Rendiconti di Matematica, Serie VII
Volume 11 , Roma (1991), 283-294

Conservation Laws for the von Karman Equations of a Thin Plate

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RIASSUNTO – *Si considerano le equazioni di von Karman che approssimano le vibrazioni di una piastra sottile. Sfruttando il fatto che esse sono autoaggiunte, tramite una versione generalizzata del teorema di Noether, si calcolano tutte le leggi di conservazione in corrispondenza alle trasformazioni puntuali di queste equazioni.*

ABSTRACT – *The equations of motion and the associated Lagrangian density of one of the systems approximating the large deflection of plates known as the von Karman equations are considered. A generalized form of Noether's theorem is applied and a systematic approach is developed which allows the derivations of all the conservation laws in correspondence of the punctual transformations of such equations.*

KEY WORDS – *Conservation laws - von Karman equations.*

A.M.S. CLASSIFICATION: 35L25 - 80A20

1 – Introduction

In recent years there has been a renewed interest in the exploitation of Noether's theorem in continuum mechanics and especially in elasticity [1]. The reasons for this are the paucity of systematic and complete results [2] and the fact that conservation laws constitute a basic tool in the analysis of solution properties for any given differential system [3], [4].

The aim of this paper is to apply Noether's theorem, as generalized by

BESSEL-HAGEN [3], to compute all conservation laws relate to geometric symmetries of von Karman equations.

Considering the full Lie group of point transformations, computed by K.A. AMES and W.F. AMES in [5], for the above equations and exploiting the fact that these arise from a variational formulation [6], we compute all the transformations which leave the Lagrangian in the variational integral invariant only up to a divergence. In such a way we provide a detailed classification of all the conservation laws.

Untill now this invariance group has been exploited only to determine some exact solutions of these equations. To find this kind of solutions is not an easy task as it is pointed out in [5] and [7]; for this reason these conservation laws may be fundamental for the qualitative study of von Karman equations.

2 – The equations

The von Karman equations are a non linear treatment of a thin plate under normal pressure that takes into account the bending stresses, under the following assumptions: displacements are small compared to large dimensions of the plate, the normal displacement is constant throughout the thickness of the plate, the normal to the underformed midle surface remains the normal to the deformed middle surface, lateral load is perpendicular to the plate and there are no body forces in the plane of the plate [8]. To have an idea on the range of applicability of these equations the reader is referred to [9].

The following notation will be used: w is the normal displacement; ϕ the Airy stress function; E the modulus of elasticity; δ the flexural rigidity; h^* the thickness of the plate; q the lateral load intensity.

With this notation the equations read:

$$\phi_{xxxx} + 2\phi_{xxyy} + \phi_{yyyy} = E\left\{(w_{xy})^2 - w_{xx}w_{yy}\right\}, \tag{2.1}$$

$$w_{xxxx} + 2w_{xxyy} + w_{yyyy} = \frac{q}{\delta} + \frac{h^*}{\delta}\left\{\phi_{yy}w_{xx} + \phi_{xx}w_{xx} - 2\phi_{xy}w_{xy}\right\}. \tag{2.2}$$

We shall be interested in two cases: the first one which concerns the vibration of a plate whose deflections are large in comparison with h^*, i.e. $q = -\rho w_{tt}$ and the second one in which there is no lateral loading, i.e. $q = 0$.

It is important for our purpose to note that the equations (2.1) and (2.2) are the Euler-Lagrange equations of the functional [6]:

$$\mathcal{F} = \int_V L dV \,, \tag{2.3}$$

where

$$\begin{aligned} L =& E\left\{\frac{1}{2}(w_{xx})^2 + \frac{1}{2}(w_{yy})^2 + (w_{xx})^2\right\} + \\ & - \frac{h^*}{\delta}\left\{\frac{1}{2}(\phi_{xx})^2 + \frac{1}{2}(\phi_{yy})^2 + (\phi_{xy})^2\right\} + \\ & + \frac{Eh^*}{\delta}\phi\left\{(w_{xy})^2 - w_{xx}w_{yy}\right\} - \frac{1}{2}\frac{E\rho}{\delta}(w_{tt})^2 \,. \end{aligned} \tag{2.4}$$

3 – Symmetry Groups and Noether's Theorem

The theorem of Noether establishes a correspondence between transformations which leave invariant an action functional, up to a divergence term, and conservation laws of Euler-Lagrange equations associated with this functional [3]. Here we adopt the assumption to consider only smooth, (at least C^4), solutions of the von Karman equations and we introduce the compact notation $x^0 = t$, $x^1 = x$, $x^2 = y$.

Let:

$$\begin{aligned} \overline{x}^i &= x^i + \epsilon X^i(x^i, \phi, w) + O(\epsilon^2) \,, \\ \overline{\phi} &= \phi + \epsilon \Phi(x^i, \phi, w) + O(\epsilon^2) \,, \\ \overline{w} &= w + \epsilon W(x^i, \phi, w) + O(\epsilon^2) \,, \qquad i = 0, 1, 2, \end{aligned} \tag{3.1}$$

be a one-parameter Lie group of punctual transformations, with infinitesimal generators X^i, Φ and W. These generators determine the operator:

$$\mathcal{X} = X^i \frac{\partial}{\partial x^i} + W \frac{\partial}{\partial w} + \Phi \frac{\partial}{\partial \phi} \,.$$

The action of the group (3.1) on the first derivatives of unknown functions are determined using the formulae:

$$\tilde{\Phi}^i = D_i\Phi - \phi_k D_i X^k\,, \qquad \widetilde{W}^i = D_i W - w_k D_i X^k\,, \tag{3.2}$$

and on the second derivatives by the formulae:

$$\tilde{\Phi}^{ij} = D_j\tilde{\Phi}^i - \phi_{ki} D_j X^k\,, \qquad \widetilde{W}^{ij} = D_j\widetilde{W}^i - w_{ki} D_j X^k\,, \tag{3.3}$$

here the summation convention is in force over the range (0,1,2) and:

$$D_i = \frac{\partial}{\partial x^i} + \phi_i\frac{\partial}{\partial\phi} + w_i\frac{\partial}{\partial w} + \phi_{ij}\frac{\partial}{\partial\phi_j} + w_{ij}\frac{\partial}{\partial w_j}\,.$$

According to [3] (3.1) is a divergence variational symmetry, *DVS*, for the functional $\mathcal{F}$ if and only if:

$$\begin{aligned}&\int_V L(\bar{x}^i,\bar{\phi},\bar{w},\bar{\phi}_i,\bar{w}_i,\bar{\phi}_{ij},\bar{w}_{ij})(1+\epsilon X_i)dV+\\ &-\int_V L(x^i,\phi,w,\phi_i,w_i,\phi_{ij},w_{ij}) = \epsilon\int_V D_i\Gamma^i dV\,,\end{aligned} \tag{3.4}$$

where $\Gamma \equiv (\Gamma^0,\Gamma^1,\Gamma^2)$ denotes a vector whose components, here, are functions of x^i, ϕ, w and their derivatives up to second order. Keeping only terms up to first order in ϵ we obtain the infinitesimal invariance criterion [3]:

$$\begin{aligned}&X^i\frac{\partial L}{\partial x^i} + \Phi\frac{\partial L}{\partial\phi} + W\frac{\partial L}{\partial w} + \tilde{\Phi}^i\frac{\partial L}{\partial\phi_i} + \widetilde{W}^i\frac{\partial L}{\partial w_i}+\\ &+\tilde{\Phi}^{ij}\frac{\partial L}{\partial\phi_{ij}} + \widetilde{W}^{ij}\frac{\partial L}{\partial w_{ij}} + LD_iX^i = D_i\Gamma^i\,.\end{aligned} \tag{3.5}$$

Letting $Q^1 = \Phi - \phi_i X^i$, $Q^2 = W - w_i X^i$ and $\tilde{\Gamma} = \Gamma - LX$, we can rewrite (3.5) as:

$$\begin{aligned}&Q^1\frac{\partial L}{\partial\phi} + Q^2\frac{\partial L}{\partial w} + D_iQ^1\frac{\partial L}{\partial\phi_i} + D_iQ^2\frac{\partial L}{\partial w_i}+\\ &+D_{ij}Q^1\frac{\partial L}{\partial\phi_{ij}} + D_{ij}Q^2\frac{\partial L}{\partial w_{ij}} + = D_i\tilde{\Gamma}^i\,,\end{aligned} \tag{3.6}$$

where D_{ij} is the composition of operators D_i and D_j.

If (3.1) is a symmetry group for the functional $\mathcal{F}$ the relation (3.6) must hold. Performing an integration by parts we obtain the following identities:

$$(D_iQ^\alpha)\frac{\partial L}{\partial u_h^\alpha} = D_i\left(Q^\alpha \frac{\partial L}{\partial u_h^\alpha}\right) - Q^\alpha D_i \frac{L}{u_h^\alpha}, \tag{3.7}$$

$$\begin{aligned}(D_{ij}Q^\alpha)\frac{\partial L}{\partial u_{hk}^\alpha} =& Q^\alpha D_{ij}\frac{\partial L}{\partial u_{hk}^\alpha} + \frac{1}{2}\left\{ D_i\left(D_j(Q^\alpha)\frac{\partial L}{\partial u_{hk}^\alpha} - Q^\alpha D_j \frac{\partial L}{\partial u_{hk}^\alpha}\right) + \right.\\ &\left. + D_j\left(D_i(Q^\alpha)\frac{\partial L}{\partial u_{hk}^\alpha} - Q^\alpha D_i\frac{\partial L}{\partial u_{hk}^\alpha}\right)\right\},\end{aligned} \tag{3.8}$$

where the Greek index is understood in the range (1,2) and the Latin indices in the range (0,1,2), here $u^1 = \phi$ and $u^2 = w$. Substituting (3.7), (3.8) into (3.6), this last relation, for the particular case of von Karman equations, reads as follows:

(3.9)

$$\begin{aligned}&Q^1\left\{\frac{\partial L}{\partial \phi} + D_{xx}\frac{\partial L}{\partial \phi_{xx}} + D_{xy}\frac{\partial L}{\partial \phi_{xy}} + D_{yy}\frac{\partial L}{\partial \phi_{yy}}\right\} - Q^2\left\{D_t\frac{\partial L}{\partial w_t} - D_{xx}\frac{\partial L}{\partial w_{xx}} + \right.\\
&\left. - D_{xy}\frac{\partial L}{\partial w_{xy}} - D_{yy}\frac{\partial L}{\partial w_{yy}}\right\} + D_t\left\{Q^2\frac{\partial L}{\partial w_t}\right\} + D_x\left\{D_x(Q^1)\frac{\partial L}{\partial \phi_{xx}} - Q^1 D_x\frac{\partial L}{\partial \phi_{xx}} + \right.\\
&+\frac{1}{2}D_y(Q^1)\frac{\partial L}{\partial \phi_{xy}} - \frac{1}{2}Q^1 D_y\frac{\partial L}{\partial \phi_{xy}} + D_x(Q^2)\frac{\partial L}{\partial w_{xx}} - Q^2 D_x\frac{\partial L}{\partial w_{xx}} + \\
&\left. +\frac{1}{2}D_y(Q^2)\frac{\partial L}{\partial w_{xy}} - \frac{1}{2}Q^2 D_y\frac{\partial L}{\partial w_{xy}}\right\} + D_y\left\{D_y(Q^1)\frac{\partial L}{\partial \phi_{yy}} - Q^1 D_y\frac{\partial L}{\partial \phi_{yy}} + \right.\\
&+\frac{1}{2}D_x(Q^1)\frac{\partial L}{\partial \phi_{xy}} - \frac{1}{2}Q^1 D_x\frac{\partial L}{\partial \phi_{xy}} + D_y(Q^2)\frac{\partial L}{\partial w_{yy}} - Q^2 D_y\frac{\partial L}{\partial w_{yy}} + \\
&\left. +\frac{1}{2}D_x(Q^2)\frac{\partial L}{\partial w_{xy}} - \frac{1}{2}Q^2 D_x\frac{\partial L}{\partial w_{xy}}\right\} = D_t\tilde{\Gamma}_0 + D_x\tilde{\Gamma}_1 + D_y\tilde{\Gamma}_2 .\end{aligned}$$

When $\{\phi, u\}$ is a solution of the Euler-Lagrange equations then (3.9) shows that:

$$P^0 \equiv Q^2 \frac{\partial L}{\partial w_t} - \Gamma_0 + L X_0 , \tag{3.10}$$

$$\begin{aligned} P^1 \equiv & D_x(Q^1)\frac{\partial L}{\partial \phi_{xx}} - Q^1 D_x \frac{\partial L}{\partial \phi_{xx}} + \frac{1}{2} D_y(Q^1)\frac{\partial L}{\partial \phi_{xy}} + \\ & - \frac{1}{2} Q^1 D_y \frac{\partial L}{\partial \phi_{xy}} + D_x(Q^2)\frac{\partial L}{\partial w_{xx}} - Q^2 D_x \frac{\partial L}{\partial w_{xx}} + \\ & + \frac{1}{2} D_y(Q^2)\frac{\partial L}{\partial w_{xy}} - \frac{1}{2} Q^2 D_y \frac{\partial L}{\partial w_{xy}} - \Gamma_1 + L X_1 , \end{aligned} \tag{3.11}$$

$$\begin{aligned} P^2 \equiv & D_y(Q^1)\frac{\partial L}{\partial \phi_{yy}} - Q^1 D_y \frac{\partial L}{\partial \phi_{yy}} + \frac{1}{2} D_x(Q^1)\frac{\partial L}{\partial \phi_{xy}} + \\ & - \frac{1}{2} Q^1 D_x \frac{\partial L}{\partial \phi_{xy}} + D_y(Q^2)\frac{\partial L}{\partial w_{yy}} - Q^2 D_y \frac{\partial L}{\partial w_{yy}} + \\ & + \frac{1}{2} D_x(Q^2)\frac{\partial L}{\partial w_{xy}} - \frac{1}{2} Q^2 D_x \frac{\partial L}{\partial w_{xy}} - \Gamma_2 - L X_2 , \end{aligned} \tag{3.12}$$

are the components of the conserved current for such equations.

4 – Conservation laws

It is very important to note that if (3.1) is a *DVS* group of the functional $\mathcal{F}$ then (3.1) is a symmetry group for the Euler-Lagrange equations of this functional [3]. Then the full Lie group, which leaves (2.3) invariant up to a divergence term, is a subgroup of the full Lie group admitted by (2.1), (2.2), which has been computed by K.A. AMES and W.F. AMES in [5].

Here we reproduce the final form of the generators X_k, Φ and W which is given by:

$$(X^0, X^1, X^2) = (a_1 + 2a_4 t, a_2 + a_4 x + a_5 y, a_3 + a_4 y - a_5 x) , \tag{4.1}$$

(4.2) $(\Phi, W) = (xf_1(t)+yf_2(t)+f_3(t), a_6+a_7t+a_8x+a_9y+a_{10}tx+a_{11}ty)$,

where f_i are three arbitrary functions of t and a_i are eleven arbitrary parameters.

Starting from (4.1) and (4.2) we can establish which transformations admitted by the equations (2.1) and (2.2) are also admitted, up to a divergence term, by the functional (2.3) to compute in correspondence of such transformations the conservation laws of the von Karman equations. By using the infinitesimal criterion (3.5) and after some calculation we find that the *DVS* group of the functional (2.3) is the full Lie group (3.11), (3.12) with:

$$\begin{aligned}\Gamma = \frac{Eh^*}{\delta}\Bigg(&\frac{\rho w}{h^*}(a_7 + xa_{10} + ya_{11}), (xw_{xy}w_y - \frac{1}{2}w_y^2)f_1 + \\ &- yw_xw_{yy}f_2 - w_xw_{yy}f_3\,, -w_yw_{xx}f_1 + \\ &+ \left(yw_{xy}w_x - \frac{1}{2}w_x^2\right)f_2 + w_{xy}w_xf_3\Bigg)\,. \end{aligned} \tag{4.3}$$

This means that the transformation groups we obtain in correspondence of each parameter $a_1, a_2, a_3, a_4, a_5, a_5, a_6, a_8, a_9$ leave invariant $\mathcal{F}$ in the classical sense while the transformation groups in correspondence with the arbitrary functions and a_7, a_{10}, a_{11} are Bessel-Hagen's divergence symmetry groups.

The generators (4.1) and (4.2) in evolutionary form read:

$$\begin{aligned} Q^1 \equiv\ & xf_1(t) + yf_2(t) + f_3(t) - (a_1 + 2a_4t)\phi_t + \\ & - (a_2 + a_4x + a_5y)\phi_x - (a_3 + a_4y - a_5x)\phi_y\,, \end{aligned} \tag{4.4}$$

$$\begin{aligned} Q^2 \equiv\ & a_6 + a_7t + a_8x + a_9y + a_{10}tx + a_{11}ty - (a_1 + 2a_4t)w_t + \\ & - (a_2 + a_4x + a_5y)w_x - (a_3 + a_4y - a_5x)w_y\,. \end{aligned} \tag{4.5}$$

In this way we can now write by means of (3.10) the conserved currents P^i:

$$P^0 \equiv \rho w_t Q^2 - \frac{\delta}{E}L(a_1 + 2a_4t) - \rho w(a_7 + xa_{10} + ya_{11})\,, \tag{4.6}$$

$$
\begin{aligned}
P^1 \equiv{}& \frac{h^*}{E}\{-\phi_{xx}D_xQ^1 - \phi_{xy}D_yQ^1 + (\phi_{xxx}+\phi_{xyy})Q^1\} + \\
&+\delta\Big\{\left(w_{xx} - \frac{h^*}{\delta}\phi w_{yy}\right)D_xQ^2 + \left(1+\frac{h^*}{\delta}\phi\right)w_{xy}D_yQ^2 + \\
&-\left[w_{xxx} + w_{xyy} + \frac{h^*}{\delta}(\phi_y w_{xy} - \phi_x w_{yy})\right]Q^2\Big\} + \\
&-h^*\left((xw_{xy}w_y - \frac{1}{2}w_y^2)f_1 + yw_xw_{yy}f_2 + w_xw_{yy}f_3\right) + \\
&+\frac{\delta}{E}(a_2 + a_4x + a_5y)L,
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
P^2 \equiv{}& \frac{h^*}{E}\{-\phi_{xy}D_xQ^1 - \phi_{yy}D_yQ^1 + (\phi_{yyy}+\phi_{xxy})Q^1\} + \\
&+\delta\Big\{\left(1+\frac{h^*}{\delta}\phi\right)w_{xy}D_xQ^2 + \left(w_{yy} - \frac{h^*}{\delta}\phi w_{xx}\right)D_yQ^2 + \\
&-\left[w_{yyy} + w_{xyx} + \frac{h^*}{\delta}(\phi_x w_{xy} - \phi_y w_{xx})\right]Q^2\Big\} + \\
&+h^*\left(xw_{xx}w_yf_1 - (yw_xw_{xy} - \frac{1}{2}w_x^2)f_2 - w_xw_{xy}f_3\right) + \\
&+\frac{\delta}{E}(a_3 + a_4y - a_5x)L
\end{aligned}
\tag{4.8}
$$

The conserved currents in (4.4), (4.5) and (4.6) are the components of the full family of conservation laws related to the geometric symmetries of the von Karman equations. This family depends upon eleven arbitrary parameters and three arbitrary functions, none of these conservation laws is trivial in the sense of [3] i.e. P^i do not vanish identically on the solutions of (2.1) and (2.2) and $D_iP^i = 0$ holds exclusively on the solutions of the equations. However the vector fields X^{f_1}, X^{f_2}, X^{f_3} generate three conservation laws that differ each other for a trivial one.

In the case in which there is no lateral loading, i.e. $q = 0$ and then $w = w(x,y)$, $\phi = \phi(x,y)$, the families of conservation laws are obtained from (4.7) and (4.8) by setting:

$$a_1 = a_7 = a_{10} = a_{11} = 0\,, \qquad f_1(t) = k_1\,, \qquad f_2(t) = k_2\,, \qquad f_3(t) = k_3\,,$$

with k_i arbitrary constants and replacing the vector field with operator:

$$\mathcal{X}^4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,,$$

with its "steady" version with operator:

$$\mathcal{X}^4_s = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,.$$

The physical interpretation of conservation laws found here can be derived observing their densities:

– *t-Translations*

The vector field with operator $\mathcal{X}^1 = \frac{\partial}{\partial t}$ give rise to the conservation law which density is:

$$P^0_1 = \rho w_t^2 - \frac{\delta}{E}L\,, \tag{4.9}$$

as it is well known that corresponds to conservation of energy [10].

– *x, y-Translations*

The vector fields with operators $\mathcal{X}^2 = \frac{\partial}{\partial x}$, $\mathcal{X}^3 = \frac{\partial}{\partial y}$ generate the laws with densities:

$$P^0_2 = \rho w_t w_x\,, \qquad P^0_3 = \rho w_t w_y\,, \tag{4.10}$$

these conservation laws are the components of the energy-momentum tensor and they establish the famous Rice's path independent integral that when integrated around the tip of a crack determines the associated energy-release rate [11].

– *x, y-Rotations*

The vector field with operator $\mathcal{X}^5 = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$ is in correspondence with the density:

$$P_5^0 = \rho w_t(y w_x - x w_y)\,, \tag{4.11}$$

this conservation law is an other path independent integral related to energy-release rates associated with cavity or crack rotation and is a consequence of the isotropy of the body [12].

- *Dilatations*

The vector field with operator $\mathcal{X}^4 = 2t\dfrac{\partial}{\partial t} + x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$ gives rise to the conservation law with density:

$$P_4^0 = -\rho w_t(2t w_t + x w_x + y w_y) + 2\frac{\delta}{E} t L\,, \tag{4.12}$$

this also is a path independent integral related to energy-release rates of the expansion of cavities or cracks [12].

- *w-Translation*

The vector field with operator $\mathcal{X}^6 = \dfrac{\partial}{\partial w}$ is associated with the conservation of linear momentum; indeed the density is:

$$P_6^0 = \rho w_t\,, \tag{4.13}$$

this vector field and the one obtained in correspondence with the operator $\mathcal{X}^{f_3} = f_3\dfrac{\partial}{\partial \phi}$ allow us to rewrite in potential form the von Karman equations.

- *Rigid Body Rotations*

The natural conservation laws of angular momentum are obtained in correspondence of the vector fields with operators $\mathcal{X}^8$ and $\mathcal{X}^9$. The densities of such laws are:

$$P_8^0 = \rho x w_t\,, \qquad P_9^0 = \rho y w_t\,. \tag{4.14}$$

- *Galileian boost*

The vector field with operator $\mathcal{X}^7 = t\dfrac{\partial}{\partial w}$ gives rise to the density:

$$P_7^0 = \rho(tw_t - w)\,, \tag{4.15}$$

this law is related to the motion of the center of mass and it is known as the center-of-mass theorem [10]. In absence of external forces acting on the system this theorem is a direct consequence of Dynamic's fundamental equations.

5 – Comments and conclusions

In this paper we have obtained the full class of conservation laws related to geometric symmetries of von Karman equations, by the means of Noether's theorem. These laws are a generalization of certain path-independent integrals in linear elasticity which have been of considerable practical interest [13]. The conservation law (4.15) at first sight seems to be new and have no analogous in linear elasticity, in reality, in absence of body forces, it is enclosed in the infinite-dimensional family of conservation laws peculiar to every linear self-adjoint system of differential equations known as reciprocity.

It is interesting to compare our work with the recent paper of ŞUHUBI [14] devoted to the derivation of conservation laws associated with non-linear elastodynamics through an approach based on the calculus of exterior differential forms. The conservation laws found in [14] are a subclass of the ones found here; the laws in common are the direct consequences of the natural balance laws and the path-independent integral (4.10), all the other ones seems to be news for a nonlinear elastodynamical theory.

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Lavoro pervenuto alla redazione il 9 maggio 1990
ed accettato per la pubblicazione il 1° ottobre 1990
su parere favorevole di P. Bassanini e di R. Balli

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