Rendiconti di Matematica, Serie VII
Volume 11 , Roma (1991), 295-311

Notes on $\aleph_\alpha$-Lindelöf Spaces

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RIASSUNTO – *Vengono generalizzate ad una cardinalità più elevata delle ben note caratterizzazioni, come il teorema di Aquaro e quello di Aull, di alcuni spazi $\aleph_1$-compatti come spazi di Lindelöf.*

ABSTRACT – *Some well known characterizations such as Aquaro's theorem and Aull's theorem on being Lindelöf of certain $\aleph_1$-compact spaces are given this time in higher cardinality forms.*

KEY WORDS – *$\aleph_\alpha$-compact - $\aleph_\alpha$-Lindelöf - $\aleph_\alpha$-developable - weakly $\aleph_\alpha$-Lindelöf.*

A.M.S. CLASSIFICATION: 54D20

1 – Introduction and preliminaries

The existence of at least one complete accumulation point of certain subsets has under been guaranteed and even is the characteristic property of certain spaces whose open coverings admit certain subcoverings. The following joint statements contain two well known assertions related with this purpose:

THEOREM 1. *A topological space is countably compact (resp. compact) iff every countably infinite (resp. infinite) subset has at least one complete accumulation point.*

This characterization of countably compactness via- ω-limit points is known as Bolzano-Weierstrass theorem. The other characterization written in the parenthesis is due to KELLEY, see [12]. It is also well known that every Lindelöf space X is $\aleph_1$-compact i.e. all closed-discrete subspaces of X are countable or equivalently every uncountable subset of X has at least one accumulation i.e. limit point. Lindelöfness of certain $\aleph_1$-compact spaces were conversely established by the first two results in the following which were discovered respectively by JONES and AQUARO, see [11] and [1]. The third one in the following is due to AULL and generalizing these two, see [3].

THEOREM 2. *A developable T_1 space is Lindelöf iff it is $\aleph_1$-compact.*

THEOREM 3. *A T_1 space is Lindelöf iff it is $\aleph_1$-compact and meta-Lindelöf.*

THEOREM 4. *A T_1 space is Lindelöf iff it is $\aleph_1$-compact and $\delta\theta$-refinable.*

The following two sufficiency conditions on being $\aleph_1$-compact of separable spaces are due to JONES [11] and HEATH [9] respectively.

THEOREM 5 - [WCH]. *Separable normal spaces are $\aleph_1$-compact.*

THEOREM 6. *Separable meta-Lindelöf spaces are $\aleph_1$-compact.*

WCH in above denotes the weak continuum hypothesis (see page 171 of [16]) which asserts that $2^{\aleph_0} < 2^{\aleph_1}$ is taken as true in that theorem. Similarly GCH denotes the generalized continuum hypothesis saying $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for each ordinal number α.

We want to give in this note, analogous results of these theorems for higher cardinals beyond of some other results. No separation axiom is assumed unless otherwise is explicitly stated. A topological space X is called $\aleph_\alpha$*-Lindelöf* iff each open covering admits a subcovering containing at most $\aleph_\alpha$ number of members [8], [6]. X is called $\aleph_\alpha$*-compact* iff each subset of cardinality $\aleph_\alpha$ has at least one accumulation point or equivalently cardinalities of all closed-discrete subspaces are less than $\aleph_\alpha$. This concept is weaker than the one given by KÖTHE, see [13]. ω_α will denote the least ordinal number of cardinality $\aleph_\alpha$ as usual. The greek letters are provisioned for denoting the ordinal numbers throughout the note. A family of open coverings $(\mathcal{G}_\beta)_{\beta<\omega_\alpha}$ of X is called an $\aleph_\alpha$-*development* for X iff $\mathcal{B}_x = \left\{\mathrm{st}(x,\mathcal{G}_\beta)\right\}_{\beta<\omega_\alpha}$ is a local basis for each $x \in X$. HODEL has proved the following in [10] for these spaces:

THEOREM 7. *A regular space X has a basis which is the union of $\aleph_\alpha$ number of open-discrete families iff it has an $\aleph_\alpha$-development so that $\mathcal{B}_x = \left\{\mathrm{st}^2(x,\mathcal{G}_\beta)\right\}_{\beta<\omega_\alpha}$ is a local basis for each $x \in X$.*

Spaces possessing an $\aleph_\alpha$-development are called as $\aleph_\alpha$-*developable* in this note. Recall that *developable spaces* are noting but $\aleph_0$-developable ones. Iterated starsets are defined as $\mathrm{st}^n(x,\mathcal{G}) = \mathrm{st}(\mathrm{st}^{n-1}(x,\mathcal{G}),\mathcal{G})$ for any point x and for any family $\mathcal{G}$. X is called *meta* $\aleph_\alpha$-Lindelöf [15] iff each open covering of X admits an open refinement $\mathcal{G}$ such that $\mathrm{ord}(x,\mathcal{G}) = \mathrm{card}\left\{G \in \mathcal{G}\colon x \in G\right\} \le \aleph_\alpha$ holds for each $x \in X$. A space X is called *weakly* $\aleph_\alpha$-*Lindelöf* iff each open covering of X admits a subfamily $\mathcal{G}$ such that $\mathrm{card}\,\mathcal{G} \le \aleph_\alpha$ and the union $\bigcup\mathcal{G} = \bigcup\left\{G\colon G \in \mathcal{G}\right\}$ of its members is dense in X. Thus weakly $\aleph_\alpha$-Lindelöf spaces are nothing but weakly $\aleph_{\alpha+1}$-compact spaces of ULMER [17]. A space X is called $\aleph_\alpha\aleph_\beta$-*refinable* iff each open covering admits a refinement $\bigcup_{\mu<\omega_\alpha}\mathcal{G}_\mu$ which is the union of $\aleph_\alpha$ number of open families such that each point $x \in X$ has order $\mathrm{ord}\left(x,\mathcal{G}_{\mu(x)}\right) \le \aleph_\beta$ for an appropriate index $\mu(x) < \omega_\alpha$. The *density* (resp, *weight*) of any space X is the minimum of all cardinal numbers $\mathrm{card}\,D$ (resp. $\mathrm{card}\,\mathcal{B}$) over all dense subsets D of X (resp. over all basis $\mathcal{B}$ of X). The *cellularity* of X is the supremum of all cardinal numbers $\mathrm{card}\,\mathcal{G}$ over all families $\mathcal{G}$ of X with nonempty and pairwise disjoint open members, whereas, the *character* of X is the supremum of its local character numbers $\chi(x,X)$ where $\chi(x,X)$ is the minimum of numbers $\mathrm{card}\,\mathcal{B}_x$ over all local basis $\mathcal{B}_x$ of the point x in X. Notice that the density number and weight are always assumed in any space. See [7] for instance for these cardinal functions. Semi-regularization space, almost open functions and some other concepts are defined in the last section which they mentioned and used. The standard reference book is [12]. We give now some examples on all these.

REMARK 1. All countably compact spaces are $\aleph_0$-compact. Countable compactness and $\aleph_0$-compactness coincide on T_1 spaces. An $\aleph_\alpha$-Lindelöf space is evidently $\aleph_{\alpha+1}$-compact and also is $\aleph_\beta$-Lindelöf for each $\alpha \leq \beta$. Notice that $X = [0,\omega_\beta[$ equipped with the usual order topology is an $\aleph_\beta$-Lindelöf but non $\aleph_\alpha$-Lindelöf space, if $\alpha < \beta$, after considering the open covering with members $G_\mu = [0,\mu[$ $(\mu < \omega_\beta)$. $Y = [0,\omega_{\alpha+2}[$ is an example of $\aleph_{\alpha+1}$-compact non $\aleph_\alpha$-Lindelöf space. Any $\aleph_\alpha$-compact space is naturally $\aleph_\beta$-compact if $\alpha \leq \beta$. The topology on the *set* $Z = [0,\omega_{\alpha+1}[$ which is generated by the whole subsets $A \subset Z$ satisfying $\operatorname{card}(Z - A) \leq \aleph_\alpha$ is an $\aleph_{\alpha+1}$-compact T_1 space but it is not $\aleph_\alpha$-compact. $\aleph_\alpha$-Lindelöf spaces are evidently meta $\aleph_\alpha$-Lindelöf. The free sum space $\sum_{\mu<\omega_{\alpha+1}} X_\mu$ is evidently a meta $\aleph_\alpha$-Lindelöf non $\aleph_\alpha$-Lindelöf space, where, each X_μ is $\mathbb{R}^1 \times \{\mu\}$ and $\mathbb{R}^1$ is the one dimensional Euclidean space. All paracompact even all metacompact spaces and in particularly all pseudometrizable spaces are meta-Lindelöf but they are not necessarily Lindelöf. All non separable pseudometrizable spaces are such spaces. The paper [5] contain some examples of weakly Lindelöf, non Lindelöf spaces. Notice that all subsets of cardinality less or equal to $\aleph_{\alpha+1}$ in space $Y = [0,\omega_{\alpha+2}[$ defined in above has complete accumulation points, but none of the subsets with cardinality $\aleph_{\alpha+2}$ have such kind of accumulation points. The σ-compact space $[0,\omega_0] \cup \bigcup_{n=0}^{\infty} [\omega_n^+,\omega_{n+1}]$ is evidently a Lindelöf space, but uncountable subsets with cardinality $\aleph_{\omega_0}$ in this space has no complete accumulation points. Finally every regular space is $\omega(X)$-developable after considering the development containing all the open coverings $\mathcal{G}(B_1,B_2) = \{B_1, X - \overline{B}_2 : \overline{B}_2 \subseteq B_1, (B_1,B_2) \in \mathcal{B} \times \mathcal{B}\}$ whereas the basis $\mathcal{B}$ is satisfying $\operatorname{card}\mathcal{B} = \omega(X)$.

2 – On $\aleph_\alpha$-Lindelöfness

Being $\aleph_\alpha$-Lindelöf of certain $\aleph_{\alpha+1}$-compact spaces is studied.

PROPOSITION 1. *A topological space is $\aleph_\alpha$-Lindelöf if each subset of cardinality greater than $\aleph_\alpha$ has at least one complete accumulation point.*

PROOF. Suppose that X is not $\aleph_\alpha$-Lindelöf. Then there exists at least one open covering $\mathcal{G} = (G_\beta)_{\beta<\beta_0}$ of X with no subcovering of cardinality less or equal to $\aleph_\alpha$. One can suppose without loosing the generality that

$\mathcal{G}$ is the one with the least cardinality among all open coverings of X with this property and therefore its well ordered indexing set $I = [0, \beta_0[$ has the cardinality card $I = \aleph$ which is greater than $\aleph_\alpha$ i.e. $\omega_\alpha < \beta_0$ and each proper segments of it has cardinality less than $\aleph$. So $\aleph \leq \text{card}\,(X - \bigcup_{\beta<\mu} G_\beta)$ holds for each $\mu < \beta_0$ by this supposition and consequently the subset A of all selected points

$$x_\mu \in X - \left[\bigcup_{\beta<\mu} G_\beta \cup \{x_\gamma : \gamma < \mu\}\right] \quad , \qquad \forall\, \mu < \beta_0$$

is well defined by the axiom of choice. Its cardinality satisfies card $A = \aleph > \aleph_\alpha$ but A has no complete accumulation point in X since $G_\beta \cap A \subseteq \{x_\mu : \mu \leq \beta\}$ holds for each member G_β of $\mathcal{G}$ i.e. for each index $\beta < \beta_0$. Thus the statement follows.

COROLLARY 1. *A topological space is Lindelöf if each uncountable subset has at least one complete accumulation point.*

COROLLARY 2 - [GCH]. *A T_2 space with $\aleph_n$ character is Lindelöf iff each uncountable subset has at least one complete accumulation point.*

PROOF. Only the necessity requires a proof after Corollary 1. If a T_2 space X with character $\aleph_n$ is a Lindelöf space then its cardinality is not greater than $2^{\aleph_0 \cdot \aleph_n} = \aleph_{n+1}$ by the well known theorem of ARHANGEL'SKII under GCH, see Theorem 3 of [2]. Hence cardinalities of all uncountable subsets of X are successor cardinalities and therefore they evidently have at least one complete accumulation point since X is Lindelöf.

COROLLARY 3 - [CH]. *A first countable T_2 space is Lindelöf iff each uncountable subset has at least one complete accumulation point.*

COROLLARY 4 - [CH]. *A developable T_2 space is Lindelöf iff each uncountable subset has at least one complete accumulatioon point.*

PROPOSITION 2. *In any $\aleph_\alpha$- Lindelöf space each subset of cardinality $\aleph_{\beta+1} (\beta \geq \alpha)$ has at least one complete accumulation point.*

PROOF. Notice only that $\aleph_{\beta+1} = \aleph_\beta^+$ is always a successor cardinality.

PROPOSITION 3. *An $\aleph_\alpha$-developable T_1 space is $\aleph_\alpha$-Lindelöf iff it is $\aleph_{\alpha+1}$-compact.*

PROOF. This statement will be an easy consequence of Proposition 4 and Proposition 6 but we give here an independent proof which is a modification of JONES's arguments. Let X be a T_1 space with an $\aleph_\alpha$-development $(\mathcal{G}_\beta)_{\beta<\omega_\alpha}$. We are going to prove that X is $\aleph_\alpha$-Lindelöf if it is $\aleph_{\alpha+1}$-compact. Let us take an open covering $\mathcal{U} = (U_i)_{i\in I}$ of X. We suppose that X as well as I are well ordered by the well known Zermelo's principle. Let X_β be the subset of all points x such that $\mathrm{st}(x,\mathcal{G}_\beta)$ is contained in a suitable member of $\mathcal{U}$. Notice that X_β could be empty for some β but $X = \bigcup_{\beta<\omega_\alpha} X_\beta$. Now let us take a fixed $\beta < \omega_\alpha$ and define $x_{\beta 1} = \min X_\beta$ where $\mathrm{st}(x_{\beta 1},\mathcal{G}_\beta) \subseteq U_{\beta 1} \in \mathcal{U}$. If $X_\beta - U_{\beta 1}$ is non empty define then $x_{\beta 2} = \min(X_\beta - U_{\beta 1})$ and $U_{\beta 2} \in \mathcal{U}$ with $\mathrm{st}(x_{\beta 2},\mathcal{G}_\beta) \subseteq U_{\beta 2}$. Let all the points $x_{\beta i}$ and members $U_{\beta i} \in \mathcal{U}$ be defined for each ordinal number $i < j$. Define the point $x_{\beta j}$ and $U_{\beta j} \in \mathcal{U}$ as follows

$$x_{\beta j} = \min\left(X_\beta - \bigcup_{i<j} U_{\beta i}\right), \qquad \mathrm{st}(x_{\beta j},\mathcal{G}_\beta) \subseteq U_{\beta j}$$

if the difference set in above is still nonempty. This process continues only at most $\aleph_\alpha$ number of steps, since X is an $\aleph_{\alpha+1}$-compact T_1 space and the subset of all points $x_{\beta i}$ is a closed-discrete subset of X, for any member of the open covering $\mathcal{G}_\beta$ contains at most one $x_{\beta i}$. In fact if $i_1 < i_2$ and $x_{\beta i_1}$, $x_{\beta i_2}$ are belong to the same member of $\mathcal{G}_\beta$ then the following contradiction would be obtained:

$$x_{\beta i_2} \in \mathrm{st}(x_{\beta i_1},\mathcal{G}_\beta) - \bigcup_{i<i_2} U_{\beta i} \subseteq U_{\beta i_1} - \bigcup_{i<i_2} U_{\beta i} = \emptyset .$$

Therefore each $X_\beta(\beta < \omega_\alpha)$ is covered by at most $\aleph_\alpha$ number of $U_{\beta i} \in \mathcal{U}$ and so X is covered by at most $\aleph_\alpha$ number of members of $\mathcal{U}$.

PROPOSITION 4. *Every open covering of an $\aleph_\alpha$-developable space has a refinement which is the union of $\aleph_\alpha$ number of closed-discrete families.*

PROOF. Let the development $(\mathcal{G}_\beta)_{\beta<\omega_\alpha}$ be defined on X and the open covering $\mathcal{U} = (U_i)_{i\in I}$ be given. Let the well ordering $<$ be defined on I. Then for each $\beta < \omega_\alpha$ and $i \in I$, define the following closed sets

$$K_{\beta i} = \{x \in X : \mathrm{st}(x, \mathcal{G}_\beta) \subseteq U_i\} - \bigcup_{j<i} U_j .$$

Notice that the subset $K(A,\mathcal{G}) \doteq \{x \in X : \mathrm{st}(x,\mathcal{G}) \subseteq A\}$ is always closed for any open covering $\mathcal{G}$ of X and for any subset $A \subseteq X$, since if $x_0 \notin K(A,\mathcal{G})$ then there exists at least one $G_0 \in \mathcal{G}$ contains x_0 and not contained in A and so $G_0 \cap K(A,\mathcal{G}) = \emptyset$ is obtained. The family $\mathcal{K}_\beta = (K_{\beta i})_{i\in I}$ is discrete for each $\beta < \omega_\alpha$ since if a member G of the open covering $\mathcal{G}_\beta$ intersects $K_{\beta i_0}$ then it doesn't intersect all the other $K_{\beta i}$ sets. In fact by using any point $x \in G \cap K_{\beta i_0}$ one gets $G \subseteq \mathrm{st}(x, \mathcal{G}_\beta) \subseteq U_{i_0}$ and consequently the following inclusions hold for any $i > i_0$

$$G \cap K_{\beta i} \subseteq U_{i_0} \cap K_{\beta i} \subseteq U_{i_0} - \bigcup_{j<i} U_j = \emptyset .$$

In particularly $G \cap K_{\beta i}$ is empty for each $i < i_0$. The family $\bigcup_{\beta<\omega_\alpha} \mathcal{K}_\beta$ refines $\mathcal{U}$ and covers X since $\{\mathrm{st}(x,\mathcal{G}_\beta)\}_{\beta<\omega_\alpha}$ is a local basis for each $x \in X$ and $x \in U_{i(x)}$ where $i(x) = \min\{i \in I : x \in U_i\}$.

PROPOSITION 5. *A T_1 space is $\aleph_{\alpha+1}$-compact iff every discrete family of subsets contains at most $\aleph_\alpha$ number of members.*

PROOF. Let $\mathcal{A} = (A_\beta)$ be any discrete family of subsets in an $\aleph_{\alpha+1}$-compact T_1 space X. We naturally suppose that each A_β is nonempty. The set X_0 of the all selected points $a_\beta \in A_\beta$ is closed and discrete in X since $\mathcal{A}$ is a discrete family in X and $X_0 = \bigcup\{a_\beta\} = \bigcup\overline{\{a_\beta\}} = \overline{\bigcup\{a_\beta\}} = \overline{X_0}$. Thus card $\mathcal{A}$ = card $X_0 \le \aleph_\alpha$ is found since members of $\mathcal{A}$ are pairwise disjoint and X is an $\aleph_{\alpha+1}$-compact space. Sufficiency is easy since all the singletons of any subset without any limit point constitute a discrete family in any space.

PROPOSITION 6. *A T_1 space in which every open covering has a refinement of the union of $\aleph_\alpha$ number of closed-discrete families is $\aleph_\alpha$-Lindelöf iff it is $\aleph_{\alpha+1}$-compact.*

PROOF. Notice that the proof of sufficiency is easy since the refinement $\mathcal{K} = \bigcup_{\beta<\omega_\alpha} \mathcal{K}_\beta$ mentioned in hypothesis contains at most $\aleph_\alpha$ number of members after the previous proposition and one should only then define a unique and well defined open superset, belonging to open covering for each member of $\mathcal{K}$.

PROPOSITION 7. *The following are equivalent in any space X with an $\aleph_\alpha$-development $(\mathcal{G}_\beta)_{\beta<\omega_\alpha}$ if $\mathcal{B}_x = \{\mathrm{st}^2(x,\mathcal{G}_\beta)\}_{\beta<\omega_\alpha}$ is a local basis for each $x \in X$:*

i) *X is $\aleph_\alpha$-Lindelöf.*

ii) *X is weakly $\aleph_\alpha$-Lindelöf.*

iii) *X has $\aleph_\alpha$-cellylarity.*

iv) *X has $\aleph_\alpha$-density.*

PROOF. Let X be an $\aleph_\alpha$-developable space so that the family $\mathcal{B}_x$ written in hypothesis is a local basis for each $x \in X$. Notice first of all that, there exists an index $\mu = \mu(x,\beta) < \omega_\alpha$ for each $x \in X$ and $\beta < \omega_\alpha$ so that $\mathrm{st}^2(x,\mathcal{G}_\mu) \subseteq \mathrm{st}(x,\mathcal{G}_\beta)$. Then the common refinement $\mathcal{G}_{\mu\beta} = \{G_\mu \cap G_\beta : G_\mu \in \mathcal{G}_\mu, G_\beta \in \mathcal{G}_\beta\}$ of $\mathcal{G}_\mu$ and $\mathcal{G}_\beta$ satisfies

$$\mathrm{st}^3(x,\mathcal{G}_{\mu\beta}) = \mathrm{st}\,(\mathrm{st}^2(x,\mathcal{G}_{\mu\beta}),\mathcal{G}_{\mu\beta}) \subseteq \mathrm{st}\,(\mathrm{st}(x,\mathcal{G}_\beta),\mathcal{G}_\beta) = \mathrm{st}^2(x,\mathcal{G}_\beta)\,.$$

Hence without loosing the generality the family $\mathcal{B}_x^* = \{\mathrm{st}^3(x,\mathcal{G}_\beta)\}_{\beta<\omega_\alpha}$ could be accepted as a local basis at $x \in X$ from the very begining; otherwise one can work with the families $\mathcal{G}_{\mu\beta}$ of all common refinements as an $\aleph_\alpha$-development of X. Now let $\mathcal{U} = (U_i)_{i\in I}$ be any open covering of X. Define the following (closed) subsets for each $\beta < \omega_\alpha$ and $i \in I$.

$$K_{\beta i} = \{x \in X : \mathrm{st}^3(x,\mathcal{G}_\beta) \subseteq U_i\} - \bigcup_{j<i} U_j\,.$$

Then the family of open sets $\mathcal{W}_\beta = \{\mathrm{st}(K_{\beta i},\mathcal{G}_\beta)\}_{i\in I}$ is discrete in X for each fixed $\beta < \omega_\alpha$ since each member of the open covering $\mathcal{G}_\beta$ intersects at most one starset from $\mathcal{W}_\beta$. In fact if a member $G \in \mathcal{G}_\beta$ intersect both $\mathrm{st}(K_{\beta i},\mathcal{G}_\beta)$ and $\mathrm{st}(K_{\beta j},\mathcal{G}_\beta)$ where $i < j$, then, there exist two members G_β^i and G_β^j of $\mathcal{G}_\beta$ such that the four sets $G \cap G_\beta^i$, $G \cap G_\beta^j$, $G_\beta^i \cap K_{\beta i}$ and

$G^j_\beta \cap K_{\beta j}$ are all nonempty. So by choosing any point $x \in G^i_\beta \cap K_{\beta i}$ one yields the contradiction $G^j_\beta \cap K_{\beta j} = \emptyset$ since

$$G^j_\beta \subseteq \mathrm{st}\,(\mathrm{st}^2(x,\mathcal{G}_\beta),\mathcal{G}_\beta) \subseteq U_i \subseteq \bigcup_{k<j} U_k \subseteq X - K_{\beta j}\,.$$

Thus the discrete family $\mathcal{W}_\beta$ of open sets contains at most $\aleph_\alpha$ number of member since X has $\aleph_\alpha$-cellularity and members of a discrete family are pairwise disjoint. Therefore X is an $\aleph_\alpha$-Lindelöf space since $\bigcup_{\beta<\omega_\alpha} \mathcal{W}_\beta$ covers X and refines $\mathcal{U}$. All these considerations prove the implication iii) $\Longrightarrow$ i). The implication i) $\Longrightarrow$ ii) is clear. Now let the families $\mathcal{B}_x$ written in hypothesis be a local basis for each $x \in X$ in the weakly $\aleph_\alpha$-Lindelöf space X. Let $\mathcal{G}^*_{\mu\beta}$ be the suitable subfamily of the common refinement $\mathcal{G}_{\mu\beta}$ of $\mathcal{G}_\mu$ and $\mathcal{G}_\beta$ such that the union $\bigcup\mathcal{G}^*_{\mu\beta}$ of its members is dense and $\mathrm{card}\,\mathcal{G}^*_{\mu\beta} \le \aleph_\alpha$. Then the family $\mathcal{B} = \{\mathrm{st}(G^*,\mathcal{G}_{\mu\beta})\colon G^* \in \mathcal{G}^*_{\mu\beta}$ and $\mu,\beta < \omega_\alpha\}$ which contains at most $\aleph_\alpha$ number of members is a base for X as it was shown in the proof of Theorem 1 of [4]. So X has evidently $\aleph_\alpha$-cellularity since its density is not greater then $\aleph_\alpha$. After Proposition 10 the equivalence of iv) and i) is easy since iv) $\Longrightarrow$ iii) is already known.

REMARK 2. Notice that each open covering of a space with $\aleph_\alpha$-development $(\mathcal{G}_\beta)_{\beta<\omega_\alpha}$ has an open refinement which could be written as the union of $\aleph_\alpha$ number of open-discrete families *if* $\mathcal{B}_x = \{\mathrm{st}^2(x,\mathcal{G}_\beta)\}_{\beta<\omega_\alpha}$ is a local basis for each point $x \in X$. Therefore such particular $\aleph_\alpha$-developable spaces are clearly $\aleph_\alpha$1-refinable. WORRELL and WICKE have proved in their widely known paper [19] that all developable spaces are $\aleph_0$ finite-refinable i.e. θ-refinable. Proving of $\aleph_\alpha$1-refinability of *any* $\aleph_\alpha$-developable space could easily be derived by Proposition 4[(1)]. Thus Proposition 3 is in fact a simple consequence of Proposition 9. Notice also that special $\aleph_\alpha$-developable spaces of the above proposition are regular since, $\overline{\mathrm{st}(x,\mathcal{G}_\beta)} \subseteq \mathrm{st}^2(x,\mathcal{G}_\beta)$ holds easily for each $x \in X$ and $\beta < \omega_\alpha$ and they possess a basis $\mathcal{B} = \{\mathcal{B}_{\beta\mu}\colon \beta < \omega_\alpha, \mu < \omega_\alpha\}$ such that each $\mathcal{B}_{\beta\mu}$ is an open and discrete family and $\bigcup_{\mu<\omega_\alpha} \mathcal{B}_{\beta\mu}$ refines $\mathcal{G}_\beta$ for each $\beta < \omega_\alpha$ after the above remark (see Theorem 7 of HODEL). Notice that the above

[(1)]In fact this assertion follows easily after the proof of Proposition 4. Define the open refinement $\bigcup_{\beta<\omega_\alpha} \mathcal{U}_\beta$ of the open covering $\mathcal{U}$ of Proposition 4 whereas $\mathcal{U}_\beta = \{U_K - \bigcup(\mathcal{K}_\beta - \{K\})\colon K \in \mathcal{K}_\beta\}$. In here the uniquely determined $U_K \in \mathcal{U}$ is chosen by the axiom of choice so that $K \subseteq U_K$ holds.

proposition also yields the following well known conclusion: i) Second countability, ii) Separability, iii) Lindelöfness, iv) Weakly Lindelöfness, v) Countable chain condition are all equivalent in any pseudometrizable space since $\mathrm{st}^2(x,\mathcal{G}_n) \subseteq \{y \in X : d(x,y) < \varepsilon\} = B(x,\varepsilon)$ holds in such spaces where the positive integer n satisfy $\log_3 4 - \log_3 \varepsilon < n$ and the sequence of open coverings $\mathcal{G}_n$ of open balls with radius 3^{-n} constitutes a development in such a space. In any metrizable space vi) $\aleph_1$-compactness is a new equivalent condition with these five.

Finally notice that *any* discrete subset A in a pseudometric space (X,d) with $\aleph_\alpha$-cellularity necessarily satisfies $\mathrm{card}\, A \leq \aleph_\alpha$ since the open balls $B(x, d(x, A - x)/2)$ with centers from A are all pairwise disjoint.

PROPOSITION 8. *For any open covering $\mathcal{G}$ and for any subset A of a T_1 space X, there exists a subset $K \subseteq A$ with the following properties:* i) *K is a closed-discrete subset of X and* ii) $A \subseteq \bigcup_{x \in K} \mathrm{st}(x, \mathcal{G})$.

PROOF. There is nothing to prove if A is contained in the union $\mathrm{st}(a_1,\mathcal{G}) \cup \ldots \cup \mathrm{st}(a_n,\mathcal{G})$ by the aid of some finite number of appropriate points of A. Otherwise by using the transfinite induction, one can define the following special subset $K \subseteq A$ of the selected points $a \in A$ where each $a \in K$ is the least element of $A - \bigcup\{\mathrm{st}(a',\mathcal{G}) : a' \in K, a' < a\}$ if this difference set is yet nonempty. Here X is accepted as equipped by the well ordering $<$. Defining the least elements of these differences must be terminated after some number of steps and this number is not greater than card A; i.e. one of these difference sets must eventually be empty. The subset K of these selected points is the required one. Notice that each member of the open covering $\mathcal{G}$ contains at most one point from K, for, if the different points a and a' of K belong to the same member of $\mathcal{G}$ then by supposing $a < a'$, one yields

$$a' \in \mathrm{st}(a,\mathcal{G}) - \bigcup\{\mathrm{st}(x,\mathcal{G}) : x \in X, x < a'\} = \emptyset .$$

Thus K has no limit point in X i.e. K is a closed-discrete subset of X, since accumulation points are necessarily ω-limit points in T_1 spaces. This statement could also be proved by an equivalent method of [3] and [15] by defining the concept of *distinguished subset* with respect to an open covering of X and using the Zorn's lemma.

PROPOSITION 9. *A T_1 space is $\aleph_\alpha$-Lindelöf iff it is $\aleph_{\alpha+1}$-compact and $\aleph_\alpha\aleph_\alpha$-refinable.*

PROOF. Only the sufficiency requires a proof. Let X be an $\aleph_{\alpha+1}$-compact and $\aleph_\alpha\aleph_\alpha$-refinable T_1 space and let $\mathcal{U}$ be any open covering of X. There exists an open refinement $\bigcup_{\beta<\omega_\alpha} \mathcal{G}_\beta$ of $\mathcal{U}$ where there exists an index $\beta(x) < \omega_\alpha$ for each $x \in X$ such that $\operatorname{ord}(x, \mathcal{G}_{\beta(x)}) \leq \aleph_\alpha$. Define now

$$A_\beta = \{x \in X \colon \operatorname{ord}(x, \mathcal{G}_\beta) \leq \aleph_\alpha\}\,, \qquad \forall\, \beta < \omega_\alpha\,.$$

Then $X = \bigcup_{\beta<\omega_\alpha} A_\beta$ holds and there exist closed-discrete subsets K_β of X by Proposition 8 such that $K_\beta \subseteq A_\beta \subseteq \bigcup_{x\in K_\beta} \operatorname{st}(x, \mathcal{G}_\beta)$. Notice that $\operatorname{card} K_\beta \leq \aleph_\alpha$ since X is $\aleph_{\alpha+1}$-compact. Each starset $\operatorname{st}(x, \mathcal{G}_\beta)$ is the union of at most $\aleph_\alpha$ number of members of $\mathcal{G}_\beta$ for each $x \in K_\beta$ since $\operatorname{ord}(x, \mathcal{G}_\beta) \leq \aleph_\alpha$ for each $x \in A_\beta$. Therefore X is covered by at most $\aleph_\alpha$ number of members of $\bigcup_{\beta<\omega_\alpha} \mathcal{G}_\beta$ which gives the required result easily.

COROLLARY 5. *A T_1 space is $\aleph_\alpha$-Lindelöf iff it is $\aleph_{\alpha+1}$-compact and meta $\aleph_\alpha$-Lindelöf.*

PROOF. Notice only that meta $\aleph_\alpha$-Lindelöf spaces are evidently $\aleph_\alpha\aleph_\alpha$-refinable.

COROLLARY 6. *A meta $\aleph_\alpha$-Lindelöf T_1 space is $\aleph_{\alpha+1}$-compact iff each subset of cardinality $\aleph_{\alpha+1}$ has at least one complete accumulation point.*

PROOF. Use Corollary 5 and Proposition 2 for necessity.

PROPOSITION 10. *An $\aleph_\alpha$-developable space has $\aleph_\alpha$ weight iff it is $\aleph_\alpha$-Lindelöf.*

PROOF. It is already known that any space with $\aleph_\alpha$ weight is $\aleph_\alpha$-Lindelöf. For the converse notice that $\mathcal{B} = \bigcup_{\beta<\omega_\alpha} \mathcal{G}_\beta^*$ is a basis for an $\aleph_\alpha$-developable $\aleph_\alpha$-Lindelöf space X, where each $\mathcal{G}_\beta^*$ is the subcovering of the open covering $\mathcal{G}_\beta$ with at most $\aleph_\alpha$ number of members and $(\mathcal{G}_\beta)_{\beta<\omega_\alpha}$ is the $\aleph_\alpha$-development of X. Thus card $\mathcal{B} \leq \aleph_\alpha$.

We close this section with two results on $\aleph_{\alpha+1}$-compactness.

PROPOSITION 11. *Every normal space with $\aleph_\alpha$ density is $\aleph_{\alpha+1}$-compact if $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$.*

PROOF. Let the subset X_0 of cardinality $\aleph_{\alpha+1}$ has no limit point in a normal space X whose possesses a dense subset D with card $D = \aleph_\alpha$. Notice then that X_0 is a proper subset since, the opposite supposition yields the contradiction $\aleph_\alpha <$ card D, for each singleton $\{x\}$ of X would be an open set and necessarily intersects D. Thus all subsets of X_0 are closed in X and therefore there exists an open and nonempty $G_A \subseteq X$ for each nonempty subset A of X_0 so that $\overline{G_A} \cap (X_0 - A) = \emptyset$ and $A \subseteq G_A$. Hence the function $\varphi(A) = G_A \cap D$ from $\mathcal{P}(X_0)$ into $\mathcal{P}(D)$ is injective since $\varphi(A) = \varphi(B)$ yield $A \subseteq \overline{G_A} = \overline{\varphi(B)} = \overline{G_B} \subseteq X - (X_0 - B)$ and its dual one and therefore both of the inclusions $A \subseteq B$ and $B \subseteq A$ must be satisfied. All these considerations give $2^{\aleph_{\alpha+1}} \leq 2^{\aleph_\alpha}$, just opposite of what we have already supposed in hypothesis.

PROPOSITION 12. *Every meta $\aleph_\alpha$-Lindelöf space with $\aleph_\alpha$-density is $\aleph_{\alpha+1}$-compact.*

PROOF. Let X be a meta $\aleph_\alpha$-Lindelöf space with a dense subset D of cardinality $\aleph_\alpha$ and let X_0 be any subset with no limit point in X. Then the open covering $\mathcal{G} = \{X - X_0\} \cup \{G_x : x \in X_0\}$ where each open $G_x (x \in X_0)$ satisfies $G_x \cap X_0 = \{x\}$, has a refinement $\mathcal{U}$ with the property ord $(x, \mathcal{U}) \leq \aleph_\alpha$ for each $x \in X$. There exists an $U_x \in \mathcal{U}$ for each $x \in X$. Notice that if $x, y \in X_0$ are different points then $U_x \not\subseteq G_y$ and therefore $U_x \subset G_x$ and $U_x \neq U_y$ hold. So card $X_0 \leq$ card $\mathcal{U} \leq \aleph_\alpha$ are obtained since card $D = \aleph_\alpha$ and ord $(x, \mathcal{U}) \leq \aleph_\alpha$ for each $x \in D$.

3 – Notes on weakly $\aleph_\alpha$-Lindelöfness

PROPOSITION 13. *A space has $\aleph_\alpha$-cellularity iff each open subspace is weakly $\aleph_\alpha$-Lindelöf.*

PROOF. This statement is nothing but a modification of Comfort's lemma, see [18]. Let X_0 be any open subspace of a space X with $\aleph_\alpha$-cellularity. Then it is clear that the cellularity number of X_0 is not greater than $\aleph_\alpha$. Now if there exists an open covering $\mathcal{G}$ of X_0 such that all its subfamilies $\mathcal{G}^*$ of cardinality $\aleph_\alpha$ satisfy $X_0 \not\subseteq \overline{\bigcup \mathcal{G}^*}$ then, one could define the points x_μ and open members $G_\mu \in \mathcal{G}$ with

$$G_\mu \ni x_\mu \in X_0 - \overline{\bigcup_{\beta<\mu} G_\beta} \quad , \qquad \forall \mu < \omega_{\alpha+1}$$

by transfinite induction. This is impossible since the family of nonempty and pairwise disjoint open sets $G_\mu - \overline{\bigcup_{\beta<\mu} G_\beta}$ $(\mu < \omega_{\alpha+1})$ of cardinality $\aleph_{\alpha+1}$ would be well defined on the subspace X_0 of cellularity $\aleph_\alpha$. This proves the necessity. Sufficiency is only straightforward.

COROLLARY 7. *Spaces with $\aleph_\alpha$-cellularity are weakly $\aleph_\alpha$-Lindelöf.*

PROPOSITION 14. *An open subset with no limit point in a weakly $\aleph_\alpha$-Lindelöf space has cardinality less or equal to $\aleph_\alpha$.*

PROOF. If an open G has no limit point, then the open covering $\{X - G\} \cup \{\{x\} : x \in G\}$ has no proper subfamily with a dense union set.

PROPOSITION 15. *Regularly closed subspace of a weakly $\aleph_\alpha$-Lindelöf space are weakly $\aleph_\alpha$-Lindelöf.*

PROOF. Let X be a weakly $\aleph_\alpha$-Lindelöf space and let X_0 be any regularly closed subspace of X. For any open covering $\{X_0 \cap G : G \in \mathcal{G}\}$ of the subspace X_0, one can define a subfamily $\mathcal{G}^*$ of $\mathcal{G}$ so that card $\mathcal{G}^* \leq \aleph_\alpha$ and $X = (X - \operatorname{int} X_0) \cup \overline{\bigcup \mathcal{G}^*}$. Thus the following gives the required conclusion:

$$X_0 = \overline{\operatorname{int} X_0} = \overline{\operatorname{int} X_0 \cap \overline{\bigcup \mathcal{G}^*}} = \operatorname{cl}_{X_0} \bigcup \{X_0 \cap G : G \in \mathcal{G}^*\} \, .$$

Proposition 16. *A space is weakly $\aleph_\alpha$-Lindelöf if it has a weakly $\aleph_\alpha$-Lindelöf open-dense subspace.*

Proof. Notice only that if $X_0 = \mathrm{cl}_{X_0} \bigcup\{X_0 \cap G : G \in \mathcal{G}\}$ holds for any open-dense subspace X_0 and for any open family $\mathcal{G}$ of X, then the following is obtained:

$$X = \overline{X_0} = \overline{\overline{\bigcup_{G\in\mathcal{G}} (X_0 \cap G)} \cap X_0} = \overline{\bigcup_{G\in\mathcal{G}} G}.$$

Proposition 17. *A space has $\aleph_\alpha$-cellularity iff all its open-dense subspaces has the same cellularity.*

Proof. Left to the reader.

Proposition 18. *Weakly $\aleph_\alpha$-Lindelöfness (resp. $\aleph_\alpha$-cellularity) of X, X_s and X_α are equivalent.*

Proof. In here X_s denotes as usually the semi-regularization space of X which is generated by the whole sets of the form int $\overline{G}$ where G is open in X. Whereas the space X_α of Njastad [14] is defined on the *set* X and its topology generated by the whole sets $G - N$ where G is open and N is nowhere dense in X. Then the topology of X_s is weaker than the topology $\mathcal{T}$ of X but X_α has a finer topology than $\mathcal{T}$. Now notice that if $(G_\mu)_{\mu\in I}$ and $(N_\mu)_{\mu\in I}$ are respectively the families of open and nowhere dense members of X, then

$$\overline{\bigcup_{\mu\in I} \overline{G_\mu}} = \overline{\bigcup_{\mu\in I} G_\mu} = \mathrm{cl}_s \bigcup_{\mu\in I} \mathrm{int}\,\overline{G_\mu} = \mathrm{cl}_\alpha \bigcup_{\mu\in I} (G_\mu - N_\mu)$$

since $\mathrm{cl}_s \mathrm{int}\,\overline{G} = \overline{G} = \mathrm{cl}_\alpha(G - N)$ hold for any open G and nowhere dense N of X. Therefore all these sets are X or not X at the same time. This proves first. The second statement written in parenthesis is easy since two open sets G_1 and G_2 are disjoint iff $\mathrm{int}\,\overline{G_1}$ and $\mathrm{int}\,\overline{G_2}$ are so iff $G_1 - N_1$ and $G_2 - N_2$ are so where N_1 and N_2 are nowhere dense in X.

Proposition 19. *Almost closed-almost open surjections with $\aleph_\alpha$-Lindelöf fibers onto weakly $\aleph_\alpha$-Lindelöf spaces can only be defined on weakly $\aleph_\alpha$-Lindelöf spaces.*

PROOF. Let f be an almost closed and almost open surjection from a space X onto a weakly $\aleph_\alpha$-Lindelöf space Y so that all fibers $f^{-1}(y)$ are $\aleph_\alpha$-Lindelöf subsets. Remember that the function $f\colon X_s \longrightarrow Y$ is open since f is almost open and $f(\overline{G})$ is closed for each open G of X since f is almost closed and these are the characteristic properties of these kinds of functions. Take any open covering $\mathcal{G} = (G_\mu)_{\mu\in I}$ of X. Then there exists a subset $I(y) \subseteq I$ so that $\operatorname{card} I(y) \leq \aleph_\alpha$ and $f^{-1}(y) \subseteq \bigcup_{\mu\in I(y)} G_\mu$ hold. Then there also exists a basic neighborhood $U_y \in \mathcal{N}_y$ of $y \in Y$ so that

$$f^{-1}(U_y) \subseteq \operatorname{int}\overline{\bigcup_{\mu\in I(y)} G_\mu} \quad \text{by} \quad y \notin f\left(\overline{X - \overline{\bigcup_{\mu\in I(y)} G_\mu}}\right).$$

But $Y = \overline{\bigcup\{U_y : y \in Y_0\}}$ holds by the aid of some subset Y_0 with $\mathit{card} Y_0 \leq \aleph_\alpha$. So

$$X = f^{-1}\left(\overline{\bigcup_{y\in Y_0} U_y}\right) \subseteq \mathrm{cl}_s f^{-1}\left(\bigcup_{y\in Y_0} U_y\right) \subseteq$$

$$\subseteq \mathrm{cl}_s \bigcup_{y\in Y_0}\left(\operatorname{int}\overline{\bigcup_{\mu\in I(y)} G_\mu}\right) =$$

$$= \overline{\bigcup_{y\in Y_0}\left(\operatorname{int}\overline{\bigcup_{\mu\in I(y)} G_\mu}\right)} = \overline{\bigcup_{y\in Y_0}\bigcup_{\mu\in I(y)} G_\mu} \subseteq X$$

give the conclusion since, the subfamily $\mathcal{G}^* = \{G_\mu : \mu \in I(y), y \in Y_0\}$ of $\mathcal{G}$ has cardinality less or equal to $\aleph_\alpha$.

COROLLARY 8. *Almost closed-almost open surjections with Lindelöf fibers onto weakly Lindelöf spaces can only be defined on weakly Lindelöf spaces.*

COROLLARY 9. *Closed-open surjections with $\aleph_\alpha$ Lindelöf fibers onto weakly $\aleph_\alpha$-Lindelöf spaces can only be defined on weakly $\aleph_\alpha$-Lindelöf spaces.*

The following propositions and their natural corollaries are all left to the reader.

PROPOSITION 20. *Closed surjections with $\aleph_\alpha$-Lindelöf fibers onto $\aleph_\alpha$-Lindelöf spaces can only be defined on $\aleph_\alpha$-Lindelöf spaces.*

PROPOSITION 21. *Closed-open injections onto $\aleph_\alpha$-compact spaces can only be defined on $\aleph_\alpha$-compact spaces.*

PROPOSITION 22. *Continuous image of $\aleph_\alpha$-Lindelöf (resp. weakly $\aleph_\alpha$-Lindelöf) spaces are so.*

Acknowledgements

This modest note could be considered as a part of sincere thanks of the author to Professor C. ARF for his encouregements, for his example personality as a scholar and it is dedicated to him on occasion of his 80^{th} birthday and for his constant efforts for development of basic sciences in Turkey.

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Lavoro pervenuto alla redazione il 2 ottobre 1989
ed accettato per la pubblicazione il 12 ottobre 1990
su parere favorevole di G. Saban e di G. Aquaro

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