+(|x_0 - x|, t_0 - x) + \\ &+ g(x)v(|x_0 - x|, t_0 - x)\Big\}dx\,, \end{aligned} \tag{2.5}$$

$$\begin{aligned} u(f; x_0, t_0) &= \frac{1}{2}\int_{\Omega_0} f(x,t)v(|x_0 - x|, t_0 - t)\,dx\,dt - \\ &- \frac{1}{2}\int_{\Omega_0'} f(x,t)v(x_0 + x, t_0 - t)\,dx\,dt\,, \end{aligned} \tag{2.6}$$

is given by

$$u(x_0,t_0) = a(t_0) + u(u_0 - a, g - Ga; x_0, t_0) + u(f - La; x_0, t_0). \tag{2.7}$$

We consider now the non-linear case and prove

THEOREM 2.1. *Suppose $a \in C^2(\overline{J}_T)$, $u_0 \in C^2(\overline{I}_T)$, $g \in C^0(\overline{I}_T)$ and there exists a positive constant H such that*

$$\begin{aligned} |f(x,t,u,u_x,u_t) - f(x,t,u^*,u_x^*,u_t^*)| \le H\Big\{|u-u^*|+ \\ + |u_x - u_x^*| + |u_t - u_t^*|\Big\}. \end{aligned} \tag{2.8}$$

Moreover, if

$$a(0) = u(0) \tag{2.9}$$

and $f(x,t,u(x,t),u_x(x,t),u_t(x,t)) \in C^0(\overline{\Omega}_T)$ $\forall\, u \in C^1(\overline{\Omega}_T)$, then Problem $\mathfrak{A}$ has a unique solution $u \in C^2(\Omega_T)$ and $(u_{tt} - u_{xx})_t \in C^0(\Omega_T)$.

PROOF. In our proof we shall use (2.7) written in a simpler and more convenient form. Indeed, proceeding as in §1, from the identity

$$vLa = \partial_x[a(kv_x - \varepsilon v_{xt})] + \partial_t[a(\varepsilon v_{tt} - v_t) + a'(v - \varepsilon v_t) + \varepsilon v a'']$$

we formally obtain

$$\begin{aligned} -\int_{\Omega_0} v(|x_0 - x|, t_0 - t)La(t)dxdt + \int_{\Omega_0'} v(x_0+x, t_0 - t)La(t)dxdt = \\ = -a(t_0) + u(a, Ga; x_0, t_0) - \varepsilon v_t(x_0, x_0)a(t_0 - x_0) + \\ + \int_0^{t_0 - x_0} a(t)(k\partial_x - \varepsilon\partial_{xt})v(x_0, t_0 - t)dt. \end{aligned} \tag{2.10}$$

When f depends only on x,t, from (2.7), (2.10) we get the following expression for the solution of 𝔓𝔯𝔬𝔟𝔩𝔢𝔪 𝔄

$$u(x_0,t_0) = -\varepsilon v_t(x_0,x_0)a(t_0-x_0) + \int_0^{t_0-x_0} a(t)(k\partial_x - \varepsilon\partial_{xt})v(x_0,t_0-t)dt + u(u_0,g;x_0,t_0) + u(f;x_0,t_0). \tag{2.11}$$

In the non-linear case (2.1) we achieve again (2.11), but this represents no more the solution in explicit form.

In order to show the theorem, we put

$$w(x_0,t_0) = -\varepsilon v_t(x_0,x_0)a(t_0-x_0) + u(x_0,g;x_0,t_0) + \int_0^{t_0-x_0} a(t)(k\partial_x - \varepsilon\partial_{xt})v(x_0,t_0-t)dt\,, \tag{2.12}$$

and introduce the functional transformation $\mathcal{F}$ that maps $u \in C^1(\overline{\Omega}_T)$ into the function

$$z(x_0,t_0) = \frac{1}{2}\int_{\Omega_0} v(|x_0-x|,t_0-t)f(x,t,u(x,t),u_x(x,t),u_t(x,t))dxdt + w(x_0,t_0) - \frac{1}{2}\int_{\Omega_0'} v(x_0+x,t_0-t)f(x,t,u(x,t),u_x(x,t),u_t(x,t))dxdt\,. \tag{2.13}$$

Under the hypotheses of the theorem we can differentiate (2.13) with respect to x_0,t_0. So, we have

$$z_{x_0}(x_0,t_0) = w_{x_0}(x_0,t_0) + \frac{1}{2}\int_{\Omega_0} v_{x_0}(|x_0 - x|, t_0 - t) f(x,t,u,u_x,u_t)dxdt -$$
$$- \frac{1}{2}\int_{\Omega_0'} v_{x_0}(x_0 + x, t_0 - t) f(x,t,u,u_x,u_t)dxdt +$$
$$+ \frac{1}{2}\int_0^{x_0} v(x_0 - x, x_0 - x) f(x, x - x_0 + t_0, u, u_x, u_t)dx +$$
$$+ \frac{1}{4}\int_{x_2}^{x_0+t_0-x_2} v(x_2 - x_0, t_0 - t) f(x_2, t, u, u_x, u_t)dt +$$
$$+ \frac{1}{2}\int_{x_0}^{x_2} v(x - x_0, x - x_0) f(x, x_0 - x + t_0, u, u_x, u_t)dx +$$
$$+ \frac{1}{4}\int_{x_1}^{t_0-x_0-x_1} v(x_0 + x_1, t_0 - t) f(x_1, t, u, u_x, u_t)dt +$$
$$+ \frac{1}{2}\int_0^{x_1} v(x_0 + x, x_0 + x) f(x, t_0 - x_0 - x, u, u_x, u_t)dx\,.$$

In this formula the last five integrals vanish since either $v = 0$ on ∂Z_0 or $x_i = -x_i + t_0 + (-1)^i x_0$, $i = 1, 2$. Therefore,

$$z_{x_0}(x_0,t_0) = w_{x_0}(x_0,t_0) - \frac{1}{2}\int_{\Omega_0'} v_{x_0}(x_0 + x, t_0 - t) f(x,t,u,u_x,u_t)dxdt +$$
$$+ \frac{1}{2}\int_{\Omega_0} v_{x_0}(|x_0 - x|, t_0 - t) f(x,t,u,u_x,u_t)dxdt\,. \tag{2.14}$$

Similarly,

$$z_{t_0}(x_0,t_0) = w_{t_0}(x_0,t_0) - \frac{1}{2}\int_{\Omega_0'} v_{t_0}(x_0 + x, t_0 - t) f(x,t,u,u_x,u_t)dxdt +$$
$$+ \frac{1}{2}\int_{\Omega_0} v_{t_0}(|x_0 - x|, t_0 - t) f(x,t,u,u_x,u_t)dxdt\,. \tag{2.15}$$

From (2.13) - (2.15) we see that $\mathcal{F}$ maps $C_1(\overline{\Omega}_T)$ into itself. Then, we put $z^* = \mathcal{F}(u^*)$ and equip $C_1(\overline{\Omega}_T)$ with the norm

$$\|u\| = \max|e^{-ct_0}u(x_0,t_0)| + \max|e^{-ct_0}u_{x_0}(x_0,t_0)| + \\ + \max|e^{-ct_0}u_{t_0}(x_0,t_0)|, \quad (x_0,t_0)\in\overline{\Omega}_T,$$

where c is a positive constant we fix later. Moreover, let

$$M_1 = \max v(|x_0-x|,t_0-t) + \max|v_x(|x_0-x|,t_0-t)| + \\ + \max|v_t(|x_0-x|,t_0-t)|, \quad (x,t)\in\overline{\Omega}_0, \quad (x_0,t_0)\in\overline{\Omega}_T,$$

$$M_2 = \max v(x_0+x,t_0-t) + \max|v_x(x_0+x,t_0-t)| + \\ + \max|v_t(x_0+x,t_0-t)|, \quad (x,t)\in\overline{\Omega'}_0, \quad (x_0,t_0)\in\overline{\Omega}_T.$$

From (2.13), recalling (2.8) and the definition of norm, we get

$$|z(x_0,t_0)-z^*(x_0,t_0)| \le \\ \le (M_1/2)\int_{\Omega_0}|f(x,t,u,u_x,u_t)-f(x,t,u^*,u_x^*,u_t^*)|dxdt + \\ + (M_2/2)\int_{\Omega_0'}|f(x,t,u,u_x,u_t)-f(x,,t,u^*,u_x^*,u_t^*)|dxdt \le \\ \le \frac{1}{2}MH\|u-u^*\|\left\{\int_{\Omega_0}e^{ct}dxdt + \int_{\Omega_0'}e^{ct}dxdt\right\}, \tag{2.16}$$

where

$$M = \max\{M_1,M_2\}.$$

Let us multiply (2.16) by e^{-ct_0} and compute the integrals. One has

$$e^{-ct_0}|z-z^*| \le \frac{MH}{c^2}\|u-u^*\|\Big\{2-e^{c(x_0-x_2)}-e^{-c(x_1+x_0)} + \\ + e^{-ct_0}[2-e^{-cx_2}-e^{cx_1}]\Big\}.$$

Therefore,

$$e^{-ct_0}|z - z^*| \leq MH\|u - u^*\|/c^2. \tag{2.17}$$

Moreover, (2.14) and (2.15) imply

$$e^{-ct_0}|z_x - z_x^*| \leq MH\|u - u^*\|/c^2, \tag{2.18}$$

$$e^{-ct_0}|z_t - z_t^*| \leq MH\|u - u^*\|/c^2. \tag{2.19}$$

By taking into account the definition of norm, from (2.17) - (2.19) it follows $\|z - z^*\| \leq (3MH/c^2)\|u - u^*\|$. Assuming $c^2 > 3MH$, $\mathcal{F}$ results a contractive map. So, we can conclude that there exists a unique $u \in C^1(\overline{\Omega}_T)$ satisfying (2.11) when f is the non-linear forcing term given by the right-hand side of (2.1).

The analysis of the expression (2.11), then, points out $u \in C^2(\Omega_T)$ and that $u_{tt} - u_{xx}$ is continuously differentiable with respect to t. Besides, the boundary conditions (2.2) are verified by means of (2.9). Consequently, u is the only solution of 𝔓roblem 𝔄.

An analogous result holds also for 𝔓roblem 𝔅; indeed, one has

THEOREM 2.2. *Assume u_0, g and f satisfy the hypotheses of* Theor. 2.1. *If $b \in C^2(\overline{J}_T)$, then there exists a unique solution u of* 𝔓roblem 𝔅 *such that $u \in C^2(\Omega_T)$, $(u_{tt} - u_{xx})_t \in C^0(\Omega_T)$.*

PROOF. We proceed as in the previous theorem. Indeed, we can apply the same methodology replacing $\mathcal{F}$ with the map $\mathcal{F}'$ defined by

$$\mathcal{F}'u(x_0,t_0)=\frac{1}{2}\int_{\Omega_0'} v(x_0+x,t_0-t)f(x,t,u(x,t),u_x(x,t),u_t(x,t))dxdt+$$

$$+\,w_B(x_0,t_0)+\frac{1}{2}\int_{\Omega_0} v(|x_0-x|,t_0-t)f(x,t,u(x,t),u_x(x,t),u_t(x,t))dxdt,$$

with w_B given by the right-hand side of (1.16) for $f = 0$.

2.2 - We refer again to the semi-linear equation (2.1) and consider an initial value problem on

$$S = \{(x,t): 0 < t < 1/2, \quad t < x < 1-t\}.$$

We want to study (2.1) on S with the following initial conditions

$$(2.20) \qquad u = u_1, \quad u_t = u_2, \quad Gu = u_3 \qquad \text{for } t = 0 \text{ and } x \in I_{01},$$

where

$$(2.21) \qquad I_{01} =]0,1[.$$

For this problem, setting

$$S_0 = \{(x,t) \in S: 0 < t < t_0, x_0 - t_0 + t < x < x_0 + t_0 - t\},$$

one has

THEOREM 2.3. *Suppose $u_i \in C^{3-i}(\overline{I}_{01})$, $i = 1,2,3$, and that there exists a positive constant H such that*

$$(2.22) \qquad \begin{aligned} |f(x,t,u,u_x,u_t) - f(x,t,u^*,u_x^*,u_t^*)| \le H\{|u-u^*| + |u_x - u_x^*| + \\ + |u_t - u_t^*|\}. \end{aligned}$$

if $f(x,t,u(x,t),u_x(x,t),u_t(x,t)) \in C^0(\overline{S})$ $\forall$ $u \in C^1(\overline{S})$, then the initial value problem for (2.1) *on S has a unique solution $u \in C^2(S)$, $(u_{tt} - u_{xx})_t \in C^0(S)$, satisfying initial conditions* (2.20).

PROOF. Let us recall [10] the solution $\tilde{u}$ of the above-cited problem with $f = 0$ belongs to $C^2(S)$ and $(\tilde{u}_{tt} - \tilde{u}_{xx})_t \in C^0(S)$; moreover, $\tilde{u}$ is explicitly given by

$$\tilde{u}(x_0,t_0) = \frac{1}{2}\int_{x_0-t_0}^{x_0+t_0} (u_1(x)(\varepsilon\partial_t^2 - \partial_t) + u_2(x)(k - \varepsilon\partial_t) + u_3(x))v(|x_0 - x|, t_0)dx -$$

$$- \frac{\varepsilon}{2}v_t(t_0,t_0)[u_1(x_0 - t_0) + u_1(x_0 + t_0)], \; (x_0,t_0) \in S.$$

In the non-linear case one realizes the solution of the initial value problem can be obtained as solution of the following integral equation

(2.23)
$$u(x_0,t_0) = \frac{1}{2}\int_{S_0} f(x,t,u(x,t),u_t(x,t),u_x(x,t))v(|x_0-x|,t_0-t)dxdt+$$
$$+\tilde{u}(x_0,t_0), \qquad (x_0,t_0)\in S.$$

Now, we are ready to apply to (2.23) the same method used in Theor. 2.1 and show completely the theorem.

2.3 – We conclude this section with the analysis of the evolution problem on

$$\Delta_T = \{(x,t)\colon x\in I_{01}\,,\ 0<t<T\}\,,\ T>0\,, \tag{2.24}$$

with the initial data (2.20) and the boundary conditions

$$u = a_j(t) \quad \text{for } t>0 \quad \text{and } x=j\ ,\ j=0,1\,. \tag{2.25}$$

We want to solve this problem by successive extensions of previous considered questions. So, we need the laws of the jumps propagation along the characteristics $t\pm x=$ constant. Defining, as usually,

$$[u]=u^+-u^-\,,\ [f]=f^+-f^-\,,\ (d/dt)='$$

and applying the Rankine-Hugoniot method, we obtain

$$\begin{aligned}
&2\varepsilon[u]'+(1-k)[u]=0\,,\\
&2\varepsilon[u]_t'+(1-k)[u]_t+\frac{1}{2}\{[u]''+(1+k)[u]'\}=0\,,\\
&2[Gu]+\varepsilon[u]''+2k[u]'=0\,,\\
&2\varepsilon[u]_{tt}'+(1-k)[u]_{tt}+\frac{1-k}{2}[u_t]'-\frac{k(1-k)}{\varepsilon}[u_t]+\\
&\qquad+\frac{\varepsilon}{4}[u]'''+(1-2k)[u]''/2-k^2[u]'/\varepsilon=[f]\,.
\end{aligned} \tag{2.26}$$

The way of solving the evolution problem can be, now, rapidly sketched as follows: first, we use Theor. 2.3; then, we employ Theor. 2.1, the conditions (2.26) and the characteristic problem discussed in [15], and so on. Thus, we get

THEOREM 2.4. *Assume* u_i, $i = 1,2,3$, *and* f *satisfy the hypotheses of* Theorem 2.1, 2.3 *and that* $a_j \in C^2(\overline{J}_T)$ *with* $a_j(j) = u_1(j)$, $a'_j(j) = u_2(j)$, $\varepsilon a''_j(j) = \varepsilon u''_1(j) - (1-k)u'_2(j) + u_3(j)$, $j = 0,1$. *Then equation* (2.1) *has a unique solution* u *verifying initial and boundary conditions* (2.20), (2.25); *moreover,* $u \in C^2(\Delta_T)$ *and* $(u_{tt} - u_{xx})_t \in C^0(\Delta_T)$.

3 – Some maximum principles

At first, we consider the case studied at Section 1 in which u_x is assigned on the boundary $x = 0$ and establish the following

THEOREM 3.1. *Let the hypotheses of Theorem 1.1 be fulfilled. If in addition*

$$b < 0\,,\ (\leq 0)\,,\ (\leq 0) \text{ on } J_T\,, \tag{3.1$_1$}$$

$$u_0 \geq 0\,,\ (> 0)\,,\ (\geq 0) \text{ on } I_T\,, \tag{3.1$_2$}$$

$$g \geq 0\,,\ (\geq 0)\,,\ (> 0) \text{ on } I_T\,, \tag{3.1$_3$}$$

and, moreover,

$$f \geq 0 \qquad \text{on } \Omega_T\,, \tag{3.2}$$

$$u_0(0) = 0\,, \tag{3.3}$$

then the solution u *of the problem* (1.2) - (1.4) *is such that*

$$u > 0 \qquad \text{on } \Omega_T\,. \tag{3.4}$$

PROOF. The proof is based on the analysis of some inequalities involving the Riemann function v, as we deduce by examining the form (1.16) of the solution.

At first, we observe that from the expression (1.6) of v it follows that

$$v > 0\,,\ v_t < 0 \qquad \text{on } Z_0\,. \tag{3.5}$$

Furthermore, as it has been established in [9], one has

$$K^{\pm} v > 0 \qquad (x,t) \in Z_0\,. \tag{3.6}$$

Inequality (3.4) is, now, a consequence of hypotheses (3.1) - (3.3) and properties (3.5), (3.6).

We examine, then, the same question with reference to the non-linear case studied at §.2 and denoted as 𝔓roblem 𝔅.

THEOREM 3.2. *Assume that u_0, g, and f verify the hypotheses of* Theorem 2.1 *and that $b \in C^2(J_T)$. If* (3.1), (3.3) *subsist and, moreover,*

$$f(x,t,u,u_x,u_t) \geq 0 \text{ for } u \geq 0\,, \tag{3.7}$$

then the solution u of 𝔓roblem 𝔅 *is strictly positive on the set Ω_T.*

PROOF. Let us introduce the function

(3.8)

$$\begin{aligned}
w_B(x_0,t_0) = &-\varepsilon u_0(x_1) v_t(x_0+x_1,t_0-t_1) - \\
&- \varepsilon u_0(x_2) v_t(x_2-x_0,t_0-t_2) + u_0(0)(\varepsilon\partial_t - k)v(x_0,t_0) + \\
&+ \frac{1}{2}\int_0^{x_1}\{u_0(x)K^+v(x_0+x,t_0-x) + g(x)v(x_0+x,t_0-x)\}dx + \\
&+ \frac{1}{2}\int_0^{x_2}[u_0(x)K^+v(|x_0-x|,t_0-x) + g(x)v(|x_0-x|,t_0-x)]dx + \\
&+ \int_0^{t_0-x_0} b(t)(\varepsilon\partial_t - k)v(x_0,t_0-t)dt
\end{aligned}$$

and observe that for (3.5), (3.6) it results

$$w_B(x_0,t_0) > 0 \text{ on } \Omega_T. \tag{3.9}$$

On the other hand, from (3.7), (3.9), we derive

$$f(x_0,t_0,w_B,w_{Bx_0},w_{Bt_0}) \geq 0 \text{ on } \Omega_T. \tag{3.10}$$

We recall, now, that the solution u of the 𝔓roblem 𝔅, as it has been shown in Theorem 2.2, is approximated by the sequence

$$\begin{aligned} &u^0(x_0,t_0) = w_B(x_0,t_0)\ ,\ u^{n+1}(x_0,t_0) = w_B(x_0,t_0)+\\ &+\frac{1}{2}\int_{\Omega_0'} v(x_0+x,t_0-t)f(x,t,u^n(x,t),u^n_x(x,t),u^n_t(x,t))dxdt+\\ &+\frac{1}{2}\int_{\Omega_0} v(|x_0-x|,t_0-t)f(x,t,u^n(x,t),u^n_x(x,t),u^n_t(x,t))dxdt. \end{aligned} \tag{3.11}$$

From (3.9) - (3.11) it follows $u^{n+1}(x_0,t_0) > 0$ on Ω_T. Since (3.9) holds, one has $u^{n+1} \to u > 0$ on Ω_T.

REMARK 3.1. The restriction $a(t)$ of $u(x,t)$ to the line $x = 0$ is still strictly positive since we have

$$\begin{aligned} a(t_0) = &-2\varepsilon v_t(x_1,t_0-t_1)u_0(x_1)+\int_0^{t_0} b(t)(\varepsilon\partial_t - k)v(0,t_0-t)dt+\\ &+\int_0^{x_1}[u_0(x)K^+v(x,t_0-x)+g(x)v(x,t_0-x)]dx+\\ &+\int_{\Omega_0'} v(x,t_0-t)f(x,t,u(x,t),u_x(x,t),u_t(x,t))dxdt. \end{aligned} \tag{3.12}$$

In particular, for $f = f(x,t)$, formula (3.12) gives us the connection between a and b; finally, if $u_0 = 0$ then a is strictly increasing.

Also for the $\mathfrak{Problem}\ \mathfrak{A}$ studied at §2.1, in which u is prescribed on the boundary $x = 0$, an analogous maximum theorem subsists. Indeed, we have

THEOREM 3.3. *Suppose that the hypotheses of* Theorem 2.1 *are fulfilled. If in addition*

$$ka + \varepsilon\dot{a} > 0 \quad (\geq 0) \text{ on } J_T\,, \tag{3.13$_1$}$$

$$a(0) = 0\,, \tag{3.13$_2$}$$

$$g \geq 0 \qquad (> 0) \text{ on } I_T\,, \tag{3.13$_3$}$$

$$u_0 = 0 \text{ on } I_T\,, \tag{3.13$_4$}$$

and, moreover,

$$f(x,t,u,u_x,u_t) \geq 0 \qquad \text{for } u \geq 0\,. \tag{3.14}$$

Then the solution u of the $\mathfrak{Problem}\ \mathfrak{A}$ verifies

$$u > 0 \qquad \text{on } \Omega_T\,.$$

PROOF. In the occurrence of the proof of Theorem 2.1 we have demonstrated that the quoted solution u satisfies the integral equation (2.11) and this, by using the notation

$$u(a;x_0,t_0) = -\varepsilon v_t(x_0,x_0)a(t_0-x_0) + \int_0^{t_0-x_0} a(t)(k\partial_x - \varepsilon\partial_{xt})v(x_0,t_0-t)dt\,, \tag{3.15}$$

can be rewritten as

$$u(x_0,t_0) = u(a;x_0,t_0) + u(u_0,g;x_0,t_0) + u(f;x_0,t_0)\,. \tag{3.16}$$

A basic role to determine the sign of the right-hand side of (3.16) is played by the inequality

$$(3.17) \qquad v_r < 0 \quad \text{on} \quad Z_0, \quad r = |x_0 - x|,$$

which is a consequence of (1.6).
Integrating by parts the integral in the right-hand member of (3.15) and recalling that $v = 0$ on ∂Z_0, we attain to

$$(3.18) \qquad \begin{aligned} u(a; x_0, t_0) = &-\varepsilon a(0) v_r(x_0, t_0) - \\ &- \int_0^{t_0 - x_0} [k a(t) + \varepsilon \dot{a}(t)] v_r(x_0, t_0 - t) dt. \end{aligned}$$

Since (3.17) holds, the hypotheses $(3.13_{1,2})$ assure that

$$(3.19) \qquad u(a; x_0, t_0) > 0 \quad (\geq 0) \quad \text{on} \quad \Omega_T .$$

Furthermore, by means of the expression (2.5), we get

$$u(0, g; x_0, t_0) = \frac{1}{2} \int_0^{x_1} g(x)[v(|x_0 - x|, t_0 - x) - v(x_0 + x, t_0 - x)] dx +$$

$$+ \frac{1}{2} \int_{x_1}^{x_2} g(x) v(|x_0 - x|, t_0 - x) dx ,$$

in which it is $|x_0 - x| \leq x_0 + x$. Therefore, $(3.13_{3,4})$ and (3.17) imply

$$(3.20) \qquad u(u_0, g; x_0, t_0) \geq 0 \quad (> 0) \quad \text{on} \quad \Omega_T .$$

Afterwards, from (3.19), (3.20) we deduce

$$(3.21) \qquad w(x_0, t_0) = u(a; x_0, t_0) + u(u_0, g; x_0, t_0) > 0 .$$

Finally, we observe that from the relation (2.6) it follows

$$u(f; x_0, t_0) = \int_{\Omega_0 - \Omega_0'} f(x, t, u, u_x, u_t) v(|x_0 - x|, t_0 - t) dx dt +$$

$$+ \int_{\Omega_0'} f(x, t, u, u_x, u_t)[v(|x_0 - x|, t_0 - t) - v(x_0 + x, t_0 - t)] dx dt .$$

In this formula it is $v > 0$ on $\Omega_0 - \Omega'_0$, while $|x_0 - x| \leq x_0 + x$ and (3.17) imply

$$v(|x_0 - x|, t_0 - t) - v(x_0 + x, t_0 - t) > 0 \quad \text{on} \quad \Omega'_0 .$$

At this point reapplying to the integral equation

$$u(x_0, t_0) = w(x_0, t_0) + u(f; x_0, t_0),$$

in which f fulfils (3.14), the same argument employed in the proof of Theorem 3.2, one proves $u > 0$ on Ω_T.

REMARK 3.2 Also on the first derivatives of the solutions of the Problems $\mathfrak{A}$ and $\mathfrak{B}$ some maximum principles can be stated, under suitable hypotheses and using the same arguments.

4 – Liapunov stability

Given the function $\phi(x,t,u)$ defined on the set $A = \{x \in I_{01}, t \in I, u \in \mathbb{R}\}$, $I = [0,\infty[$ and $I_{01} =]0,1[$, together with the derivatives ϕ_t, ϕ_u, we particularize the evolution problem examined at §2.3 by considering in the set $\{x \in I_{0,1}, t > t_0 \geq 0\}$ the equation

$$\text{(4.1)} \qquad \varepsilon(u_{tt} - u_{xx})_t + u_{tt} - k u_{xx} = -(\varepsilon \partial_t + k)\phi(x, t, u(x,t)),$$

where $\partial_t \phi = \phi_t + \phi_u u_t$, with the same initial conditions and the following boundary data

$$\text{(4.2)} \qquad u(0,t) = u(1,t) = 0 \qquad \text{for } t > t_0 \geq 0 .$$

Consequently, the joint conditions that we associate to (4.1) and (4.2) are

$$\text{(4.3)} \qquad u_i(j) = 0 \quad , \quad \varepsilon u''_1(j) - (1-k)u'_2(j) + u_3(j) = 0 \qquad (i,j) \in \{1,2\} \times \{0,1\} .$$

If the right-hand member of (4.1) verifies the hypotheses of Theorem 2.4, such a problem admits a unique solution of class $C^2(I_{01} \times I)$.

We intend, now, to investigate on some qualitative properties of these solutions and in order to realize this analysis we need the following notations and definitions.

Referring to a general element $e \in C^2(I_{01} \times I)$, we consider the restriction $e(\cdot, t) \in C^2(I_{01})$ and introduce the two metrics

$$(4.4) \qquad \begin{cases} d_1^2(e)(t) = \int_0^1 [e_{xx}^2 + e_x^2 + e_t^2] dx , \\ \\ d_2^2(e)(t) = \int_0^1 [e_{xx}^2 + e_{xt}^2 + e_{tt}^2] dx + d_1^2(e)(t) . \end{cases}$$

Consequently, the following definitions are meaningful.

DEFINITION 4.1. *The solutions $u(t)$ of problem* (4.1) - (4.3) *are uniformly bounded with respect to the metrics d_2 and d_1 if for any $\alpha > 0$ and $t_0 \geq 0$ there exists a $\beta(\alpha) \geq \alpha$ such that if the initial data verify $d_2(u)(t_0) \leq \alpha$, then $d_1(u)(t) \leq \beta$ for every $t \geq t_0$.*

DEFINITION 4.2. *The solution $u(t) \equiv 0$ of problem* (4.1) - (4.3) *is uniformly stable with respect to the metrics d_2 and d_1 if for any $\sigma > 0$ and $t_0 \geq 0$ there exists a $\delta(\sigma) \in]0, \sigma[$ such that if the initial data verify $d_2(u)(t_0) < \delta$, then $d_1(u)(t) < \sigma$ for all $t \geq t_0$.*

Now, we are able to give the following

THEOREM 4.1. *Suppose that the function $\phi(t)$ satisfies the conditions*

$$(4.5) \qquad \phi(x,t,z)z \geq 0 , \quad \phi_t(x,t,z)z \leq 0 \quad on\ A .$$

Then, with respect to the metrics d_2 and d_1, for the problem (4.1) - (4.3) *we have*

a) the solutions are uniformly bounded;

b) the solution $u \equiv 0$ is uniformly stable.

PROOF. We carry out the proof considering the Liapunov functional defined by

$$
(4.6)\qquad \begin{aligned} V(t) =& \frac{1}{2}\int_0^1 [(\varepsilon(u_{tt} - u_{xx} + \phi) + (1-k)u_t)^2 + \\ &+ k(1-k)(u_x^2 + u_t^2 + 2\int_0^u \phi(x,t,z)dz)]dx\,. \end{aligned}
$$

First of all, it is useful to determine lower and upper bounds for $V(t)$. For this end, we recall that [3]

$$
(4.7)\qquad u(0,t) = 0 \implies \int_0^1 u_x^2(x,t)dx \geq u^2(x,t)\,;
$$

furthermore, $(4.5)_1$ implies

$$
(4.8)\qquad \int_0^1 \phi(x,t,z)dz \geq 0\,.
$$

Therefore, using (4.7) and (4.8), from (4.6) we derive

$$
(4.9)\qquad V(t) \geq \frac{k(1-k)}{4} d_1^2(u)(t)\,.
$$

On the other hand, it is known [2] that given n positive numbers a_i and an integer $s \geq 1$ the inequality

$$
(4.10)\qquad \left(\sum_i^{1\dots n} a_i\right)^2 \leq n^{s-1}\sum_i^{1\dots n} a_i^s
$$

is fulfilled. By employing (4.10) we can increase the first term of the integral (4.6). Since it is $0 < k < 1$ too, we have

$$
(4.11)\qquad \begin{aligned} V(t) \leq& \frac{\max(1,\varepsilon)}{2}\int_0^1 [4(u_{xx}^2 + u_{tt}^2 + \phi^2 + u_t^2) + \\ &+ u_x^2 + u_t^2 + 2\int_0^u \phi dz]dx\,. \end{aligned}
$$

As regards the last term of (4.6), we note that from (4.5) it follows $\phi\phi_t \leq 0$ and, therefore, $|\phi(x,t,z)|$ is a decreasing function of t for each fixed $x \in I_{01}$ and $z \in \mathbb{R}$; consequently

$$|\phi(x,t,z)| \leq |\phi(x,0,z)|, \quad x \in I_{01}, \quad z \in \mathbb{R}. \tag{4.12}$$

Furthermore, since $(4.5)_1$ implies $\phi(x,t,0) = 0$ too, it results

$$\phi(x,0,z) = \int_0^z \phi_\zeta(x,0,\zeta)d\zeta; \tag{4.13}$$

indicating by

$$m(|z|) = \max\{|\phi_\zeta(x,0,\zeta)|, \quad x \in I_{01}, \quad |\zeta| \leq |z|\},$$

(4.13) can be considered together with (4.12), obtaining

$$|\phi(x,t,z)| \leq m(|z|)|z|. \tag{4.14}$$

Inequality (4.14) allows us to determine an upper bound for the third and the last term of (4.11); all things considered we can write

$$V(t) \leq \max(1,\varepsilon)\{3d_2^2(u)(t) + 2m^2(|u|)u^2 + m(|u|)u^2\}. \tag{4.15}$$

By virtue of $(4.4)_2$ and (4.7), the relation (4.15) give rises to

$$V(t) \leq 4\max(1,\varepsilon)[1 + m[d_2(u)(t)]]^2 d_2^2(u)(t). \tag{4.16}$$

Now, we estimate the derivative $\dot{V}$ of V along the solutions of (4.1) - (4.3). It results

$$\dot{V}(t) = \int_0^1 \Big\{ [\varepsilon(u_{tt} - u_{xx} + \phi) + (1-k)u_t]\, k(u_{xx} - u_{tt} - \phi) +$$

$$+ k(1-k)\left(u_x u_{xt} + u_t u_{tt} + \phi(x,t,u)u_t + \int_0^u \phi_t(x,t,z)dz\right)\Big\}dx$$

and integrating by parts with the conditions (4.2) and executing the necessary simplifications, one obtains

$$\dot{V}(t) = \int_0^1 \left[-\varepsilon k(u_{tt} - u_{xx} + \phi)^2 + k(1-k) \int_0^u \phi_t(x,t,z)dz \right] .$$

Hypothesis $(4.5)_2$ assure that

$$\dot{V}(t) \leq 0 \qquad \forall\, t \geq 0 .$$

Inequalities (4.9) and (4.16), considered together with the decrease of the function $V(t)$, imply that for any time $t \geq t_0$ it results

$$d_1(u)(t) \leq a[1 + m(d_2(u)(t_0))]d_2(u)(t_0) , \tag{4.17}$$

where $a^2 = 16\max(1,\varepsilon)/k(1-k)$.

At this point, exploiting (4.17), we observe that, given $\alpha > 0$ and $\beta(\alpha) = (1 + d_2(\alpha))a\alpha$, the solutions of (4.1) - (4.3), starting from initial data such that $d_2(u)(t_0) \leq \alpha$, verifies the condition $d_1(u)(t) \leq \beta$, for all $t \geq t_0$.

At last, if we assume $\delta = \sigma/(1 + m(\sigma))a$, $\sigma > 0$, and the initial data fulfil $d_2(u)(t_0) \leq \delta$, then the corresponding solutions of (4.1) - (4.3) verify $d_1(u)(t) < \sigma$, for all $t \geq t_0$.

Thus the Theorem is completely proved.

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