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The Fourth-Order Differential Equation Satisfied by the Associated Orthogonal Polynomials

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RIASSUNTO – *Partendo dall'equazione differenziale di Riccati soddisfatta dalla funzione di Stieltjes di un funzionale lineare, viene costruito un algoritmo che permette di costruire l'equazione differenziale di quarto ordine soddisfatta dai polinomi ortogonali associati (di grado arbitrario) della classe di Laguerre-Hahn. Nella situazione classica (Hermite, Laguerre, Bessel, Jacobi) inoltre, queste equazioni differenziali vengono fornite in modo esplicito.*

ABSTRACT – *Starting from the Riccati Differential equation satisfied by the Stieltjes function of a linear functional, we work out an algorithm which enables us to write the fourth-order differential equation satisfied by the associated (any order) orthogonal polynomial of the Laguerre-Hahn class. Moreover in the classical situation (Hermite, Laguerre, Bessel, Jacobi) we give explicitely these differntial equations.*

KEY WORDS – *Riccati differential equation - Laguerre-Hahn class - Asociated orthogonal polynomials.*

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1 – Introduction

The associated polynomials of any order (order r) of a given Laguerre-Hahn orthogonal sequence belong to the Laguerre-Hahn class [1] and

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therefore satisfy a 4th order linear differential equation.

For the classical families: Jacobi, Laguerre, Hermite, Bessel, the 4th order differential equation is known in the case $r = 1$, as given by GROSJEAN [2,3] for Legendre and Jacobi polynomials, and by RONVEAUX [4] in the general situation. For any r (integer or not) the differential equation was obtained by WIMP [5] for the Jacobi polynomials after heavy computation using MACSYMA. For all the classical polynomials and for an arbitrary integer r (r fixed) a REDUCE package was written by BELMEHDI and RONVEAUX [6] from which a conjecture was put forward about the differential equation satisfied by the associated polynomials, of any order, of the Hermite and the Laguerre sequences.

In this work, using systematically the properties of the linked Stieltjes functions of associated polynomials, we obtain the differential equation for the associated of any order of a Laguerre-Hahn orthogonal polynomial sequence. The application to the classical case confirms the conjecture already indicated and recover, via a change of interval, the differential equation of WIMP [5].

2 – Riccati differential equation for the Stieltjes functions and consequences

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials with respect to the regular linear functional $\mathcal{L}$.
$\{P_n\}_{n\geq 0}$ satisfies the following second order recurrence relation:

$$P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \qquad n \geq 0 \tag{1}$$

$$P_1(x) = x - \beta_0, \qquad P_0(x) = 1$$

with

$$(\beta_n, \gamma_n) \in C \times C^*$$

. $\gamma_0 = \langle \mathcal{L}, 1\rangle$ is the first moment of $\mathcal{L}$; we will take it equal to unity.
$\langle \mathcal{L}, P_n\rangle$ denotes the value of the linear functional $\mathcal{L}$ applied to $P_n(x)$.

We define the associated orthogonal polynomials of order r with regard to $\mathcal{L}^{(r-1)}$ as follows:

$$\gamma_{r-1}P_n^{(r)}(x) = \langle \mathcal{L}^{(r-1)}, \frac{P_{n+1}^{(r-1)}(x) - P_{n+1}^{(r-1)}(t)}{x-t}\rangle, \quad r \geq 1, n \geq 0 \tag{2}$$

assuming that $P_n^{(0)} = P_n$ and $\mathcal{L}^{(0)} = \mathcal{L}$, where $\mathcal{L}^{(r-1)}$ is the linear functional with respect to which $\{P_n^{(r-1)}\}(n \geq 0)$ is orthogonal, and it is understood that $\mathcal{L}^{(r-1)}$ acts on the variable t. Using the three-term recurrence relation satisfied by $\{P_n\}_{n\geq 0}$, and by induction on r we get: [7]

$$(3) \qquad \begin{aligned} &P_{n+2}^{(r)}(x) = (x - \beta_{r+n+1})P_{n+1}^{(r)}(x) - \gamma_{r+n+1}P_n^{(r)}(x)\,, && n \geq 0 \\ &P_1^{(r)}(x) = x - \beta_r\,, \quad P_0^{(r)}(x) = 1\,, && r \geq 0 \end{aligned}$$

Let us define the formal Stieltjes function of $\mathcal{L}^{(r)}$:

$$(4) \qquad S_r(z) = -\sum_{k\geq 0} \frac{(\mathcal{L}^{(r)})_k}{z^{k+1}}$$

where

$$\left(\mathcal{L}^{(r)}\right)_k = \langle \mathcal{L}^{(r)}, x^k \rangle\,, \quad k \geq 0\,, \quad r \geq 0\,.$$

It is a well known result that: [8,9]

$$(5) \qquad S_0(z) = -\frac{(\mathcal{L})_0}{z - \beta_0 + S_1(z)}\,.$$

One can easily show that:

$$\left(\mathcal{L}^{(r)}\right)_0 = \gamma_r$$

and

$$(6) \qquad S_r(z) = -\frac{\gamma_r}{z - \beta_r + S_{r+1}(z)}\,.$$

LEMMA 1 - [1]. *If S_r satisfies the Riccati differential equation i.e.:*

$$(7) \qquad \phi S_r' = \frac{D_{r-1}}{\gamma_{r-1}}\, S_r^2 + C_r S_r + D_r$$

where ϕ, C_r and D_r are polynomials, then the same property holds for S_{r+1}, without increase of the degree of the polynomial coefficients; i.e.:

$$\phi S_{r+1}' = \frac{Dr}{\gamma_r} S_{r+1}^2 + C_{r+1}S_{r+1} + D_{r+1}$$

with

$$C_{r+1} = -C_r + 2\frac{D_r}{\gamma_r}(x-\beta_r)$$

(8) $$D_{r+1} = -\phi + \frac{\gamma_r}{\gamma_{r-1}}D_{r-1} + \frac{D_r}{\gamma_r}(x-\beta_r)^2 - C_r(x-\beta_r)\,.$$

An important consequence is:

(9) $$\frac{C_{r+1}^2 - C_r^2}{4} = \frac{D_r D_{r+1}}{\gamma_r} - \frac{D_{r-1}D_r}{\gamma_{r-1}} + \phi\frac{D_r}{\gamma_r}\,.$$

The key to the relation between S_r and the associated orthogonal polynomials is given by the basic identity [10].

(10) $$S_r = \gamma_r \frac{P_n^{(r+1)} - S_{r+n+1}P_{n-1}^{(r+1)}}{P_{n+1}^{(r)} - S_{r+n+1}P_n^{(r)}}\,.$$

Entering (10) in (7) gives a Riccati equation for S_{r+n+1} to be identified with:

$$\phi S'_{r+n+1} = \frac{D_{r+n}}{\gamma_{r+n}}S_{r+n+1}^2 + C_{r+n+1}S_{r+n+1} + D_{r+n+1}$$

and using the identity [10,1]

(11) $$P_n^{(r+1)}P_n^{(r)} - P_{n+1}^{(r)}P_{n-1}^{(r+1)} = \prod_{k=r}^{r+n}\gamma_k\,, \quad r \geq 0\,, \quad n \geq 1$$

we get:

LEMMA 2.

(12) $$\prod_{k=r}^{r+n}\gamma_k \frac{D_{r+n}}{\gamma_{r+n}} = \gamma_r^2\left(P_{n-1}^{(r+1)}\right)^2 - \gamma_r C_r P_n^{(r)}P_{n-1}^{(r+1)} + $$
$$+ D_r\left(P_n^{(r)}\right)^2 + \gamma_r\phi W\left(P_n^{(r)}, P_{n-1}^{(r+1)}\right)$$

$$
(13)\quad
\begin{aligned}
\prod_{k=r}^{r+n}\gamma_k C_{r+n} &= 2\gamma_r\phi\left[\left(P_n^{(r+1)}\right)' P_n^{(r)} - P_{n-1}^{(r+1)}\left(P_{n+1}^{(r)}\right)'\right]\\
&\quad + 2\frac{\gamma_r^2}{\gamma_{r-1}}D_{r-1}P_n^{(r+1)}P_{n-1}^{(r+1)}\\
&\quad + 2D_rP_n^{(r)}P_{n+1}^{(r)} - \gamma_rC_r\left[P_n^{(r)}P_n^{(r+1)} + P_{n+1}^{(r)}P_{n-1}^{(r+1)}\right]
\end{aligned}
$$

$$
(14)\quad
\begin{aligned}
\prod_{k=r}^{r+n}\gamma_k D_{r+n+1} &= \frac{\gamma_r^2}{\gamma_{r-1}}D_{r-1}\left(P_n^{(r+1)}\right)^2 - \gamma_rC_rP_n^{(r+1)}P_{n+1}^{(r)}\\
&\quad + D_r\left(P_{n+1}^{(r)}\right)^2 + \gamma_r\phi W\left(P_{n+1}^{(r)}, P_n^{(r+1)}\right) \qquad n \geq 0
\end{aligned}
$$

where:

$$W(f,g) = fg' - f'g\,.$$

REMARK. If we set $n = 0$ in (13) and (14) we obtain (8).

By eliminating $(P_n^{(r+1)})'$ (resp $(P_{n+1}^{(r)})'$) between (13) and (14), and using (11) we get a differential relation for $P_{n+1}^{(r)}$ (resp $P_n^{(r+1)}$).

LEMMA 3 [1,12].

$$(15)\quad \phi\left(P_{n+1}^{(r)}\right)' = \frac{C_{r+n+1} - C_r}{2}P_{n+1}^{(r)} - D_{r+n+1}P_n^{(r+1)} + \frac{\gamma_r}{\gamma_{r-1}}D_{r-1}P_n^{(r+1)}$$

$$(16)\quad \phi\left(P_n^{(r+1)}\right)' = \frac{C_{r+n+1} + C_r}{2}P_n^{(r+1)} - D_{r+n+1}P_{n-1}^{(r+1)} - \frac{D_r}{\gamma_r}P_{n+1}^{(r)}\,.$$

An alternative expression of Lemma 3 can be obtained by substituting $n-1$ for n, using the second order recurrence formula (3) and applying the relation (8); that is to say:

LEMMA 4.

$$(17)\quad \phi\left(P_n^{(r)}\right)' + \frac{C_{r+n+1} + C_r}{2}P_n^{(r)} = \frac{D_{r+n}}{\gamma_{r+n}}P_{n+1}^{(r)} + \frac{\gamma_r}{\gamma_{r-1}}D_{r-1}P_{n-1}^{(r+1)}$$

$$\phi\left(P_{n-1}^{(r+1)}\right)' + \frac{C_{r+n+1} - C_r}{2} P_{n-1}^{(r+1)} = \frac{D_{r+n}}{\gamma_{r+n}} P_n^{(r+1)} - \frac{D_r}{\gamma_r} P_n^{(r)} \tag{18}$$

We substitute for $\phi(P_{n+1}^{(r)})'$ from (15) into a new equation obtained after differentiation and multiplication of the equation (17) throughout by ϕ, and we get:

$$\begin{aligned} &\left\{\phi^2 D_{r+n}\frac{d^2}{dx^2} + \phi[C_r D_{r+n} - W(\phi, D_{r+n})]\frac{d}{dx}\right. \\ &+ \left[D_{r+n}\left(\frac{D_{r+n}D_{r+n+1}}{\gamma_{r+n}} - \frac{C_{r+n+1}^2 - C_r^2}{4}\right)\right. \\ &\left.\left. - \phi W\left(\frac{C_{r+n+1}+C_r}{2}, D_{r+n}\right)\right] I_d\right\}\left[P_n^{(r)}\right] \\ &= \frac{\gamma_r}{\gamma_{r-1}}\left\{\phi D_{r-1}D_{r+n}\left(P_{n-1}^{(r+1)}\right)' - \left[\phi W(D_{r-1}, D_{r+n})\right.\right. \\ &\left.\left. + \frac{C_{r+n+1} - C_r}{2} D_{r-1}D_{r+n}\right] P_{n-1}^{(r+1)} + \frac{D_{r-1}D_{r+n}^2}{\gamma_{r+n}} P_n^{(r+1)}\right\} \end{aligned} \tag{19}$$

Using (9), (19) can be written as follows:

$$\begin{aligned} &\left\{\phi D_{r+n}\frac{d^2}{dx^2} + \phi[C_r D_{r+n} - W(\phi, D_{r+n})]\frac{d}{dx}\right. \\ &+ \left[\frac{D_{r-1}D_r D_{r+n}}{\gamma_{r-1}} - \phi W\left(\frac{C_{r+n+1}+C_r}{2}, D_{r+n}\right)\right. \\ &\left.\left. - \phi D_{r+n}\sum_{k=r}^{r+n}\frac{D_k}{\gamma_k}\right] I_d\right\}\left[P_n^{(r)}\right] = \frac{\gamma_r}{\gamma_{r-1}}\left\{\phi D_{r-1}D_{r+n}\left(P_{n-1}^{(r+1)}\right)'\right. \\ &- \left[\phi W(D_{r-1}, D_{r+n}) + \frac{C_{r+n+1} - C_r}{2} D_{r-1}D_{r+n}\right] P_{n-1}^{(r+1)} \\ &\left. + \frac{D_{r-1}D_{r+n}^2}{\gamma_{r+n}} P_n^{(r+1)}\right\} \end{aligned} \tag{20}$$

We substitute for $\frac{D_{r+n}}{\gamma_{r+n}} P_n^{(r+1)}$ from (16) into (20) and we obtain:

PROPOSITION 1.

$$\mathcal{D}_{r,n}\left[P_n^{(r)}\right] = N_{r+1,n-1}\left[P_{n-1}^{(r+1)}\right] \tag{21}$$

where $\mathcal{D}_{r,n}$ and $N_{r+1,n-1}$ are the following differential operators:

$$\begin{aligned}\mathcal{D}_{r,n} &= \phi D_{r+n}\frac{d^2}{dx^2} + \left[C_r D_{r+n} - W(\phi, D_{r+n})\right]\frac{d}{dx} \\ &\quad - \left[W\left(\frac{C_{r+n+1}+C_r}{2}, D_{r+n}\right) + D_{r+n}\sum_{k=r}^{r+n}\frac{D_k}{\gamma_k}\right] I_d \\ N_{r+1,n-1} &= \frac{\gamma_r}{\gamma_{r-1}}\left\{2D_{r-1}D_{r+n}\frac{d}{dx} - W\left(D_{r-1}, D_{r+n}\right) I_d\right\}.\end{aligned}$$

If we star from (18) and apply the same process as we did for (17) we get a differential relation involving only $P_{n-1}^{(r+1)}$ and $P_n^{(r)}$, i.e.:

PROPOSITION 2.

$$\overline{\mathcal{D}}_{r+1,n-1}\left[P_{n-1}^{(r+1)}\right] = \overline{N}_{r,n}\left[P_n^{(r)}\right] \tag{22}$$

where

$$\begin{aligned}\overline{\mathcal{D}}_{r+1,n-1} &= \phi D_{r+n}\frac{d^2}{dx^2} - \left[C_r D_{r+n} + W(\phi, D_{r+n})\right]\frac{d}{dx} \\ &\quad - \left[W\left(\frac{C_{r+n+1}-C_r}{2}, D_{r+n}\right) + D_{r+n}\sum_{k=r}^{r+n}\frac{D_k}{\gamma_k}\right] I_d \\ \overline{N}_{r,n} &= -\frac{2D_r D_{r+n}}{\gamma_r}\frac{d}{dx} + W\left(D_r, D_{r+n}\right) I_d .\end{aligned}$$

3 – Differential equations

The following steps now give the fourth-order differential equation satisfied by each $P_n^{(r)}$. For the sake of expedience, we rewrite (21) in the following form:

$$\mathcal{D}_{r,n}\left[P_n^{(r)}\right] = a_1 D_{r+n}\left(P_{n-1}^{(r+1)}\right)' + b_1 P_{n-1}^{(r+1)}$$

with

$$a_1 = 2\frac{\gamma_r}{\gamma_{r-1}} D_{r-1}\,, \qquad b_1 = -\frac{\gamma_r}{\gamma_{r-1}} W\left(D_{r-1}, D_{r+n}\right)$$

we substitute for $\phi D_{r+n}\left(P_{n-1}^{(r+1)}\right)''$ from (22) into a new equation obtained after differentiation and multiplication of the equation (23) throughout by ϕ.

We get:

$$\phi\left(\mathcal{D}_{r,n}\left[P_n^{(r)}\right]\right)' - a_1\overline{N}_{r,n}\left[P_n^{(r)}\right] = a_2\left(P_{n-1}^{(r+1)}\right)' + b_2 P_{n-1}^{(r+1)}\,.$$

By applying the same process to the previous equation, we can write:

$$\phi D_{r+n}\left[\phi\left(\mathcal{D}_{r,n}\left[P_n^{(r)}\right]\right)'\right]' - \phi D_{r+n}\left(a_1\overline{N}_{r,n}\left[P_n^{(r)}\right]\right)' - a_2\overline{N}_{r,n}\left[P_n^{(r)}\right]$$
$$= a_3\left(P_{n-1}^{(r+1)}\right)' + b_3 P_{n-1}^{(r+1)}$$

where

$$a_2 = a_1\left[C_r D_{r+n} + W\left(\phi, D_{r+n}\right)\right] + \phi\left[\left(a_1 D_{r+n}\right)' + b_1\right]$$

$$b_2 = a_1\left[W\left(\frac{C_{r+n+1} - C_r}{2}, D_{r+n}\right) + D_{r+n}\sum_{k=r}^{r+n}\frac{D_k}{\gamma_k}\right] + \phi b_1'$$

$$a_3 = a_2\left[C_r D_{r+n} + W\left(\phi, D_{r+n}\right)\right] + \phi D_{r+n}(a_2' + b_1)$$

$$b_3 = a_2\left[W\left(\frac{C_{r+n+1} - C_r}{2}, D_{r+n}\right) + D_{r+n}\sum_{k=r}^{r+n}\frac{D_k}{\gamma_k}\right] + \phi D_{r+n} b_2'\,.$$

The equation:

$$\begin{vmatrix} \mathcal{D}_{r,n}\left[P_n^{(r)}\right] & a_1 D_{r+n} & b_1 \\ \phi\left(\mathcal{D}_{r,n}\left[P_n^{(r)}\right]\right)' - a_1\overline{N}_{r,n}\left[P_n^{(r)}\right] & a_2 & b_2 \\ \phi D_{r+n}\left[\phi\left(\mathcal{D}_{r,n}\left[P_n^{(r)}\right]\right)'\right]' & & \\ -\phi D_{r+n}\left(a_1\overline{N}_{r,n}\left[P_n^{(r)}\right]\right)' & & \\ -a_2\overline{N}_{r,n}\left[P_n^{(r)}\right] & a_3 & b_3 \end{vmatrix} = 0$$

now gives the expected fourth order differential equation for $P_n^{(r)}$.

The scheme described here can be applied to all orthogonal polynomials provided that their formal Stieltjes function satisfies a Riccati differential equation; the same condition is true for the Laguerre-Hahn class [12,1,13] which includes the semiclassical [12] and the classical families.

4 – Applications

The very peculiar situation corresponding to the classical case (C_k: polynomial of degree one, D_k: constant, $k \geq 0$, $D_{-1} = 0$ cf. Table 1) simplifies considerably the previous developments. After simplification by D_{r+n} the differential operators are reduced to :

$$\mathcal{D}_{r,n} = \phi\frac{d^2}{dx^2} + (C_r + \phi')\frac{d}{dx} + \left(\frac{C'_{r+n+1} + C'_r}{2} - \sum_{k=r}^{r+n}\frac{D_k}{\gamma_k}\right) I_d$$

$$\overline{\mathcal{D}}_{r+1,n-1} = \phi\frac{d^2}{dx^2} - (C_r - \phi')\frac{d}{dx} + \left(\frac{C'_{r+n+1} - C'_r}{2} - \sum_{k=r}^{r+n}\frac{D_k}{\gamma_k}\right) I_d$$

$$N_{r+1,n-1} = 2\frac{\gamma_r}{\gamma_{r-1}} D_{r-1}\frac{d}{dx}$$

$$\overline{N}_{r,n} = -2\frac{D_r}{\gamma_r}\frac{d}{dx}$$

and (21), (22) can be written as follows:

$$
(23)\qquad \left\{\phi\frac{d_2}{dx^2} + (C_r + \phi')\frac{d}{dx} + \left(\frac{C'_{r+n+1} + C'_r}{2} - \sum_{k=r}^{r+n}\frac{D_k}{\gamma_k}\right) I_d\right\}[P_n^{(r)}] = 2\frac{\gamma_r}{\gamma_{r-1}} D_{r-1}\left(P_{n-1}^{(r+1)}\right)'
$$

$$
(24)\qquad \left\{\phi\frac{d_2}{dx^2} - (C_r - \phi')\frac{d}{dx} + \left(\frac{C'_{r+n+1} - C'_r}{2} - \sum_{k=r}^{r+n}\frac{D_k}{\gamma_k}\right) I_d\right\}[P_{n-1}^{(r+1)}] = -2\frac{D_r}{\gamma_r}\left(P_n^{(r)}\right)'
$$

By differentiating (24) and multiplying the obtained equation throughout by $2\frac{\gamma_r}{\gamma_{r-1}} D_{r-1}$, and using (23) we get:

$$
\phi\left(\mathcal{D}_{r,n}[P_n^{(r)}]\right)'' + \left(2\phi' - C_r\right)\left(\mathcal{D}_{r,n}[P_n^{(r)}]\right)' + \left(\frac{C'_{r+n+1} - 3C'_r}{2} - \sum_{k=r}^{r+n}\frac{D_k}{\gamma_k} + \phi''\right)\mathcal{D}_{r,n}[P_n^{(r)}] = -\frac{4D_{r-1}D_r}{\gamma_{r-1}}(P_n^{(r)})''
$$

in other words:

$$
\begin{aligned}
&\left\{\phi^2\frac{d^4}{dx^4} + 5\phi\phi'\frac{d^3}{dx^3} + \left[2\phi\left(\frac{C'_{r+n+1} + C'r}{2} - \sum_{k=r}^{r+n}\frac{D_k}{\gamma_k} + 2\phi''\right)\right.\right.\\
&\left. + 4\phi'^2 - C_r^2 + \frac{4D_{r-1}D_r}{\gamma_{r-1}}\right]\frac{d^2}{dx^2}\\
&+ 3\left[\phi'\left(\frac{C'_{r+n+1} + C'_r}{2} - \sum_{k=r}^{r+n}\frac{D_k}{\gamma_k} + \phi''\right) - C'_r C_r\right]\frac{d}{dx}\\
&\left.+\left(\frac{C'_{r+n+1} - 3C'_r}{2} - \sum_{k=r}^{r+n}\frac{D_k}{\gamma_k} + \phi''\right)\left(\frac{C'_{r+n+1} + C'_r}{2} - \sum_{k=r}^{r+n}\frac{D_k}{\gamma_k}\right) I_d\right\}[P_n^{(r)}] = 0\,.
\end{aligned}
$$

Thus we can write explicitly the coefficients of this differential equation for the four classical sequences; cf. Table 2 and Table 3.

Table 1: Data for the classical orthogonal polynomials

	Hermite	Laguerre	Bessel	Jacobi
$\psi(x)$	$2x$	$x-\alpha-1$	$-2(ax+1)$	$-(\alpha+\beta+2)x+\beta-\alpha$
$\phi(x)$	1	x	x^2	x^2-1
$C_0(x)=-(\psi+\phi')(x)$	$-2x$	$-x+\alpha$	$2[(a-1)x+1]$	$(\alpha+\beta)x+\alpha-\beta$
$D_0=-\left(\psi'+\frac{1}{2}\phi''\right)$	-2	-1	$2a-1$	$\alpha+\beta+1$
$C_k(x)$	$-2x$	$-x+\alpha+2k$	$2\left[(a+k-1)x+\dfrac{a-1}{a+k-1}\right]$	$(\alpha+\beta+2k)x+\dfrac{\alpha^2-\beta^2}{\alpha+\beta+2k}$
$\dfrac{D_k}{\gamma_k}=\frac{2k-1}{2}\phi''-\psi'$	-2	-1	$2(a+k)-1$	$\alpha+\beta+2k+1$
$\sum_{k=r}^{r+n}\dfrac{D_k}{\gamma_k}$	$-2(n+1)$	$-(n+1)$	$(n+1)(2a+n+2r-1)$	$(n+1)(\alpha+\beta+n+2r+1)$

Table 2: Coefficients of the 4th order differential equation (associated Hermite and Laguerre)

$P_n \rightarrow$ $P_n^{(r)} = y \downarrow$	Hermite	Laguerre
Coef. of y''''	1	x^2
Coef. of y'''	0	$5x$
Coef. of y''	$4(-x^2+n+2r)$	$-x^2+2(\alpha+n+2r)x-\alpha^2+4$
Coef. of y'	$-12x$	$3(-x+\alpha+n+2r)$
Coef. of y	$4n(n+2)$	$n(n+2)$

Table 3: Coefficients of the 4th order differential equation (associated Bessel and Jacobi)

$P_n \rightarrow$ $P_n^{(r)} = y \downarrow$	Bessel	Jacobi
Coef. of y''''	x^4	$(x^2-1)^2$
Coef. of y'''	$10x^3$	$10x(x^2-1)$
Coef. of y''	$2[12-n(2a+n+2r-1)-2(a+r-1)^2]x^2$ $-8(a-1)x-4$	$[24-2n(\alpha+\beta+n+2r+1)-(\alpha+\beta+2r)^2]x^2-2(\alpha^2-\beta^2)x$ $+4r^2+4(\alpha+\beta+n)r+2n(\alpha+\beta+n+1)-(\alpha-\beta)^2-8$
Coef. of y'	$3[2-n(2a+n+2r-1)-2(a+r-1)^2]x$ $-12(a-1)$	$3[4-2n(\alpha+\beta+n+2r+1)-(\alpha+\beta+2r)^2]x-3(\alpha^2-\beta^2)$
Coef. of y	$n(n+2)(2a+n+2r-1)(2a+n+2r-3)$	$n(n+2)[(\alpha+\beta+n+2r)^2-1]$

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