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Classification of Almost Parahermitian Manifolds

P.M. GADEA - J.M. MASQUE(*)

RIASSUNTO – *Si ottengono: una classificazione delle varietà quasi parahermitiane e le proprietà caratteristiche delle 136 classi essenzialmente diverse, si studiano, infine, esempi delle classi primitive.*

ABSTRACT – *A classification of almost parahermitian manifolds, a characterization of the 136 essentially different classes, and examples of the primitive ones are given.*

KEY WORDS – *Almost parahermitian manifolds - Parakählerian manifolds - Horizontal lifts of tensor fields - Representations of the full linear group.*

A.M.S. CLASSIFICATION: 53C15 - 53C50

1 – Introduction

As is well known, GRAY and HERVELLA [9] have classified the almost hermitian manifolds (see also [14]), and thus inaugurated a method for the classification of structures given by a 0-deformable (1,1) tensor field J and a metric g compatible with J, defined on a differentiable manifold. The method uses the decomposition of the space W of tensors satisfying the same symmetries as the covariant derivative (with respect to the Levi-Civita connection of g) of the fundamental 2-form, in the invariant and irreducible subspaces of W under the action of the structural group.

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NAVEIRA [18] has obtained a similar classiffication of the riemannian almost product manifolds, which has been studied by several authors (see [2], [8], [15], [16]).

The case of almost complex manifolds with a Norden metric has been solved in a similar way by GANCHEV and BORISOV [7].

In the present paper we give the corresponding classification for the almost parahermitian manifolds, obtaining a characterization of the 136 essentially different classes, and showing examples of the primitive ones, and also of parakählerian manifolds. This classification completes, together with the earlier ones, that of the structures (J,g) on a $2n$-dimensional differentiable manifold M, such that $J \in \mathcal{T}_1^1(M)$ fulfils $J^2 = \pm I$, and J is an isometry of the riemannian metric g, or an anti-isometry of the (necessarily pseudoriemannian of signature (n,n)) metric g.

2 – The classification

2.1 – The classification at the tangent space level

Let (M,g,J) be an almost parahermitian manifold. That is, M is a differentiable manifold, g a pseudoriemannian metric on M, and J an almost product structure on M such that $g(JX,JY) = -g(X,Y)$ for all $X,Y \in \mathcal{X}(M)$, i.e., J is an anti-isometry of g. Then (see [10]) $\dim M = 2n$, and the structural group of the tangent bundle TM can be reduced to the group of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \qquad A \in G\ell(n,\mathbb{R}).$$

Let F be the fundamental 2-form on M, defined by $F(X,Y) = g(JX,Y)$, and ϕ the covariant derivative of F with respect to the Levi-Civita connection of g. We can then write

$$\phi(X,Y,Z) = (\nabla F)(X,Y,Z) = (\nabla_X F)(Y,Z) = g\big((\nabla_X J)Y, Z\big),$$

and it is immediate to prove the following:

$$\phi(X,Y,Z) = -\phi(X,Z,Y), \tag{2.1.1}$$

$$\phi(X,JY,JZ) = \phi(X,Y,Z). \tag{2.1.2}$$

The tangent space $T = T_xM$ at each point $x \in M$ splits as the direct sum of n-dimensional subspaces $T = \mathcal{V} \oplus \mathcal{H}$, such that we can choose a basis $\{A_1, \ldots, A_n, U_1, \ldots, U_n\}$ of T, where $\{A_1, \ldots, A_n\}$ and $\{U_1, \ldots, U_n\}$ are bases of $\mathcal{V}$ and $\mathcal{H}$ respectively, in which the expressions of g and J are

$$g = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, J = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

So, $\mathcal{V}$ and $\mathcal{H}$ are the eigenspaces J_+ and J_- of T corresponding to the eigenvalues $+1$ and -1 of J, respectively; and both spaces being maximally isotropic with respect to g.

Let us now consider the vector space

$$W = \left\{\alpha \in \otimes^3 T^*; \alpha(X,Y,Z) = -\alpha(X,Z,Y),\ \alpha(X,JY,JZ) = \alpha(X,Y,Z)\right\}$$

of tensors in $\otimes^3 T^*$ satisfying the symmetries (2.1.1) and (2.1.2), and let us study the decomposition of W in invariant and irreducible subspaces under the natural action of the group $G\ell(n, \mathbb{R})$. According to the symmetries of W we have

(2.1.3) $$W = (\mathcal{V}^* \otimes \Lambda^2 \mathcal{V}^*) \oplus (\mathcal{V}^* \otimes \Lambda^2 \mathcal{H}^*) \oplus (\mathcal{H}^* \otimes \Lambda^2 \mathcal{V}^*) \oplus (\mathcal{H}^* \otimes \Lambda^2 \mathcal{H}^*).$$

Firstly, we study the decomposition of the summand $\mathcal{V}^* \otimes \Lambda^2 \mathcal{V}^*$. It follows from the theory of representations of the full linear group (see [1], [4], [7], [20]) that this decomposition is $\mathcal{V}^* \otimes \Lambda^2 \mathcal{V}^* = \Lambda^3 \mathcal{V}^* \oplus \mathcal{Y}$, where $\mathcal{Y}$ is the subspace of $\otimes^3 \mathcal{V}^*$ corresponding to the Young element $e+$ (12) - (23) - (132); that is,

$$\mathcal{Y} = \Big\{\alpha \in \otimes^3 \mathcal{V}^*; \alpha(A,B,C) = \beta(A,B,C) + \beta(B,A,C) - \beta(A,C,B) - \beta(C,A,B), \text{ for all } A,B,C \in \mathcal{V} \text{ and some } \beta \in \otimes^3 \mathcal{V}^*\Big\}.$$

Thus, as it is easily proved: $\mathcal{V}^* \otimes \Lambda^2 \mathcal{V}^* = W_1 \oplus W_2$, where

$$W_1 = \Lambda^3 \mathcal{V}^* = \Big\{\alpha \in \mathcal{V}^* \otimes \Lambda^2 \mathcal{V}^*; \alpha(A,B,C) = = \frac{1}{3} \underset{ABC}{\mathfrak{S}} \alpha(A,B,C) \text{ for all } A,B,C \in \mathcal{V}\Big\}$$

and

$$W_2 = \mathcal{Y} = \left\{\alpha \in \mathcal{V}^* \otimes \Lambda^2\mathcal{V}^*;\ \underset{ABC}{\mathfrak{S}}\,\alpha(A,B,C) = 0 \text{ for all } A,B,C \in \mathcal{V}\right\},$$

where $\underset{ABC}{\mathfrak{S}}$ stands for the cyclic sum with respect to A, B and C.

In order to study the decomposition of the second summand $\mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*$ of (2.1.3), we can use the musical isomorphism $\sharp\colon \mathcal{H}^* \longrightarrow \mathcal{V}$ associated to g, and hence it suffices to study the decomposition of the space $\mathcal{V}^* \otimes \Lambda^2\mathcal{V}$, which in turn is $S\ell(n,\mathbb{R})$-isomorphic to $\mathcal{V}^* \otimes \Lambda^{n-2}\mathcal{V}^*$ by the usual isomorphism μ. It can be proved that we have the decomposition in invariant and irreducible subspaces under $G\ell(n,\mathbb{R})$: $\mathcal{V}^* \otimes \Lambda^2\mathcal{H}^* = W_3 \oplus W_4$, where W_3 and W_4 are isomorphic, respectively, to the subspaces $\mathcal{N} = \ker c_1^1$ and $\mathcal{V} = c_1^1(\mathcal{V}^* \otimes \Lambda^2\mathcal{V})$ (c_1^1 being the contraction $c_1^1(\theta \otimes A \wedge B) = c_1^1(\theta(A)B - \theta(B)A$, A, $B \in \mathcal{V}$, $\theta \in \mathcal{V}^*$) of $\mathcal{V}^* \otimes \Lambda^2\mathcal{V}$, and to the subspaces $\mathcal{Y}$ and $\Lambda^{n-1}\mathcal{V}^*$ of $\mathcal{V}^* \otimes \Lambda^{n-2}\mathcal{V}^*$, such that $\mathcal{V}^* \otimes \Lambda^2\mathcal{V} = \mathcal{N} \oplus \mathcal{V}$ and $\mathcal{V}^* \otimes \Lambda^{n-2}\mathcal{V}^* = \mathcal{Y} \oplus \Lambda^{n-1}\mathcal{V}^*$ are decompositions in invariant and irreducible subspaces under $G\ell(n,\mathbb{R})$. The fact that μ is a $S\ell(n,\mathbb{R})$-isomorphism does not cause any problem. In fact, let $\mathcal{N}' = \mu(\mathcal{N})$. Obviously, $\mathcal{N}$ is $G\ell(n,\mathbb{R})$-invariant. If we prove that $\mathcal{N}'$ is $S\ell(n,\mathbb{R})$-irreducible, then $\mathcal{N}$ will also be $S\ell(n,\mathbb{R})$-irreducible and, hence, $G\ell(n,\mathbb{R})$-irreducible. If this were not so, it would exist a $G\ell(n,\mathbb{R})$-invariant proper subspace $\mathcal{N}_1$ of $\mathcal{N}$; thus, $\mathcal{N}_1$ would be $S\ell(n,\mathbb{R})$-invariant. But then $\mathcal{N}$ would not be $S\ell(n,\mathbb{R})$-irreducible.

We can write

$$W_3 = \left\{\alpha \in \mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*;\ \sum_{i=1}^{n} \alpha(A_i, U_i, U_j) = 0,\ 1 \le j \le n\right\},$$

$$\{A_1, \dots, A_n, U_1, \dots, U_n\}$$

being an adapted basis of T, and

$$W_4 = \Big\{\alpha \in \mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*;\ \alpha(A,U,V) = \beta(U)g(A,V) - \beta(V)g(A,U)$$

$$\text{for all } A \in \mathcal{V}, U, V \in \mathcal{H}, \text{ and some } \beta \in \mathcal{H}^*\Big\}.$$

The subspaces $W_5, \dots, W_8$ are defined as $W_1, \dots, W_4$, respectively, under interchange of $\mathcal{V}$ and $\mathcal{H}$.

In low dimension, some of these subspaces are zero. Thus, for $n = 1$, it is immediate from the symmetries of W that $W = \{0\}$. For $n = 2$, since $\Lambda^3\mathcal{V}^* = \Lambda^3\mathcal{H}^* = 0$, we have $W_1 = W_5 = \{0\}$. If $\alpha \in \mathcal{V}^* \otimes \Lambda^2\mathcal{V}^*$, it is immediate that $\underset{ABC}{\mathfrak{S}}\alpha(A,B,C) = 0$, for all $A,B,C \in \mathcal{V}$; hence $W_2 = \mathcal{V}^* \otimes \Lambda^2\mathcal{V}^*$. Similarly, $W_6 = \mathcal{H}^* \otimes \Lambda^2\mathcal{H}^*$. The space $\mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*$ admits the basis $\{\theta_1 \otimes \omega_1 \wedge \omega_2\, ,\, \theta_2 \otimes \omega_1 \wedge \omega_2\}$, $\{\theta_1, \theta_2, \omega_1, \omega_2\}$ being an adapted basis of T^*. Thus, $\mathcal{V}^* \otimes \Lambda^2\mathcal{H}^* \simeq \mathcal{V}^*$. Hence $W_3 = W_7 = \{0\}$ and $W_4 = \mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*$, $W_8 = \mathcal{H}^* \otimes \Lambda^2\mathcal{V}^*$.

For $n \geq 3$, the eight subspaces are not trivial.

We have thus proved:

THEOREM 2.1. *The space W splits into the direct sum $W = \bigoplus_{i=1}^{8} W_i$ of the following invariant and irreducible subspaces under $Gl(n,\mathbb{R})$:*

$$W_1 = \Lambda^3\mathcal{V}^*\, ;\, W_2 = \left\{\alpha \in \mathcal{V}^* \otimes \Lambda^2\mathcal{V}^*\, ;\, \underset{ABC}{\mathfrak{S}}\alpha(A,B,C) = 0 \right.$$

$$\left. \textit{for all } A,B,C \in \mathcal{V} \right\};$$

$$W_3 = \left\{\alpha \in \mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*\, ;\, \sum_{i=1}^{n}\alpha(A_i,U_i,U_j) = 0\, ,\ 1 \leq j \leq n\, , \right.$$

$$\left. \textit{where } \{A_i, U_i\}\, , 1 \leq i \leq n\, , \textit{is an adapted basis of } T \right\};$$

$$W_4 = \left\{\alpha \in \mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*\, ;\, \alpha(A,U,V) = \beta(U)g(A,V) - \beta(V)g(A,U)\, , \right.$$

$$\left. \textit{for all } A \in \mathcal{V}\, , U,V \in \mathcal{H}\, , \textit{and some } \beta \in \mathcal{H}^* \right\}.$$

The subspaces $W_5,\ldots,W_8$ are defined as $W_1,\ldots,W_4$, respectively, interchanging $\mathcal{V}$ and $\mathcal{H}$. If $n = 1$, then $W = \{0\}$. If $n = 2$, then $W_1 = W_3 = W_5 = W_7 = \{0\}$. If $n \geq 3$, then $\dim W_i \geq 1$, $i = 1,\ldots,8$.

Thus, we have 2^8 classes of almost parahermitian manifolds. However, since $\mathcal{V}$ and $\mathcal{H}$ are equivalent (it is enough to change J by $-J$), we have only 136 essentially different classes if $n \geq 3$. If $n = 2$, that figure is reduced to 10.

2.2 – The primitive classes of almost parahermitian manifolds

We say that an almost parahermitian manifold M is of class $\mathcal{W}_i, \mathcal{W}_i \oplus \mathcal{W}_j$, etc., $i = 1, \ldots, 8$, if for every $x \in M$ we have $\phi_x \in W_i$, $\phi_x \in W_i \oplus W_j$, etc. We denote the class corresponding to $\{0\}$ by $\mathcal{PK}$ (parakählerian) and the one corresponding to W by $\mathcal{W}$. Since T is the model of each tangent space T_xM, $x \in M$, the tensor field ϕ corresponding to a given class satisfies the conditions of the tensor α of the invariant subspace of W corresponding to this class. We can thus establish the conditions for the class $\mathcal{W}_1, \mathcal{W}_2, \ldots$, in terms of an adapted local basis $\{A_i, U_i\}$, $1 \leq i \leq n$. From now on, we denote by $\mathcal{V}$ and $\mathcal{H}$ the eigenbundles of TM corresponding to the eigenvalues $+1$ and -1 of the almost product structure J on M, respectively.

Thus, for the class $\mathcal{W}_1$ we have: $\phi \in \mathcal{V}^* \otimes \Lambda^2\mathcal{V}^*$ and $\phi(A, A, B) = 0$ for all $A, B \in \mathcal{V}$. Hence, $\phi(X, X, Y) = 0$ for all $X, Y \in \mathcal{X}(M)$. Moreover, from $\phi(U, A, B) = 0$ we obtain $\nabla_U A \in \mathcal{V}$, for all $A, B \in \mathcal{V}$, $U \in \mathcal{H}$. That is, $\mathcal{V}$ is parallel along $\mathcal{H}$. From $\phi(A, U, V) = 0$ we also obtain $\nabla_A U \in \mathcal{H}$. That is, $\mathcal{H}$ is parallel along $\mathcal{V}$. Finally, from $\phi(U, V, W) = 0$ we conclude that $\nabla_U V \in \mathcal{H}$. That is, $\mathcal{H}$ is autoparallel. In particular, $\mathcal{H}$ is integrable, and the connected maximal integral submanifold through each point is totally geodesic.

Conversely, if $(\nabla_X F)(X, Y) = 0$, $\nabla_A U \in \mathcal{H}$, $\nabla_U A \in \mathcal{V}$, $\nabla_U V \in \mathcal{H}$ for all $A \in \mathcal{V}, U, V \in \mathcal{H}, X, Y \in \mathcal{X}(M)$, from the last three conditions it follows that $\phi(U, A, B) = \phi(A, U, V) = \phi(U, V, W) = 0$, and from the symmetries of ϕ we obtain: $\phi(A, B, U) = \phi(A, U, B) = \phi(U, A, V) = \phi(U, V, A) = 0$. Hence $\phi \in \mathcal{V}^* \otimes \Lambda^2\mathcal{V}^*$, and since $\phi(X, X, Y) = 0$, we also have $\phi(A, A, B) = 0$. We thus conclude that the characteristic conditions of the class $\mathcal{W}_1$ are the following:

$$(\nabla_X F)(X, Y) = 0, \; \nabla_U A \in \mathcal{V}, \; \nabla_X U \in \mathcal{H},$$
$$\text{for all } A \in \mathcal{V}, \; U \in \mathcal{H}, \; X, Y \in \mathcal{X}(M).$$

In $\mathcal{W}_1$, $\mathcal{H}$ is parallel. Moreover, the condition $(\nabla_X F)(X,Y) = 0$ is equivalent to adding to the other ones the following: $\nabla_A A \in \mathcal{V}$ for all $A \in \mathcal{V}$. That is, every geodesic with initial tangent vector in $\mathcal{V}$ maintains its tangent vector in $\mathcal{V}$.

Similarly, for the class $\mathcal{W}_5$ we have the characteristic conditions:

$$(\nabla_X F)(X,Y) = 0\,,\ \nabla_X A \in \mathcal{V}\,,\ \nabla_A U \in \mathcal{H}\,,$$
$$\text{for all } A \in \mathcal{V}\,,\ U \in \mathcal{H}\,,\ X,Y \in \mathcal{X}(M)\,.$$

We now consider the class $\mathcal{W}_2$. We have

$$dF(X,Y,Z) = \underset{XYZ}{\mathfrak{S}}\, g\big((\nabla_X J)Y, Z\big) = \tag{2.2.1}$$
$$= \underset{XYZ}{\mathfrak{S}}\, \phi(X,Y,Z) (X,Y,Z \in \mathcal{X}(M))\,.$$

Since $\phi \in \mathcal{V}^* \otimes \Lambda^2 \mathcal{V}^*$, it follows that $\phi(U,X,Y) = \phi(X,U,Y) = \phi(X,Y,U) = 0$ for all $U \in \mathcal{H}$, $X,Y \in \mathcal{X}(M)$. But from (2.2.1) we obtain $dF(A,B,C) = 0$, and thus $dF = 0$. As in the case $\mathcal{W}_1$ we also have $\nabla_X U \in \mathcal{H}$, $\nabla_U A \in \mathcal{V}$, for all $A \in \mathcal{V}$, $U \in \mathcal{H}$, $X \in \mathcal{X}(M)$, and $\phi \in \mathcal{V}^* \otimes \Lambda^2 \mathcal{V}^*$.

Furthermore, from $dF = 0$ and (2.2.1) it follows that $\underset{ABC}{\mathfrak{S}}\, \phi(A,B,C) = 0$ for all $A,B,C \in \mathcal{V}$. Consequently, the conditions

$$dF = 0\,, \nabla_X U \in \mathcal{H}\,, \nabla_U A \in \mathcal{V} \text{ for all } A \in \mathcal{V}\,, U \in \mathcal{H}\,, X \in \mathcal{X}(M)$$

characterize the class $\mathcal{W}_2$. Similarly, the conditions

$$dF = 0\,, \nabla_X A \in \mathcal{V}\,, \nabla_A U \in \mathcal{H} \text{ for all } A \in \mathcal{V}\,, U \in \mathcal{H}\,, X \in \mathcal{X}(M)$$

characterize the class $\mathcal{W}_6$.

As for the class $\mathcal{W}_3$, since in an adapted local basis $\{A_i, U_i\}$ the metric is expressed by the standard matrix

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

we have:

$$\begin{aligned}\delta F(X) &= -\sum_{i=1}^{n}\left\{\left(\nabla_{A_i}F\right)\left(U_i,X\right)+\left(\nabla_{U_i}F\right)\left(A_i,X\right)\right\}\\ &= -\sum_{i=1}^{n}\left\{\phi\left(A_i,U_i,X\right)+\phi\left(U_i,A_i,X\right)\right\} \text{ for all } X\in\mathcal{X}(M).\end{aligned}$$

In $\mathcal{W}_3$, $\phi \in \mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*$ and $\Sigma_i\phi(A_i,U_i,U_j) = 0$ for every fixed j. We therefore have $\delta F(X) = -\Sigma_{i,j}(hX)^j\phi(A_i,U_i,U_j) = 0$, $hX = (hX)^jU_j$ being the component of X in $\mathcal{H}$. Furthermore, since $\phi \in \mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*$, we have $\nabla_X A \in \mathcal{V}$, $\nabla_U V \in \mathcal{H}$ for all $A \in \mathcal{V}$, $U,V \in \mathcal{H}$, $X \in \mathcal{X}(M)$. Conversely, from these conditions and the symmetries of ϕ we deduce that $\phi \in \mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*$. But since $\delta F = 0$ as well, we obtain $\Sigma_i\phi(A_i,U_i,U_j) = 0$, $1 \leq j \leq n$. Hence, the characteristic conditions of $\mathcal{W}_3$ are

$$\delta F = 0, \nabla_X A \in \mathcal{V}, \nabla_U V \in \mathcal{H} \text{ for all } A \in \mathcal{V}, U,V \in \mathcal{H}, X \in \mathcal{X}(M),$$

and thus $\mathcal{V}$ and $\mathcal{H}$ are autoparallel, the integral submanifolds of both distributions are totally geodesic and J is integrable.

Similarly, the characteristic conditions for $\mathcal{W}_7$ are:

$$\delta F = 0, \nabla_A B \in \mathcal{V}, \nabla_X U \in \mathcal{H} \text{ for all } A,B \in \mathcal{V}, U \in \mathcal{H}, X \in \mathcal{X}(M),$$

and the geometric properties are similar.

With regard to $\mathcal{W}_4$, we have $\phi \in \mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*$ and

$$\phi(A,U,V) = \beta(U)g(A,V) - \beta(V)g(A,U) \quad \text{for all } A \in \mathcal{V}, U,V \in \mathcal{H} \text{ and some } \beta \in \mathcal{H}^* . \tag{2.2.2}$$

Hence, writing $vX = \frac{1}{2}(X + JX)$, $hX = \frac{1}{2}(X - JX)$, we obtain:

$$\begin{aligned}\phi(X,Y,Z) &= \beta(hY)g(vX,hZ) - \beta(hZ)g(vX,hY) =\\ &= \frac{1}{4}\{\beta(Y-JY)g(X,Z-JZ) - \beta(Z-JZ)g(X,Y-JY)\},\end{aligned}$$

for all $X, Y, Z \in \mathcal{X}(M)$. Thus,

$$\begin{aligned}\delta F(U_j) &= -\sum_{i=1}^{n} \phi(A_i, U_i, U_j) = \\ &= -\sum_{i=1}^{n} \frac{1}{4}\left\{\beta(2U_i)g(A_i, 2U_j) - \beta(2U_j)g(A_i, 2U_i)\right\} = \\ &= -\beta(U_i)\delta_{ij} + n\beta(U_j) = (n-1)\beta(U_j).\end{aligned}$$

Consequently,

$$\tag{2.2.3} \begin{aligned}(\nabla_X F)(Y, Z) = \frac{1}{4(n-1)}\Big\{&\delta F(Y - JY)g(X, Z - JZ) \\ &- \delta F(Z - JZ)g(X, Y - JY)\Big\}\end{aligned}$$

Moreover, as in the case $\mathcal{W}_3$, we have

$$\tag{2.2.4} \nabla_X A \in \mathcal{V}, \nabla_U V \in \mathcal{H} \text{ for all } A \in \mathcal{V}, U, V \in \mathcal{H}, X \in \mathcal{X}(M),$$

and geometric properties as in $\mathcal{W}_3$.

Conversely, that ϕ belongs to $\mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*$ is obtained in the usual way, and property (2.2.2) is immediate from (2.2.3). Thus, the characteristic conditions for the class $\mathcal{W}_4$ are given by (2.2.3) and (2.2.4). The corresponding conditions for $\mathcal{W}_8$ are

$$\tag{2.2.5} \begin{aligned}(\nabla_X F)(Y, Z) = \frac{1}{4(n-1)}\Big\{&\delta F(Y + JY)g(X, Z + JZ) \\ &- \delta F(Z + JZ)g(X, Y + JY)\Big\}\end{aligned}$$

and

$$\tag{2.2.6} \nabla_A B \in \mathcal{V}, \nabla_X U \in \mathcal{H} \text{ for all } A, B \in \mathcal{V}, U \in \mathcal{H}, X \in \mathcal{X}(M).$$

Since $\phi \in \mathcal{V}^* \otimes \Lambda^2\mathcal{H}^*$ for the class $\mathcal{W}_4$ and for $\mathcal{W}_8$ is $\phi \in \mathcal{H}^* \otimes \Lambda^2\mathcal{V}^*$, we can add the right hand sides in (2.2.3) and (2.2.5) in order to obtain for both classes the following characterization:

$$(\nabla_X F)(Y,Z) = \frac{1}{2(n-1)}\Big\{\delta F(Y)g(X,Z) - \delta F(Z)g(X,Y) + \delta F(JY)g(X,JZ) - \delta F(JZ)g(X,JY)\Big\}, \tag{2.2.7}$$

with (2.2.4) for $\mathcal{W}_4$, and with (2.2.6) for $\mathcal{W}_8$. In both cases, as in $\mathcal{W}_3$ and $\mathcal{W}_7$, $\mathcal{V}$ and $\mathcal{H}$ are autoparallel, the integral submanifolds are totally geodesic, and J is integrable.

In a similar way to [9], we give the following

DEFINITION 2.2. *The almost parahermitian manifolds (M, g_1, J_1) and (M, g_2, J_2) are said to be locally conformally related if $J_1 = J_2$ and for every $x \in M$ there exists an open neighbourhood N of x such that g_1 and g_2 are conformally related in N.*

DEFINITION 2.3. *The Lee form θ of the almost parahermitian manifold (M, g, J) is the 1-form on M defined by*

$$\theta(X) = -\frac{1}{n-1}\delta F(JX) \qquad (X \in \mathcal{X}(M)).$$

Then the following can be proved:

PROPOSITON 2.4. *The almost parahermitian manifold (M, g, J) is locally conformally related to a parakählerian manifold (M, g_0, J) if and only if: $N = 0$, $dF + \theta \wedge F = 0$ and $d\theta = 0$, where N denotes the Nijenhuis tensor of J and θ the Lee form of (M, g, J).*

REMARK. Let ψ be the tensor field on (M,g,J) defined by $g(\psi(X,Y),Z) = (d\theta + \theta \wedge F)(X,Y,Z)$. Then the three tensor fields N, ψ and $d\theta$ are global conformal invariants of the structure.

Moreover, it should be also noted that if we define a tensor field μ on M by the formula (cf. [9]):

$$g(\mu(X,Y),Z) = (\nabla_X F)(Y,Z) - \frac{1}{2(n-1)}\Big\{\delta F(Y)g(X,Z)$$

$$-\delta F(Z)g(X,Y) + \delta F(JY)g(X,JZ) - \delta F(JZ)g(X,JY)\Big\},$$

for all $X,Y,Z \in \mathcal{X}(M)$, then (2.2.7) is equivalent to say that $\mu = 0$, and the following can be proved:

PROPOSITION 2.5. *i) The tensor field μ is a global conformal invariant.*

ii) $\mu = 0$ if and only if $N = 0$ and $dF + \theta \wedge F = 0$, or equivalently, if and only if $\nabla_A B \in \mathcal{V}$, $\nabla_U V \in \mathcal{H}$ for all $A,B \in \mathcal{V}$, $U,V \in \mathcal{H}$, and $dF + \theta \wedge F = 0$.

From Propositions 2.4 and 2.5 it follows that any almost parahermitian manifold locally conformally equivalent to a parakählerian manifold is in $\mathcal{W}_4 \oplus \mathcal{W}_8$. We also note that the pair $(\mu, d\theta)$ is the analogous to the Weyl conformal tensor of riemannian geometry. The examples which we shall consider in §3 provide in particular examples of manifolds satisfying $\mu = 0$ and which however are not locally conformally equivalent to any parakählerian manifold (see the last remark of the paper).

The following equivalence will be used in §3.

PROPOSITION 2.6. *The condition*

$$(\nabla_A F)(U,V) = \frac{1}{n-1}\Big\{\delta F(U)g(A,V) - \delta F(V)g(A,U)\Big\}$$

for all $A \in \mathcal{V}, U,V \in \mathcal{H}$

for $\mathcal{W}_4$ is equivalent to the condition

(2.2.8) $$\underset{UVW}{\mathfrak{S}}\Big(\nabla_{g(A,U)A}F\Big)(V,W) = 0 \text{ for all } A \in \mathcal{V}\ U,V,W \in \mathcal{H}.$$

The similar equivalence for $\mathcal{W}_8$ is also satisfied.

The proof follows from a simple computation and thus is omitted.

In view of the foregoing, we have the following table of properties of the primitive classes $\mathcal{W}_1, \dots, \mathcal{W}_8$:

Table 1: Primitive classes of almost parahermitian manifolds of dimension ≥ 6

	$\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}_3$	$\mathcal{W}_4$	$\mathcal{W}_5$	$\mathcal{W}_6$	$\mathcal{W}_7$	$\mathcal{W}_8$
$(\nabla_X F)(X,Y) = 0$	*				*			
$dF = 0$		*				*		
$\delta F = 0$			*				*	
$(\nabla_X F)(Y,Z) = [1/2(n-1)]\{\delta F(Y)g(X,Z) - \delta F(Z)g(X,Y) + \delta F(JY)g(X,JZ) - \delta F(JZ)g(X,JY)\}$				*				*
$\nabla_A B \in \mathcal{V}$			*	*	*	*	*	*
$\nabla_U A \in \mathcal{V}$	*	*	*	*	*	*		
$\nabla_A U \in \mathcal{H}$	*	*			*	*	*	*
$\nabla_U V \in \mathcal{H}$	*	*	*	*			*	*

If $\dim M = 4$, the classes $\mathcal{W}_1, \mathcal{W}_3, \mathcal{W}_5$ and $\mathcal{W}_7$ do not exist. According to the fact that $\mathcal{V}$ and $\mathcal{H}$ are, respectively, the $(+1)$-eigenbundle and the (-1)-eigenbundle associated to J, we will call the manifolds M in the earlier classes:

$(+)$ -*nearly parakählerian*, if $M \in \mathcal{W}_1$,

$(-)$ -*nearly parakählerian*, if $M \in \mathcal{W}_5$,

$(+)$ -*almost parakählerian*, if $M \in \mathcal{W}_2$,

$(-)$ -*almost parakählerian*, if $M \in \mathcal{W}_6$,

$(+)$ -*parahermitian semi-parakählerian*, if $M \in \mathcal{W}_3$,

$(-)$ -*parahermitian semi-parakählerian*, if $M \in \mathcal{W}_7$.

We have named the last two classes according to the first factor in the space to which ϕ belongs.

We have not found an appropriate name for the manifolds in the classes $\mathcal{W}_4$ and $\mathcal{W}_8$.

2.3 – Characteristic conditions of the 136 essentially different classes of almost parahermitian manifolds

In order to obtain the characteristic conditions for each of the 136 essentially different classes, it suffices to consider the following 8 properties:

1) $\underset{ABC}{\mathfrak{S}}(\nabla_A F)(B,C) = 0$ for all $A, B, C \in \mathcal{V}$.
2) $\nabla_A A \in \mathcal{V}$ for all $A \in \mathcal{V}$.
3) $(\nabla_A F)(U,V) = \theta(V)g(A,U) - \theta(U)g(A,V)$ for all $A \in \mathcal{V}$, $U, V \in \mathcal{H}$.
4) $\sum_{i=1}^{n}(\nabla_{A_i}F)(U_i, U) = 0$ for all $U \in \mathcal{H}$, $\{A_i, U_i\}$ being a local adapted frame.
5) $\underset{UVW}{\mathfrak{S}}(\nabla_U F)(V,W) = 0$ for all $U, V, W \in \mathcal{H}$.
6) $\nabla_U U \in \mathcal{H}$ for all $U \in \mathcal{H}$.
7) $(\nabla_U F)(A,B) = \theta(A)g(U,B) - \theta(B)g(U,A)$ for all $A, B \in \mathcal{V}$, $U \in \mathcal{H}$.
8) $\sum_{i=1}^{n}(\nabla_{U_i}F)(A_i, A) = 0$ for all $A \in \mathcal{V}$, $\{A_i, U_i\}$ being a local adapted frame.

Then the class $\mathcal{PK}$ is the class of manifolds which satisfy the above 8 properties; the class $\mathcal{W}_i (i = 1, \dots, 8)$ is characterized by all these properties except the corresponding to i); the class $\mathcal{W}_i \oplus \mathcal{W}_j$, by the 6 properties different from the corresponding to i) and j), and so on. Thus, in particular, it is immediate that the class $\mathcal{PK}$ is characterized by the condition $(\nabla_X F)(Y,Z) = 0$ for all $X, Y, Z \in \mathcal{X}(M)$, and the primitive classes $\mathcal{W}_i$ by the conditions in Table 1. We shall see some examples of those classes in §3. The study of the geometric properties of the classes different from $\mathcal{PK}$, $\mathcal{W}_i$ and $\mathcal{W}$, and examples of them remains to be done.

3 – Examples

3.1 – Examples of parakählerian manifolds

3.1.a) *The paracomplex projective space of* Libermann [13].

This is the product $P_n(\mathbb{R}) \times P_n(\mathbb{R})$ of two real projective spaces of the same dimension, endowed with the structure specified in [13, p. 89].

3.1.b) *The "paracomplex projective space" $P_n(B)$ and the "reduced paracomplex projective space" $P_n(B)/\mathbb{Z}_2$ of* GADEA and MONTESINOS AMILIBIA [5], [6].

These are the spaces $P_n(B) \simeq G\ell_0(n+1,\mathbb{R})/\big(G\ell_0(1,\mathbb{R})\times G\ell_0(n,\mathbb{R})\big)$ and $P_n(B)/\mathbb{Z}_2 \simeq G\ell_0(n+1,\mathbb{R})/\big((G\ell(1,\mathbb{R})\times G\ell(n,\mathbb{R})\big)_0$, endowed with the structure specified in [6]. They are diffeomorphic to TS^n and $TP_n(\mathbb{R})$, respectively, and they are spaces of constant paraholomorphic sectional curvature, which are called paracomplex projective spaces by its analogy as symmetric spaces with the complex projective spaces $P_n(\mathbb{C})$.

REMARKS. 1) Of course, all the parahermitian symmetric spaces corresponding to the symmetric pairs in the infinitesimal classification of KANEYUKI and KOZAI [11] belong to the class $\mathcal{PK}$. Specifically, the space $P_n(B)$ and $P_n(B)/\mathbb{Z}_2$ correspond to the case $m = 1$ of the symmetric pair $(\mathfrak{sl}(n+m,\mathbb{R}), \mathfrak{sl}(n,\mathbb{R}) + \mathfrak{sl}(m,\mathbb{R}) + \mathbb{R})$. For more examples, see [11, p. 92-93].

2) The relation between examples 3.1.a) and 3.1.b) considered as symmetric spaces will be studied in a forthcoming paper.

3.1.c) *The parakählerian tangent bundle of* CRUCEANU [3]

This is a particular case of a general structure studied by Cruceanu on the tangent bundle TM of a riemannian manifold M endowed with a linear connection ∇, which we shall consider in the next section. When ∇ is the Levi-Civita connection of the riemannian metric g of M, and g has no curvature, then TM is parakählerian.

3.2 – The structure of Cruceanu. Examples of manifolds in the primitive classes

Let M be a riemannian n-manifold with metric g, and let ∇ be a linear connection on M. Let X^V be the vertical lift of $X \in \mathcal{X}(M)$ and X^H, g^H the horizontal lifts of X and g with respect to ∇ to the tangent bundle TM, in the sense of [21]. Then (see [3]), we have that (TM, g^H, J) is an almost parahermitian manifold, where J is defined by

$$J(X^V) = X^V, \qquad J(X^H) = -X^H. \tag{3.2.1}$$

From [21, pp. 12, 98, 100, 101] we have:
i) The expression of g^H in an adapted local frame on TM is:

$$g^H = \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix}, \tag{3.2.2}$$

ii) We have the following formulas:

$$\begin{aligned} &[X^V, Y^V] = 0, \\ &[X^V, Y^H] = -(\nabla_X Y)^V, \\ &[X^H, Y^H] = [X,Y]^H H - \gamma \widehat{R}(X,Y), \end{aligned} \tag{3.2.3}$$

where $\widehat{R}$ denotes the curvature tensor of the linear connection on M defined by $\widehat{\nabla}_X Y = \nabla_Y X + [X,Y] (X, Y \in \mathcal{X}(M))$.

Let $\tilde{\nabla}$ be the Levi-Civita connection of g^H. We first consider the properties of parallelism with respect to $\tilde{\nabla}$ of the distributions $\mathcal{V}$ and $\mathcal{H}$ on TM corresponding to the eigenvalues $+1$ and -1 of J, respectively. They will be called vertical and horizontal distributions, respectively.

From (3.2.2), (3.2.3) and Koszul formula ([12, p. 160]), it follows that

$$\tilde{\nabla}_A B \in \mathcal{V} \quad \text{for all} \quad A, B \in \mathcal{V}. \tag{3.2.4}$$

In a similar way we obtain

$$\tilde{\nabla}_U A \in \mathcal{V} \quad \text{for all} \quad A \in \mathcal{V}, \quad U \in \mathcal{H}. \tag{3.2.5}$$

By virtue of (3.2.4) we have that $\mathcal{V}$ is autoparallel, and accordingly, it is integrable. The integral manifolds are totally geodesic. Taking into account (3.2.5), it follows that $\mathcal{V}$ is parallel. Consequently (see Table 1), the structure of Cruceanu provides examples of the classes $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$ and $\mathcal{W}_6$.

Let $\overset{0}{\nabla}$ be the Levi-Civita connection of g, and $\mathcal{A}$ the difference tensor $\mathcal{A} = \nabla - \overset{0}{\nabla}$. Computing as in the earlier cases, and from [12, p. 160, Cor. 2.4], we obtain: $\tilde{\nabla}_A U \in \mathcal{H}$ for all $A \in \mathcal{V}$, $U \in \mathcal{H}$ if and only if

$g(\mathcal{A}(X,Y),Z) = g(\mathcal{A}(X,Z),Y)$, for all $X,Y,Z \in \mathcal{X}(M)$. Similarly, it can be proved that

$$\text{(3.2.6)} \qquad \tilde{\nabla}_U V \in \mathcal{H} \qquad \text{for all} \quad U,V \in \mathcal{H}$$

if and only if $\widehat{R} = 0$.

For (TM, g^H, J) to belong to $\mathcal{W}_3$ we must have: 1) That $\mathcal{V}$ be parallel, which is always the case; 2) that $\mathcal{H}$ be autoparallel, i.e., according to (3.2.6), that $\widehat{R} = 0$; 3) that $\delta F = 0$. It suffices to evaluate δF on $A \in \mathcal{V}$ and on $U \in \mathcal{H}$. In an adapted frame, from (3.2.2), we have

$$\left(g^H\right)^{-1} = \begin{pmatrix} 0 & g^{-1} \\ g^{-1} & 0 \end{pmatrix}.$$

As a calculation shows, for (TM, g^H, J) we always have: $\delta F(A) = 0$ for all $A \in \mathcal{V}$. Moreover, from the symmetries of $\tilde{\nabla} F$, (3.2.3), the expression for hX and [12, p. 160, Cor. 2.4], we obtain $\delta F(U) = 0$ for all $U \in \mathcal{H}$ if and only if $\flat(c_{12}\mathcal{A}) = c_1^1\mathcal{A}$, where $c_1^1\mathcal{A}$ denotes the contraction of the contravariant index with the first covariant index of $\mathcal{A}$, $c_{12}\mathcal{A}$ the metric contraction (see [19,p.82]) and $\flat$ denotes the musical isomorphism $\flat : TM \longrightarrow T^*M$ associated to g. We note that the linear connections on M fulfilling the above condition constitute an affine subspace.

We now consider the class $\mathcal{W}_4$. In this case, condition (2.2.8) becomes: $\underset{UVW}{\mathfrak{S}}(\tilde{\nabla}_{g^H(U,A)}F)(V,W) = 0$ for all $A \in \mathcal{V}$, $U,V,W \in \mathcal{H}$, which can be written in terms of $\mathcal{A}$ as

$$\underset{YZW}{\mathfrak{S}} g(X,Y)\{g(\mathcal{A}(X,W),Z) - g(\mathcal{A}(X,Z),W)\} = 0,$$
$$X,Y,Z,W \in \mathcal{X}(M).$$

As for the classes $\mathcal{W}_5$ and $\mathcal{W}_6$, from Table 1 we obtain the common conditions: i) $\nabla_A U \in \mathcal{H}$ for all $A \in \mathcal{V}$, $U \in \mathcal{H}$, or equivalently $g(\mathcal{A}(X,Y),Z) = g(\mathcal{A}(X,Z),Y)$ for all $X,Y,Z \in \mathcal{X}(M)$; ii) there exist $U,V \in \mathcal{H}$ such that $\nabla_U V \notin \mathcal{H}$, or equivalently $\widehat{R} \neq 0$. Furthermore, for $\mathcal{W}_5$ we have $(\tilde{\nabla}_X F)(X,Y) = 0$ for all $X,Y,Z \in \mathcal{X}(TM)$, and $dF = 0$ for $\mathcal{W}_6$.

According to CRUCEANU [3, Th. 3], we know that (TM, g^H, J) is almost parakählerian (i.e., $dF = 0$) if and only if the cotorsion τ of

(g, ∇) vanishes. In terms of $\mathcal{A}$, $\tau = 0$ is equivalent to $g(\mathcal{A}(X,Y),Z) = g(\mathcal{A}(Z,Y),X)$ for all $X,Y,Z \in \mathcal{X}(M)$. But it is immediate that this condition together with the earlier i) on $\mathcal{A}$ is equivalent to $\mathcal{A}(X,Y) = \mathcal{A}(Y,X)$.

The condition $(\tilde{\nabla}_X F)(X,Y) = 0$ for $\mathcal{W}_6$ is equivalent to $\tilde{\nabla}_U U \in \mathcal{H}$ for all $U \in \mathcal{H}$, or even to $g(\hat{R}(X,Y)Z,T) = g(\hat{R}(T,X)Z,Y)$ for all $X,Y,Z,T \in \mathcal{X}(M)$. Hence, we can give the following table of properties of manifolds (TM, g^H, J) in order to belong to $\mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_5$ or $\mathcal{W}_6$:

Table 2

		$\mathcal{W}_3$	$\mathcal{W}_4$	$\mathcal{W}_5$	$\mathcal{W}_6$
(a)	$c_1^1\mathcal{A} = \flat(c_{12}\mathcal{A})$	*			
(b)	$\underset{YZW}{\mathfrak{S}} g(X,Y)\{g(\mathcal{A}(X,W),Z) - g(\mathcal{A}(X,Z),W)\} = 0$		*		
(c)	$\mathcal{A}(X,Y) = \mathcal{A}(Y,X)$			*	
(d)	$g(\hat{R}(X,Y)Z,T) = g(\hat{R}(T,X)Z,Y)$				*
(e)	$g(\mathcal{A}(X,Y),Z) = g(\mathcal{A}(X,Z),Y)$			*	*
(f)	$\hat{R} = 0$	*	*		

REMARKS. 1) Let us suppose the manifold (TM, g^H, J) belongs to $\mathcal{W}_3 \cap \mathcal{W}_4$. The property $g(\mathcal{A}(X,Y),Z) = g(\mathcal{A}(X,Z),Y)$ for all $X,Y,Z \in \mathcal{X}(M)$ implies (a) and (b). Conversely, if we assume properties (a) and (b), then $g(\mathcal{A}(X,Y),Z) = g(\mathcal{A}(X,Z),Y)$. In fact, we polarize (b) with respect to X in (b); we then make the transvection with g^{-1} (summing in the indices corresponding to X and W), and finally we apply (a). Thus, the families of manifolds (TM, g^H, J) in $\mathcal{W}_3$ in $\mathcal{W}_4$ are disjoint, because they have the common property that there exist $X,Y,Z \in \mathcal{X}(M)$ such that $g(\mathcal{A}(X,Y),Z) \neq g(\mathcal{A}(X,Z),Y)$.

2) Let us suppose the manifold (TM, g^H, J) belongs to $\mathcal{W}_5 \cap \mathcal{W}_6$. From property (c) for $\mathcal{W}_5$ we have $\hat{T} = -T = 0$, T and $\hat{T}$ being the torsions of ∇ and $\hat{\nabla}$, respectively, and hence we obtain $\underset{XYZ}{\mathfrak{S}} \hat{R}(X,Y)Z = 0$ (Bianchi identity). Moreover, $\hat{R}$ also satisfies $g(\hat{R}(X,Y)Z,T) = -g(\hat{R}(Y,X)Z,T)$ and property (d). Taking into account these properties, it can be proved that $\hat{R} = 0$, which is not possible because of the common property $\hat{R} \neq 0$, thus showing that $\mathcal{W}_5$ and $\mathcal{W}_6$ are disjoint classes indeed.

3) We note that by changing the roles of $\mathcal{V}$ and $\mathcal{H}$ in the structures of Cruceanu (see (3.2.1)) so that $J(X^H) = X^H$, $J(X^V) = -X^V$, we obtain examples of the "dual" classes $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_7$ and $\mathcal{W}_8$. The structures (TM, g^H, J) thus furnish examples of all the primitive classes, and also of the class $\mathcal{PK}$, as we have seen in 3.1.c).

4) Finally, we consider some examples of Cruceanu's structures such that $\mu = 0$ but which are not locally conformally parakählerian. For this, consider manifolds in the class $\mathcal{W}_4$. Then $\mu = 0$. Let θ be the Lee form of (TM, g^H, J). Since in this case we always have $\theta(A) = 0$ and $[A, U] \in \mathcal{V}$ for all $A \in \mathcal{V}$, $U \in \mathcal{H}$, we deduce that $d\theta(A, U) = A\theta(U)$. By a calculation we prove that the condition $A\theta(U) = 0$ is equivalent to the following

$$\text{(3.2.7)} \qquad \sum_{i=1}^{n}\Big\{g(\mathcal{A}(\partial/\partial x_i, \partial/\partial x_k), \partial/\partial x_i) - g(\mathcal{A}(\partial/\partial x_i, \partial/\partial x_i), \partial/\partial x_k)\Big\} = 0$$

for every $k = 1, \ldots, n$.

But $\widehat{R} = 0$ is equivalent to say that there exist a local frame $e_i = \Sigma_j f_{ji}(\partial/\partial x_j)(i = 1, \ldots, n)$ such that $\widehat{\nabla}_{e_i} e_j = 0$. If we take adequate functions f_{ji}. it can be proved that at a point $x \in M$ the conditions (3.2.7) are not satisfied.

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INDIRIZZO DEGLI AUTORI:

P.M. Gadea - J.M. Masqué - Unidad de Matemáticas - Consejo Superior de Investigaciones Científicas - C. Serrano, 123 - 28006 Madrid - Spain