Rendiconti di Matematica, Serie VII
Volume 11 , Roma (1991), 397-404

Generalized Kaluza-Klein Theory and Supersymmetry

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RIASSUNTO – *Utilizzando i metodi spinoriali di [2,b], che evitano l'uso dell'analisi anticommutativa, si mostra come si possa costruire una teoria di Kaluza-Klein generalizzata con supersimmetria. L'introduzione di una torsione totalmente antisimmetrica permette di eliminare una costante cosmologica.*

ABSTRACT – *Using our spinor methods like in [2,b], avoiding graduate anticommutative analysis, but using an enlarged triality principle [2,a], we point out that we can construct a generalized Kaluza-Klein theory with supersymmetry. Torsion with totally antisymmetric tensor is convenient to kill a big cosmological constant.*

KEY WORDS – *Kaluza-Klein - Rarita-Schwinger - Yang-Mills - Supersymmetry - Spinor - Vector-Spinor.*

A.M.S. CLASSIFICATION: 53C80 - 53C50 - 83D05 - 81E10

– Introduction

This paper follows the previous article [2.b] "Supergravity, supersymmetry: a geometric unitary spinor theory", and uses analogous definitions, notations and methods. A peculiarity of these papers is that anticommuting analysis is not used at all, supersymmetry is coming from convenient spinor properties.

In the previous article we developed Rarita-Schwinger theory, we obtain supersymmetry and showed that the geometric frame for supersymmetry is the enlarged triality principle and the interaction algebra [2.a].

We introduced also in this context the super-gravitation theory.

Generalized Kaluza-Klein theory is a particular case of Yang-Mills theory. Also in the first paragraph we give some recalls about the geometric frame. In the second paragraph are given our results. We have mixed Rarita-Schwinger, Kaluza-Klein theory and we have recupered the supersymmetry.

Coupling constants are not considered here, because we intend construct a geometric theory only. We are aware of the existence in this theory of a very big physical objection against a new coupling constant which appears. Some authors claimed that linear connection with torsion tensor gives a theory where the new coupling constant is killed. Unfortunatelly these theories are very intricate, get destroyed some supersymmetry, and get spolied the Rarita-Schwinger theory. We showed in the previous paper that torsion with totally antisymmetric tensor is the good tool to generalize the R.S. theory. In the Kaluza-Klein generalized context we show that this situation is good for preserve supersymmetry and kill the enormous constant. We note that this result about torsion is also got in the Einstein-Dirac-Cartan [2.c].

Like in the previous paper we suppose that the reader knows general relativity, spinor structures and also principal bundles theory.

1 – The geometric frame in generalized Kaluza-Klein theory

We recall some results and the methods, developed particularly by J. RAYSKI [6], B. DEWITT [3], R. KERNER [4], A. TRAUTMAN [7], Y.M. CHO [1], W. KOPCZYNSKI [5], etc...

Over the space-time manifold $V = V_{3.1}$, we construct a Yang-Mills bundle of which the standard fibre is a real p-dimensional Lie group Γ, Γ is generally a semi-simple group, because we needs a non degenerate Killing form. We are following nearly the Y.M. Cho's recapitulative version.

Let $(\hat{\xi}_\mu, \hat{\xi}_i^*)$ be a local frame of a principal bundle with structure group Γ and base $V_{3.1} = V(\mu = 0,1,2,3,\ i = 1,2,\dots p)$, the vector $\hat{\xi}_\mu$ are horizontal according a principal Γ-connection, and the vector ξ_i^* are vertical. Let $(g_{\mu\nu})$ the natural components of the pseudo-metric of $V_{3.1}$, and (g_{ik}) the components of the pseudo-metric tensor over the vertical subspace, evaluated in left-invariant frames and corresponding, modulo a constant scalar λ, to the Killing form if $p \neq 1$.

Let (e_a), $a = 1,2,\ldots p+4$, any local frame in the principal bundle, we defined (ξ_i^{*a}), $(\hat{\xi}_\mu^a)$:

$$\xi_i^* = \xi_i^{*a} e_a\,, \qquad \hat{\xi}_\mu = \hat{\xi}_\mu^a e_a\,,$$

and we introduce over the principal bundle manifold P, a pseudometric (γ_{ab}) such that:

$$
\begin{aligned}
&g_{\mu\nu} = \gamma_{ab}\hat{\xi}_\mu^a \hat{\xi}_\nu^b \\
(1)\qquad &g_{ik} = \gamma_{ab}\xi_i^{*a}\xi_k^{*b} \\
&g_{\mu k} = \gamma_{ab}\hat{\xi}_\mu^a \xi_b^{*b} = 0
\end{aligned}
$$

The important property is that the projection of the tangent space T_V onto any horizontal subspace is an isometry, horizontal and vertical subspaces are orthogonal according (γ_{ab}).

If we define local product frames according:

$$\xi_\mu = \hat{\xi}_\mu + A_\mu^i \xi_i^*\,, \qquad \xi_i = \xi_i^*\,,$$

the components of the pseudo-metric are now:

$$(2)\qquad g_{\mu\nu} + A_\mu^i A_\nu^k g_{ik}\,, \qquad A_\mu^i g_{ik}\,, \qquad g_{ik}$$

$\{G_i, i = 1,2,\ldots p\}$, define a particular frame of $\mathcal{L}(\Gamma)$, the Lie algebra of Γ, the classical Yang-Mills connection owns a form:

$$\omega = A_\mu^i \theta^\mu \otimes G_i + A_k^i \theta^k \otimes G_i\,,$$

if we suppose $\omega(\xi_i^*) = G_i$, $A_i^k = \delta_i^k$,and we can choose a local cross-section σ, such that:

$$(3)\qquad \sigma^*(\omega) = A_\lambda^k(x)(\theta^\lambda \otimes G_k)\,, \quad x \in V_{3.1}$$

The (A_λ^k) define locally, the Y-M-connection et the Y-M curvature admits the following components:

$$(4)\qquad F_{\alpha\beta}^k = \partial_\alpha A_\beta^k - \partial_\beta A_\alpha^k + C_{ij}^k A_\alpha^i A_\beta^j$$

The C_{ij}^k are structural constants of $\mathcal{L}(\Gamma)$. Then we consider over P, considered as the base manifold a *torsion-free pseudo-euclidean connection.*

Using vertical an horizontal frames, according Y.M. CHO [1], the non null components of the connection are:

$$\hat{\Gamma}^i_{jk} = \frac{1}{2}c^i_{jk}, \qquad \hat{\Gamma}^\alpha_{\beta k} = \frac{1}{2}g^{\alpha\lambda}g_{k\ell}F^i_{\beta\lambda}$$

(5)

$$\hat{\Gamma}^k_{\alpha\beta} = -\frac{1}{2}F^k_{\alpha\beta}, \quad \hat{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)$$

we will note $\hat{D}$ the associated derivation.

The riemannian scalar curvature of P is the sum of the riemannian scalar curvature of V with pseudo-metric $(g_{\mu\nu})$ of Γ with Killing pseudo-metric, and a Yang-Mills term, according the formula:

$$R_P = R_V + R_\Gamma - \frac{1}{4}F^2 \tag{6}$$

$$F^2 = g_{ik}g^{\mu\alpha}g^{\nu\beta}F^i_{\mu\nu}F^k_{\alpha\beta} \tag{7}$$

$R_\Gamma = \frac{\varrho}{4}$, with a particular choice of the scalar λ, such that (g_{ij}) is positive definite if Γ is compact.

The process of General Relativity (exterior case) coming from P and (γ_{ab}), by varying $g^{\lambda\mu}$ and A^i_μ (or (g_{ik})), gives the equations:

$$\overset{(V)}{R}_{\mu\nu} - \frac{1}{2}\overset{(V)}{R}\, g_{\mu\nu} = \frac{p}{8}g_{\mu\nu} + T_{\mu\nu} \tag{8}$$

$$g^{\lambda\mu}\left(\nabla_\lambda F^i_{\mu\nu} + C^i_{jk}A^j_\lambda F^k_{\mu\nu}\right) = 0 \tag{9}$$

with:

$$T_{\mu\nu} = \frac{1}{2}\left(F^i_{\mu\rho}F_{i\nu}{}^\rho - \frac{1}{4}g_{\mu\nu}F_{i\rho\sigma}F^{i\rho\sigma}\right. \tag{10}$$

$\overset{(V)}{R}_{\mu\nu}$ is the Ricci tensor of V, and $\overset{(V)}{R} = R_V$ the riemannian scalar curvature of V.

∇ is the covariant derivation of the Riemann-Christoffel connexion over V, with the pseudo-metric $(g_{\mu\nu})$.

Formula (9) could be written:

$$g^{\lambda\mu}(D_\lambda F^i_{\mu\nu}) = 0\,, \tag{9bis}$$

D corresponding to a principal connection for a bundle with group obtained by direct product of Γ and Lorentz group. Note that $g^{\mu\nu}\mathcal{T}_{\mu\nu} = 0$.

If $p = 1$, $\Gamma = U(1)$, and as the Killing form is null, we choose over the rank one associated fiber bundle, a constant metric ($g_{55} = 1$).

In the formula (8), $\frac{p}{8}$ does not appear, we get:

$$\overset{(V)}{R}_{\mu\nu} - \frac{1}{2}\overset{(V)}{R}\, g_{\mu\nu} = \mathcal{T}_{\mu\nu} \tag{11}$$

$$g^{\lambda\mu}\nabla_\lambda F_{\mu\nu} = 0 \tag{12}$$

$$\mathcal{T}_{\mu\nu} = \frac{1}{2}\Big(F_{\mu\nu}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big) \tag{13}$$

$$F_{\mu\nu} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{14}$$

(11), (12), (13) correspond to a pure electromagnetic relativiste schema (with null current) giving a provisory and incomplete character to that K.K. theory.

2 – Rarita-Schwinger-Kaluza-Klein theory and supersymmetry

We can apply on results given in [2.b], with a R-S field because we can use a variational principle with the same character exterior relativistic case and R-S terms ([2.b], paragraph 5).

But what about the R-S field?

The Y-M connection determines P as the direct sum of two bundles, the vertical and the horizontal ones. There exists an isometry between $T(V)$ and the horizontal bundle. The vertical bundle is trivializable, also there exist over P considered as a pseudo-riemannian manifold (pseudo-metric γ_{ab}), a spinor structure if V admits such that structure.

We can define over P, spinor cross-sections, direct sum of two spinor cross-sections, one corresponding to the tangent manifolds $T(V)$, another is trivial over the vertical bundle with Killing pseudo-metric.

We must use the covariant derivation $\widehat{D}$ of which coefficients are given by (5) above, which does not act separately over the spinor two terms.

a, b, c... are indices going from 1 to $p+4$ (dim $V = 4$, din $\Gamma = p$), we choose, like in [2.b], paragraph 5:

$$B^a = (e^a e^b e^c - e^c e^b e^a)\widehat{D}_b \psi_c \tag{15}$$

$\psi = \psi_a \otimes e^a$ is a vector-spinor, (the condition $e^a \psi_a = 0$, is useless here, but lawfull) and the variational principles leads to:

$$B^a = 0, \qquad \overset{(P)}{R}_{ab} - \frac{1}{2} \overset{(P)}{R} \gamma_{ab} = 0$$

and the last equations splits into (8) and (9). Taking $\delta\psi_a = \widehat{D}_a \varepsilon$, where ε is a spinor field, and:

$$\delta\gamma^{ab} = \frac{1}{\sqrt{|h|}} \operatorname{Im}\,(\varepsilon; e^a \psi^b + e^b \psi^a)$$

h here is connected to γ_{ab}, the scalar product was defined in [2.b], we recupere $\delta\mathcal{L} = 0$, and get again the supersymmetry. Note that supersymmetry involves the Y-M fields through the connection, according (5) above.

The triviality of the vertical bundle permits to consider ψ as a pair of spinor fields $\psi_\lambda \otimes e^\lambda$, and $\psi_i \otimes e^i$ (the first is a spin 3/2 field), mixed with Y-M fields and gravitation by the supersymmetry.

If $p = 1$, we have a particular Y-M fields which defines an electromagnetic fields with null current (cf. (12) and (14)).

Remark about the equations (8)

It is known result that in (8), the term $\frac{g}{8} g_{\mu\nu}$ is not convenient as a cosmological term [5]. Also we must throw out it by modifying a little the approach explained in the 1° above. The Rarita-Schwinger field joined with a totally skewsymmetric torsion tensor permits to realize this programm.

a, b, c... are indices going from 1 to $p+4$, and torsion tensor admits components S_{abc}, the variational principles gives coming from S_{abc}, the unique term $\frac{1}{2}Sg_{\mu\nu}$ for (8)

$$\overset{(V)}{R}_{\mu\nu} - \frac{1}{2}\overset{(V)}{R}\, g_{\mu\nu} + Sg_{\mu\nu} = \frac{p}{8}g_{\mu\nu} + T_{\mu\nu}$$

$S = S_{abc}S^{abc}$ is a scalar [2,c]. This simple results come from (6) which is now: $R_V + R_\gamma - \frac{1}{4}F^2 + \widehat{D}_a S^a_{bc} - S^a_{bd}S^d_{ac}$; by integration, $\widehat{D}_a S^a_{bc}$ gives always zero, $\widehat{D}$ is the pseudo-riemannian connection over P and R_V and F^2 was defined above.

However we can determines S_{abc} with the Rarita-Schwinger field, like in [2.b] $S_{abc} = (e_a e_b e_c \psi_d; \psi^d)+$ c.c. modulo a constant scalar factor.

If we want cancel $\frac{p}{8}g_{\mu\nu}$, we have to suppose: $p = 4S$.

This condition appears as possible, because S is not zero a priori, and this condition imposes algebraic relation only between the components of the R-S-field.

If $p = 1$, we dont need a torsion tensor, because p does not appears in the equations. Naturally the supersymmetry is preserved, because the lagrangian is obtained from a Ricci tensor (over P), and the R-S-term, like in [2.b].

Finally we obtain a generalized Rarita-Schwinger-Kaluza-Klein theory, with supersymmetry and we can cancel the big coupling constant with a completely antisymmetric torsion tensor if $p > 1$.

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Lavoro pervenuto alla redazione il 2 novembre 1990
ed accettato per la pubblicazione il 23 novembre 1990
su parere favorevole di G. Ferrarese e di P. Benvenuti

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