Rendiconti di Matematica, Serie VII
Volume 11 , Roma (1991), 405-416

Some Estimates of Integral Operators with respect to the Multidimensional Vitali Φ-variation, and Applications in Fractional Calculus

C. BARDARO - G. VINTI

RIASSUNTO – *Si considerano disuguaglianze per composizioni di operatori integrali con nuclei omogenei rispetto alla φ- variazione di Vitali in $\mathbb{R}^+_N$. L'applicazione principale è una disuguaglianza rispetto alla φ-variazione frazionaria in $\mathbb{R}^+$.*

ABSTRACT – *We consider inequalities for composition of integral operators with homogeneous kernels, with respect to the Vitali φ-variation in $\mathbb{R}^+_N$. The main application is an inequality with respect to the fractional φ-variation in $\mathbb{R}^+$.*

KEY WORDS – *Vitali φ-variation - Homogeneous kernels - Fractional φ-variation - Moment operators.*

A.M.S. CLASSIFICATION: 26D10 - 26A33 - 46E30

– Introduction

In this paper we continue the study started in [4] on inequalities for integral operators with homogeneous kernels. These inequalities can be used in approximation theorems, because they give "boundedness" properties for the operators.

Here we consider inequalities for composition of integral operators with respect to Musielak-Orlicz φ-variation [14], [16], using a new concept of homogeneity for the kernel of one of the component operator.

The main application is an interesting inequality with respect to the "fractional φ-variation of order α" of $f \in L^1_{loc}(\mathbb{R}^+)$.
This concept, which is based on the Riemann-Liouville fractional integral of f, was introduced in [17], [18], in the special case of $\varphi(u) = u$.

Thus our theorem 2 below extends the "α-variation diminishing property" for the moment operator, proved in [18].

For a sake of generality we work on multidimensional case and use the Vitali φ-variation of f in $\mathbb{R}^+_N$, $N \geq 1$, (see section 1).

For the variation we adopt the construction given in [11], [12], and for a sake of simplicity, we will distinguish the functions which are equivalent. That is a function $f \in L^1_{loc}(\mathbb{R}^+_N)$ is not considered as an "equivalence class", but just a function which is defined everywhere in $\mathbb{R}^+_N$.

1 – Notations and definitions

In the following we will put $\mathbb{R}^+_N = (]0, +\infty[)^N$, $N \in \mathbb{N}$, $N \geq 1$. Moreover by $L^1_{loc}(\mathbb{R}^+_N)$ we will denote the class of the (real) locally integrable functions on $\mathbb{R}^+_N$, (i.e. the functions which are integrable on every *"bounded"* set in $\mathbb{R}^+_N$). $\mathbb{R}^+$stands for $\mathbb{R}^+_1$ and $\mathbb{R}^+_0$ denotes the interval $[0, +\infty[$.
If $x = (x_1, \cdots, x_N), t = (t_1, \cdots, t_N) \in \mathbb{R}^+_N$, we set $xt = (x_1 t_1, \cdots, x_N t_N)$, $< x >= \prod_{i=1}^N x_i = x_1 x_2 \cdots x_N$, and $t^{-1} = (t_1^{-1}, \cdots, t_N^{-1})$.
Let $\varphi : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ be a function verifying the following assumptions:

i) φ is convex and non decreasing in $\mathbb{R}^+_0$;

ii) $\varphi(0) = 0$ and $\varphi(u) > 0$ for $u > 0$.

We introduce now the φ-variation in the sense of Vitali for functions $f \in L^1_{loc}(\mathbb{R}^+_N)$. We use the construction given in [11], [12].

For $a = (a_1, \cdots, a_N), b = (b_1, \cdots, b_N) \in \mathbb{R}^+_N$ we define $a \leq b \Leftrightarrow a_i \leq b_i$, $i = 1, \cdots, N$ and $[a, b] = \{x \in \mathbb{R}^+_N : a \leq x \leq b\}$; for example for $N = 2$, [a, b] is a rectangle.

Moreover, we set:

$$\mathrm{Cor}[a,b] = \{x \in \mathbb{R}_N^+ : x_i = a_i \text{ or } x_i = b_i,\ 1 \leq i \leq N\},$$

$$\gamma(x,a) = \{i \in \{1,\cdots,N\} : x_i = a_i\}, \quad \text{for} \quad x,a \in \mathbb{R}_N^+,$$

$$\Delta(f,[a,b]) = \sum_{x \in \mathrm{Cor}[a,b]} (-1)^{\gamma(x,a)} f(x).$$

Next, we define the Vitali φ-variation of f in $\mathbb{R}_N^+$by means of the equation

$$V_\varphi^N(f) = \sup \sum_{i=1}^{r} \varphi(|\Delta(f,[a^{(i)},b^{(i)}])|)$$

where the sup is taken over all the finite collections of non overlapping intervals $[a^{(i)},b^{(i)}] \subset \mathbb{R}_N^+$.

We remark that for $N = 1$ we have $\Delta(f,[a,b]) = f(b) - f(a)$, $a,b \in \mathbb{R}^+$, and so V_φ^Nreduces to the one-dimensional Musielak-Orlicz φ-variation $V_\varphi(f)$ in $\mathbb{R}^+$.

For $\varphi(u) = u, N > 1$ we obtain the Vitali variation in $\mathbb{R}_N^+$.

If $V_\varphi^N(f) < +\infty$, we say that f is a function with bounded Vitali φ-variation on $\mathbb{R}_N^+$.

Let now $\mathcal{K}^N$be the class of all the measurable functions $K : \mathbb{R}_N^+ \times \mathbb{R}_N^+ \to \mathbb{R}_0^+$ such that

$(K.1)_N$ There exists a measurable function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ (depending on K) such that for every $\lambda \in \mathbb{R}_N^+$ and $(x,t) \in \mathbb{R}_N^+ \times \mathbb{R}_N^+$, we have

$$K(\lambda x, \lambda t) = \frac{\eta(<\lambda t>)}{\eta(<t>)} K(x,t),$$

$(K.2)_N$ The integral operator,

$$(Tf)(x) = \int_{\mathbb{R}_N^+} K(x,t) f(t)\,dt,$$

maps D_T into $L^1_{\mathrm{loc}}(\mathbb{R}_N^+)$, where

$$D_T =: \{f \in L^1_{\mathrm{loc}}(\mathbb{R}_N^+) : T(|f|)(x) < +\infty, \text{ for every } x\}.$$

REMARK 1

(a) As a particular case, we can consider $\eta(z) = z^\gamma$, $z \in \mathbb{R}^+, \gamma \in \mathbb{R}$, to obtain a definition of homogeneous kernel of degree γ, i.e.:

$$K(\lambda x, \lambda t) = <\lambda>^\gamma K(x,t).$$

(b) A typical example of Tf is the one-dimensional "moment operator" or "average operator" with kernel

$$\mathcal{M}_\lambda(x,t) = \lambda x^{-\lambda} t^{\lambda-1} \chi_{]0,x[}(t), \lambda > 1, (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

which is homogeneous of degree -1. In this case $\eta(t) = t^{-1}$. For references on this operator see e.g. [5], [8], [9], [6], [17], [18], [1], [2], [3], [4].

2 – An inequality for Tf

The result below extends theorem 1 in [4] in various directions. Here we assume $N \geq 1$.

THEOREM 1. *Suppose that $K \in \mathcal{K}^N$ and*

$$0 < A^K_\beta =: \int_{\mathbb{R}^+_N} \{\eta(<z>)\}^{-1} <z>^{-(\beta+1)} K(1,z)dz < +\infty,$$

for a fixed $\beta \in \mathbb{R}$. Then for $f \in D_T$, it results:

$$V^N_\varphi(<\cdot>^\beta (Tf)(\cdot)) \leq V^N_\varphi(A^K_\beta <\cdot>^{\beta+1} \eta(<\cdot>)f(\cdot)). \tag{1}$$

PROOF. We can suppose that $V^N_\varphi[A^K_\beta <\cdot>^{\beta+1}\eta(<\cdot>)f(\cdot)) < +\infty$. By $(K.1)_N$ we have

$$<x>^\beta (Tf)(x) = <x>^{\beta+1} \int_{\mathbb{R}^+_N} \eta(<zx>)\{\eta(<z>)\}^{-1} K(1,z) f(zx)dz =$$

$$= \int_{\mathbb{R}^+_N} \{\eta(<z>)\}^{-1} <z>^{-(\beta+1)} K(1,z) g(zx)dz,$$

where $g(t) < t >^{\beta+1} \eta(<t>)f(t), t \in \mathbb{R}_N^+$.

Now, let $[a^{(i)}, b^{(i)}]$, $i = 1, \cdots, r$, a finite collection of non overlapping intervals in $\mathbb{R}_N^+$; we have by Jensen inequality

$$\sum_{i=1}^{r} \varphi\Big(\Big| \sum_{x \in \mathrm{Cor}[a^{(i)}, b^{(i)}]} (-1)^{\gamma(x, a^{(i)})} \int_{\mathbb{R}_N^+} \{\eta(<z>)\}^{-1} <z>^{-(\beta+1)} K(1,z) g(zx) dz\Big|\Big)$$

$$\leq \sum_{i=1}^{r} \varphi\Big(\int_{\mathbb{R}_N^+} K(1,z) <z>^{-(\beta+1)} \{\eta(<z>)\}^{-1} |\Delta(g(z\cdot), [a^{(i)}, b^{(i)}])| dz\Big)$$

$$= \sum_{i=1}^{r} \varphi\Big(\int_{\mathbb{R}_N^+} K(1,z) <z>^{-(\beta+1)} \{\eta(<z>)\}^{-1} |\Delta(g, [za^{(i)}, zb^{(i)}])| dz\Big)$$

$$\leq \frac{1}{A_\beta^K} \int_{\mathbb{R}_N^+} K(1,z) <z>^{-(\beta+1)} \{\eta(<z>)\}^{-1} V_\varphi^N(A_\beta^K g) dz = V_\varphi^N(A_\beta^K g)$$

and so the assertion follows.

As a particular case, for $\beta = 0$, (1) becomes

$$V_\varphi^N((Tf)(\cdot)) \leq V_\varphi^N(A_0^K < \cdot > \eta(< \cdot >) f(\cdot)). \tag{2}$$

Moreover, if $N = 1$, (1) becomes

$$V_\varphi((\cdot)^\beta (Tf)(\cdot)) \leq V_\varphi(A_\beta^K (\cdot)^{\beta+1} \eta(\cdot) f(\cdot)), \tag{3}$$

where $A_\beta^K =: \int_0^{+\infty} \{\eta(z)\}^{-1} z^{-(\beta+1)} K(1,z) dz < +\infty$.

If $N = 1, \beta = 0$, K homogeneous of degree $\gamma \in \mathbb{R}$, theorem 1 reduces to theorem 1 in [4].

REMARK 2 Condition $(K.2)_N$is not used in theorem 1; we can consider all the domain of T (the set of the measurable functions for which Tf exists as Lebesgue integral for every x).

3 – Composition of integral operators

Here we consider the following integral operators:

$$(Tf)(x) = \int_{\mathbb{R}_N^+} K(x,t)f(t)dt$$

$$(Uf)(x) = \int_{\mathbb{R}_N^+} H(x,t)f(t)dt.$$

We will assume that $H, K \in \mathcal{K}^N$, and that H is homogeneous of degree $\delta \in \mathbb{R}$. We also assume $T(|f|) \in D_U$.

We use the following

LEMMA. *For $f \in D_T$, let us put $g(t) = <t> \eta(<t>)f(t)$; then we have*

$$(4) \quad (U \circ T)f(x) = <x>^{1+\delta} \int_{\mathbb{R}_N^+} \{\eta(<t>)\}^{-1} <t>^{-(2+\delta)} K(x,t)(Ug)(t)dt$$

whenever $g \in D_U$.

PROOF. By property $(K.1)_N$, we have

$$(Tf)(x) = <x> \int_{\mathbb{R}_N^+} \eta(<zx>)\{\eta(<z>)\}^{-1}K(1,z)f(zx)dz$$

$$= \int_{\mathbb{R}_N^+} \{\eta(<z>)\}^{-1} <z>^{-1} K(1,z)g(zx)dz.$$

By property $(K.2)_N$, since $T(|f|) \in D_U$, it is possible to apply the Fubini-Tonelli theorem and thus we write

$$(U \circ T)f(x) = \int_{\mathbb{R}_N^+} H(x,t)(Tf)(t)dt =$$

$$= \int_{\mathbb{R}_N^+} \left\{ \int_{\mathbb{R}_N^+} \{\eta(<z>)\}^{-1} <z>^{-1} K(1,z)H(x,t)g(tz)dt \right\} dz.$$

Making use of the substitution $tz = v$ (i.e. $t_i z_i = v_i$, $i = 1, \cdots, N$) in the inner integral, we deduce

$$(U \circ T)f(x) = \int_{\mathbf{R}_N^+}\left\{\int_{\mathbf{R}_N^+} \{\eta(<z>)\}^{-1}<z>^{-2}K(1,z)H(x,vz^{-1})g(v)dv\right\}dz.$$

As $H(zx,v) = <z>^{\delta}H(x,vz^{-1})$ we have also

$$(U \circ T)f(x) = \int_{\mathbf{R}_N^+}\left\{\int_{\mathbf{R}_N^+} \{\eta(<z>)\}^{-1}<z>^{-(2+\delta)}K(1,z)H(zx,v)g(v)dv\right\}dz$$

$$= \int_{\mathbf{R}_N^+} \{\eta(<z>)\}^{-1}<z>^{-(2+\delta)}K(1,z)(Ug)(zx)dz.$$

Next, we put $zx = t$, and obtain

$$(U \circ T)f(x) = \int_{\mathbf{R}_N^+}\{\eta(<tx^{-1}>)\}^{-1}<t>^{-(2+\delta)}<x>^{1+\delta}K(1,tx^{-1})(Ug)(t)dt.$$

By $(K.1)_N$, we have

$$\{\eta(<tx^{-1}>)\}^{-1}K(1,tx^{-1}) = \eta(<t>)^{-1}K(x,t),$$

and so

$$(U \circ T)f(x) = <x>^{1+\delta}\int_{\mathbf{R}_N^+} \eta(<t>)^{-1}<t>^{-(2+\delta)}K(x,t)(Ug)(t)dt.$$

Now, we are ready to prove the main theorem of this paper.

THEOREM 2. *Under the previous conditions and notations, suppose that*

(i) $0 < A_{\delta+1}^K =: \int_{\mathbf{R}_N^+} \{\eta(<z>)\}^{-1}<z>^{-(2+\delta)}K(1,z)dz < +\infty$,

(ii) $0 < A_\delta^H =: \int_{\mathbf{R}_N^+} <z>^{-(1+\delta)}H(1,z)dz < +\infty$.

Then, for every $f \in D_T$, such that $g \in D_U$, we have

(5)
$$V_\varphi^N((U \circ T)f) \leq V_\varphi^N(A_{\delta+1}^K(Ug)(\cdot)) \leq V_\varphi^N(A_\delta^H A_{\delta+1}^K < \cdot >^{2+\delta}\eta(< \cdot >)f(\cdot)),$$

where $g(t) = <t> \eta(<t>)f(t)$.

Proof. As in theorem 1, we can suppose that

$$V_\varphi^N(A_\delta^H A_{\delta+1}^K < \cdot >^{2+\delta}\eta(< \cdot >)f(\cdot)) < +\infty.$$

Let us put $g(t) = <t> \eta(<t>)f(t)$, $h(t) = \{\eta(<t>)\}^{-1} <t>^{-(2+\delta)}(Ug)(t)$. By lemma 1, h belongs to the domain of T, and by theorem 1 and remark 2, we have (for $\beta = \delta + 1$):

$$\begin{aligned} V_\varphi^N((U \circ T)f) &= V_\varphi^N(< \cdot >^{1+\delta}(Th)(\cdot)) \leq \\ &\leq V_\varphi^N(A_{\delta+1}^K < \cdot >^{2+\delta}\eta(< \cdot >)h(\cdot)) = \\ &= V_\varphi^N(A_{\delta+1}^K(Ug)(\cdot)). \end{aligned}$$

Moreover, again by theorem 1 applied with $\beta = 0, T = U, f = A_{\delta+1}^K g$ and taking into account that the kernel of U is homogeneous of degree δ, we have

$$V_\varphi^N(A_{\delta+1}^K(Ug)(\cdot)) \leq V_\varphi^N(A_\delta^H A_{\delta+1}^K < \cdot >^{2+\delta}\eta(< \cdot >)f(\cdot)).$$

For the special case $N = 1$, (5) reduces to the following:

(6) $$V_\varphi((U \circ T)f) \leq V_\varphi(A_{\delta+1}^K(Ug)(\cdot)) \leq V_\varphi(A_\delta^H A_{\delta+1}^K (\cdot)^{2+\delta}\eta(< \cdot >)f(\cdot))$$

where
$$A_{\delta+1}^K =: \int_0^{+\infty} \{\eta(z)\}^{-1} z^{-(2+\delta)} K(1,z)dz,$$

and
$$A_\delta^H =: \int_0^{+\infty} z^{-(1+\delta)} H(1,z)dz.$$

4 – Applications to the fractional calculus

Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a locally integrable function and let $\alpha \in]0,1[$ be fixed.

Following [17], [18], we define the "generalized fractional primitive of order $1-\alpha$ of f" by means of the equation

$$f_{(1-\alpha)}(x) = \{\Gamma(1-\alpha)\}^{-1} \int_0^x \frac{f(t)}{(x-t)^\alpha} dt, x \in \mathbb{R}^+. \tag{7}$$

We remark that (7) is well-defined for each $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ and $D_U \subset L^1_{\mathrm{loc}}(\mathbb{R}^+)$, where we will put $Uf = f_{(1-\alpha)}$.

The following result is an easy consequence of theorem 1 and relates the φ-variations of f and $f_{(1-\alpha)}$.

Corollary 1. *For every $f \in D_U$, we have*

$$V_\varphi(f_{(1-\alpha)}) \leq V_\varphi(\Gamma(\alpha)(\cdot)^{1-\alpha} f(\cdot)). \tag{8}$$

Proof. Putting $H(x,t) = \{\Gamma(1-\alpha)\}^{-1}(x-t)^{-\alpha}\chi_{]0,x[}(t)$, we can write

$$f_{(1-\alpha)}(x) = \int_0^{+\infty} H(x,t) f(t) dt.$$

As H is homogeneous of degree $-\alpha$, the result follows by theorem 1 with $N = 1, \delta = -\alpha$,taking into account that $A^H_\delta = \Gamma(\alpha)$.

According to the definition given in [17], [18], we define the "fractional φ-variation of order α" of $f \in D_U$ by means of the equation

$$V^\alpha_\varphi(f) =: V_\varphi(f_{(1-\alpha)}). \tag{9}$$

With this definition, (8) of corollary 1 becomes

$$V^\alpha_\varphi(f) \leq V_\varphi(\Gamma(\alpha)(\cdot)^{1-\alpha} f(\cdot)), f \in D_U.$$

Now, using theorem 2, we can obtain estimates of the integral operator Tf with respect to the fractional φ-variation. We will use the notations of theorem 2, for $N = 1$.

COROLLARY 2. *Let* $\alpha \in]0,1[$, $K \in \mathcal{K}$; *if*

$$A_{1-\alpha} =: \int_0^{+\infty} \{\eta(z)\}^{-1} z^{\alpha-2} K(1,z)dz < +\infty,$$

then, for every $f \in D_T$, *such that* $(\cdot)\eta(\cdot)f(\cdot) \in D_U$, *we have*

$$\begin{aligned} V_\varphi^\alpha(Tf) &\leq V_\varphi^\alpha(A_{1-\alpha}(\cdot)\eta(\cdot)f(\cdot)) \\ &\leq V_\varphi(\Gamma(\alpha)A_{1-\alpha}(\cdot)^{2-\alpha}\eta(\cdot)f(\cdot)). \end{aligned}$$

PROOF. Putting $H(x,t) = \{\Gamma(1-\alpha)\}^{-1}(x-t)^{-\alpha}\chi_{]0,x[}(t)$, we have $A_i^H = \Gamma(\alpha)$, $A_{i+1}^K = A_{1-\alpha}$, and the result is an easy consequence of theorem 2.

For the moment type operators (see section 1):

$$(\mathcal{M}_\lambda f)(x) = \int_0^{+\infty} M_\lambda(x,t)f(t)dt,$$

we obtain in particular $D_{\mathcal{M}_\lambda} = L^1_{\text{loc}}(\mathbb{R}^+)$, and

$$\begin{aligned} V_\varphi^\alpha(\mathcal{M}_\lambda f) &\leq V_\varphi^\alpha(\frac{\lambda}{\lambda+\alpha-1}f(\cdot)) \\ &\leq V_\varphi(\Gamma(\alpha)\frac{\lambda}{\lambda+\alpha-1}(\cdot)^{1-\alpha}f(\cdot)), \end{aligned} \tag{10}$$

for every $f \in D_U$, and $\alpha \in]0,1[$. Indeed, in this case we have $\eta(t) = t^{-1}$ and $A_{1-\alpha} = \dfrac{\lambda}{\lambda+\alpha-1}$.

REMARK 3.

(a) If $\varphi(u) = u, \forall u \in \mathbb{R}_0^+$, the first inequality in (10) is a result proved in [18].

(b) For the "modified moment kernel" (see [18]):

$$M^*_{\lambda+\gamma}(x,t) = (\lambda+\gamma)x^{-(\lambda+1)}t^\lambda\chi_{]0,x[}(t), \quad \lambda > 0, \quad \gamma > 0,$$

we obtain $A_{1-\alpha} = \dfrac{\lambda+\gamma}{\lambda+\alpha}$ and so, for $\gamma < \alpha$, we have the "α-variation diminishing property" for $\mathcal{M}^*_{\lambda+\gamma}$, i.e.

$$V^\alpha_\varphi(\mathcal{M}^*_{\lambda+\gamma}f) \le V^\alpha_\varphi(Const\, f(\cdot)), \qquad Const < 1.$$

If $\varphi(u) = u$, we obtain in particular a result of [18], namely

$$V^\alpha(\mathcal{M}^*_{\lambda+\gamma}f) \le Const V^\alpha(f), \qquad Const < 1.$$

Obviously, it is possible to introduce other concepts of fractional variation by introducing some generalization of the integral $f_{(1-\alpha)}$(Riemann-Liouville fractional integral). For example, we may consider (see [7], [10]):

$$f^*_{(1-\alpha)}(x) = \frac{x^{\alpha-\eta}}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}t^{\eta-1}f(t)dt, \quad \eta > 1, \quad \alpha \in]0,1[,$$

or (see [15]),

$$f^{**}_{(1-\alpha)}(x) = \frac{2x^{2(\alpha-\eta-1)}}{\Gamma(1-\alpha)}\int_0^x (x^2-u^2)^{-\alpha}u^{2\eta+1}f(u)du, \quad \alpha > 0, \quad \eta > -\frac{1}{2}.$$

The kernels of $f^*_{(1-\alpha)}$ and $f^{**}_{(1-\alpha)}$, are homogeneous of degree -1. For these different types of fractional variations, we can state similar inequalities.

REFERENCES

[1] F. BARBIERI: *Approssimazione mediante nuclei momento*, Atti Sem. Mat. Fis. Univ. Modena, **32** (1983), 308- 328.

[2] C. BARDARO - G. VINTI: *Modular convergence in generalized Orlicz spaces for moment type operators*, Applicable Analysis, **32** (1989), 265-276.

[3] C. BARDARO - G. VINTI: *On convergence of moment operators with respect to φ-variation*, to appear in Applicable Analysis.

[4] C. BARDARO - G. VINTI: *Modular estimates of integral operators with homogeneous kernel in Orlicz type spaces*, to appear in Resultate der Mathematik.

[5] C. BARDARO: *On approximation properties for some classes of linear operators of convolution type*, Atti Sem. Mat. Fis. Univ. Modena, **33** (1984), 329-356.

[6] F. DEDGANI CATTELANI: *Nuclei di tipo "distanza" che attutiscono i salti in una o più variabili*, Atti Sem. Mat. Fis. Univ. Modena, 30 (1981), 299-321.

[7] A. ERDÉLYI: *On fractional integration and its application to the Theory of Hankel trasforms*, Quart. J. Math., 11 (1940), 293.

[8] F. FEHÉR: *A note on a paper of E.R. Love*, Bull. Austral. Math. Soc., 19 (1978), 67-75.

[9] F. FEHÉR: *A generalized Schur-Hardy inequality on normed Kothe spaces*, General Inequalities II (Proc. 2nd. Int. Conf. on General Inequalities, Oberwolfach, 1978), Birkhäuser, Basel, 1980, 277-285.

[10] H. KOBER: *On fractional integrals and derivatives*, Quart. J. Math., 11 (1940), 193.

[11] B. LENZE: *A hyperbolic modulus of smoothness for multivariate functions of bounded variation*, J. of Approximation Theory, to appear.

[12] B. LENZE: *On constructive one-sided approximation of multivariate functions of bounded variation*, to appear.

[13] E.R. LOVE: *Some inequalities for fractional integrals*, Linear spaces and approximation (Proc. Conf. Mathematical Research Institute, Oberwolfach, 1977, 177-184. International Series of Numerical Mathematics, 40. Birkhäuser Verlag, Basel, Stuttgart, 1978).

[14] J. MUSIELAK - W. ORLICZ: *On generalized variation I*, Studia Math., 18 (1959), 11-41.

[15] I.N. SNEDDON: *The use in mathematical physics of Erdély-Kober operators and of some of their generalizations*, Proceedings of the International Conference University of New Haven, 1974, Lecture Notes in Math. 457, (1975), 37-79.

[16] J. SZELMECZKA: *On convergence of singular integrals in the generalized variation metric*, Functiones et Approximatio 15 (1986), 53-58.

[17] V. ZANELLI: *I polinomi di Stieltjes approssimanti in variazione di ordine non intero*, Atti Sem. Mat. Fis. Univ. Modena 30 (1981), 151-175.

[18] V. ZANELLI: *Funzioni momento convergenti dal basso in variazione di ordine non intero*, Atti Sem. Mat. Fis. Univ. Modena, 30 (1981), 355-369.

Lavoro pervenuto alla redazione il 10 ottobre 1990
ed accettato per la pubblicazione il 26 novembre 1990
su parere favorevole di B. Firmani e di P.E. Ricci

INDIRIZZO DEGLI AUTORI:

Carlo Bardaro - Gianluca Vinti - Dipartimento di Matematica - Università degli Studi di Perugia - Via Vanvitelli, 1 - 06100 Perugia