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Almost-Coercive Matrix Transformations

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RIASSUNTO – *La classe* $(m : f)$ *di matrici quasi-coercive è stata caratterizzata da* EIZEN *e* LAUSH *[2]. Con* m *ed* f*, vengono rispettivamente indicati gli spazi lineari di sequenze limitate e quasi convergenti. Scopo principale di questo articolo è la caratterizzazione delle classi* $(bs: f)$ *e* $(bs: fs)$ *di matrici quasi-coercive dove* bs *e* fs *indicano gli spazi lineari delle serie le cui sequenze di somme parziali appartengono ad* m *ed* f*, rispettivamente. Inoltre è stato dimostrato un teorema di tipo Steinhaus, che afferma che, nel caso di una trasformazione da serie a sequenza, "una matrice non può essere al contempo* f*-regolare e quasi-coerciva".*

ABSTRACT – *The class* $(m: f)$ *of almost-coercive matrices was characterized by* EIZEN *and* LAUSH *[2]. By* m *and* f*, we respectively denote the linear spaces of real bounded and almost convergent sequences. The main object of the present paper is to characterize the classes* $(bs: f)$ *and* $(bs: fs)$ *of almost-coercive matrices, where* bs *and* fs *denote the linear spaces of the series whose sequence of partial sums belongs to* m *and* f*, respectively. Further, a Steinhaus type theorem, which asserts for a series-to-sequence transformation that "a matrix can not be both* f*-regular and almost-coercive" has been proved.*

KEY WORDS – *Infinite matrices - Almost convergence and coercivity.*

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1 – Introduction

An infinite matrix $A = (a_{nk})$; $n, k = 0, 1, \dots,$ defines a transformation from the sequence space X into the sequence space Y provided for each $x \in X$, the matrix product $Ax = \left(\sum_{k=0}^{\infty} a_{nk}x_k\right)$ exists and is in Y. The class of all such matrices is denoted by $(X : Y)$. When in X and Y there is some notion of limit or sum, then the subclass of $(X : Y)$ consisting of matrices which preserves this limit or sum will be denoted by $(X : Y; P)$. The sequence $Ax = \big((Ax)_n\big)$ is called the A-transform of the sequence $x = (x_n)$. A sequence x is said to be A-summable to x_0 if Ax converges to x_0.

Let m and m^* respectively denote the space of real bounded sequences and the algebraic dual of m. $L \in m^*$ is said to be a Banach limit ([1], pp.32) if it has the following properties:

(i) $L(x) \geq 0$ if $x \geq 0$,

(ii) $L(e) = 1$ where $e = (1, 1, 1, \ldots)$,

(iii) $L(Sx) = L(x)$ where S is the shift operator defined by $(Sx)_n = x_{n+1}$.

DEFINITION 1.1. *A bounded sequence x is said to be almost convergent to the generalized limit x_0 if each Banach limit of x is x_0* [4]. *This is denoted by $f - \lim x = x_0$.*

The class f of almost convergent sequences was introduced by LORENTZ [4]. Now, let us write

$$d_{ij}(x) = \frac{1}{i+1} \sum_{k=0}^{i} x_{j+k} \, ,$$

for any sequence x. LORENTZ proved in [4] that a sequence x is almost convergent if and only if $d_{ij}(x)$ tends to a limit as $i \to \infty$, uniformly in j. It is well-known that a convergent sequence is almost convergent such that its limit and its generalized limit are identical. A sequence x is said to be almost A-summable to x_0, if the A-transform of x almost converges to x_0.

DEFINITION 1.2. *A series $\sum x_k$ is said to be bounded if its sequence of partial sums is bounded. Analogously, a series $\sum x_k$ is said to be almost convergent to the generalized sum x_0 if its sequence of partial sums almost converges to x_0. By bs and fs, we denote the spaces of bounded and almost convergent series, respectively. Also, by f_0 and f_0s, we indicate the space of sequences which are almost convergent to zero and the space* *of series whose sequence of partial sums is almost convergent to zero, respectively.*

DEFINITION 1.3. *A matrix A is said to be almost-coercive if it belongs to one of the classes $(m\colon f)$, $(bs\colon f)$ or $(bs\colon fs)$.*

Throughout the paper, all summations will be taken from 0 to ∞ and the sequence $s = (s_k)$ will be formed by taking the partial sums of the series $u = \sum u_n$. So, it is clear by Definition 1.2 that $s \in m$ (or $s \in f$) whenever $u \in bs$ (or $u \in fs$).

2 – Almost-coercive series-to-sequence transformations

Firstly, we will give the following theorem due to EIZEN and LAUSH [2] characterized the class of $(m\colon f)$.

THEOREM 2.1. *An infinite matrix A transforms m into f if and only if*

$$\sup_n \sum_k \left|a_{nk}\right| < \infty, \tag{2.1}$$

$$f - \lim a_{nk} = a_k \quad ; \quad (k = 0, 1, \ldots), \tag{2.2}$$

$$\lim_q \sum_k \frac{1}{q+1} \sum_{j=n}^{n+q} (a_{jk} - a_k) = 0, \ \textit{uniformly in } n. \tag{2.3}$$

The condition (2.3) is given by DURAN [3] as follows;

$$\lim_q \sum_k \frac{1}{q+1} \left| \sum_{j=0}^{q} a_{n+j,k} - a_k \right| = 0, \text{ uniformly in } n. \tag{2.4}$$

Now, we can give the following theorem which characterizes the class of $(bs\colon f)$;

THEOREM 2.2. *An infinite matrix B transforms bs into f if and only if*

$$\sup_n \sum_k |\Delta b_{nk}| < \infty, \quad \text{where } \Delta b_{nk} = b_{nk} - b_{n,k+1}, \tag{2.5}$$

$$\lim b_{nk} = 0 \quad ; \quad (n = 0, 1, \ldots), \tag{2.6}$$

$$f - \lim b_{nk} = b_k \quad ; \quad (k = 0, 1, \ldots), \tag{2.7}$$

$$\lim_q \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} \Delta(b_{n+i,k} - b_k) \right| = 0, \text{ uniformly in } n. \tag{2.8}$$

PROOF. **Necessity.** Let $B \in (bs\colon f)$ and $u \in bs$. Now, to show that the necessity of (2.6), we assume that (2.6) is not satisfied for some n and obtain a contradiction as in Theorem 2.1 of [6]. Indeed, under this assumption we can find some $u \in bs$ such that Bu does not almost convergent. For example, if we choose $u = \left((-1)^n\right) \in bs$ then $(Bu)_n = \sum_k b_{nk}(-1)^k$. However, that series $\sum_k b_{nk}(-1)^k$ does not converge for each n. That is to say that the B-transform of the series $\sum(-1)^n$ which belongs to bs, does not even exist. But this contradicts the fact that B is almost-coercive. Hence, (2.6) is necessary.

The necessity of (2.7) is easily obtained by taking $u = e^k$, where e^k is the sequence whose only non-zero term is a 1 in the k^{th} place, since $e^k \in bs$ for each k.

Let us consider the following connection

$$\sum_{k=0}^{p} b_{nk} u_k = \sum_{k=0}^{p-1} \Delta b_{nk} s_k + b_{np} s_p \quad ; \quad (n = 0, 1, \ldots) \tag{2.9}$$

obtained by applying Abel's partial summation to the p^{th} partial sums of Bu. Letting $p \to \infty$ in (2.9), we have

$$\sum_k b_{nk} u_k = \sum_k \Delta b_{nk} s_k \tag{2.10}$$

because the second term on the right hand side of (2.9) tends to zero by (2.6). It follows by passing to the f-limit in (2.10) that $D = (d_{nk}) \in (m\colon f)$, where $d_{nk} = \Delta b_{nk}$ for all n, k. Therefore (2.1) and (2.4) are satisfied by the matrix D and these are equivalent to (2.5) and (2.8), respectively.

Sufficiency. Let us suppose that the matrix B satisfies (2.5)-(2.8) and $u \in bs$. Let us reconsider the matrix D in (2.10). Therefore the statement: "(2.1), (2.2) and (2.4) are satisfied by the matrix D if and only if (2.5), (2.7) and (2.8) are satisfied by the matrix B, respectively" is true. Hence, $D \in (m\colon f)$ and this yields by passing to the f-limit in (2.10) that $Bu \in f$. This means that every element of bs is almost B-summable, i.e. $B \in (bs\colon f)$.

Thus, the proof is completed.

As an immediate consequence of Theorem 2.2, we have the following corollary:

COROLLARY 2.3. *An infinite matrix B transforms bs into f_0 if and only if* (2.5), (2.6) *hold and* (2.7), (2.8) *also hold with $b_k = 0$ for each k.*

We conclude this section by giving a Steinhaus type theorem for f-regular and almost-coercive matrix classes.

THEOREM 2.4. *The classes of f-regular and almost-coercive matrices are disjoint.*

PROOF. Let us suppose that $(fs\colon f; P) \cap (bs\colon f) \neq \emptyset$ and B be an element of this intersection. Since $B \in (fs\colon f; P)$, the series $\sum_k \Delta b_{nk}$ and also $\sum_k 1/(q+1) \sum_{i=0}^{q} \Delta b_{n+i,k}$ are uniformly convergent in n. Now the condition iii) of Theorem 3.1 of [5] shows that

$$\text{(2.11)} \quad \lim_q \sum_k \frac{1}{q+1} \sum_{i=0}^{q} \Delta b_{n+i,k} = \lim_q \frac{1}{q+1} \sum_{i=0}^{q} b_{n+i,0} = f - \lim b_{n0} = 1$$

and if the same condition is considered in (2.8), then we get

$$\text{(2.12)} \qquad \lim_q \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} \Delta b_{n+i,k} \right| = 0\,, \text{ uniformly in } n\,.$$

Therefore, it is immediately seen by (2.12) that

$$\lim_q \left| \sum_k \frac{1}{q+1} \sum_{i=0}^{q} \Delta b_{n+i,k} \right| = 0, \text{ uniformly in } n$$

which contradicts (2.11) and this completes the proof.

3 – Almost-coercive series-to-series transformations

In the present section, we will give the characterization of the class $(bs : fs)$. Aftermore, we will give a corollary which corresponds to Corollary 2.3 for series-to-series transformations.

THEOREM 3.1. *An infinite matrix C transforms bs into fs if and only if*

$$\sup_n \sum_k \left| \sum_{i=0}^{n} \Delta c_{ik} \right| < \infty, \tag{3.1}$$

$$\lim c_{nk} = 0 \quad ; \quad (n = 0, 1, \ldots), \tag{3.2}$$

$$f - \lim \sum_{i=0}^{n} c_{ik} = c_k \quad ; \quad (k = 0, 1, \ldots), \tag{3.3}$$

$$\lim_q \sum_k \frac{1}{q+1} \left| \sum_{i=0}^{q} \sum_{j=0}^{n+i} \Delta(c_{jk} - c_k) \right| = 0, \text{ uniformly in } n. \tag{3.4}$$

PROOF. **Necessity.** Let $C \in (bs\colon fs)$ and $u \in bs$. The necessities of (3.2) and (3.3) may be established by the analogous argument of (2.6) and (2.7), respectively. Now, consider the following connection which is obtained in a similar way of (2.9);

$$(3.5)\qquad \sum_{i=0}^{n}\sum_{k=0}^{p} c_{ik}u_k = \sum_{k=0}^{p-1}\left(\sum_{i=0}^{n}\Delta c_{ik}\right)s_k + \sum_{i=0}^{n} c_{ip}s_p \quad ; \quad (n = 0,1,\ldots).$$

Letting $p \to \infty$ in (3.5), we get

$$(3.6)\qquad \sum_{i=0}^{n}\sum_{k} c_{ik}u_k = \sum_{k}\left(\sum_{i=0}^{n}\Delta c_{ik}\right)s_k \quad ; \quad (n = 0,1,\ldots).$$

Thus, it is seen by passing to f-limit in (3.6) that $E = (e_{nk}) \in (m\colon f)$, where $e_{nk} = \sum_{i=0}^{n}\Delta c_{ik}$ for all n, k. So, (2.1) and (2.4) are satisfied by the matrix E and these are equivalent to (3.1) and (3.4), respectively.

Sufficiency. Let us suppose that the matrix C satisfies (3.1) - (3.4) and $u \in bs$. Now, consider the matrix E in (3.6). Therefore, it is immediate that "(2.1), (2.2) and (2.4) are satisfied by the matrix E if and only if (3.1), (3.3) and (3.4) are satisfied by the matrix C, respectively." Hence, $E \in (m\colon f)$ and this yields by passing to f-limit in (3.6) that $Cu \in fs$. Then, $C \in (bs\colon fs)$ and this completes the proof.

Now, we may give the following corollary:

COROLLARY 3.2. *An infinite matrix C transforms bs into f_0s if and only if* (3.1), (3.2) *hold and* (3.3), (3.4) *also hold with $c_k = 0$ for each k.*

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