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Certain Classes of Analytic Functions of Complex Order and Type Beta

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RIASSUNTO – *Sia* $S(1-b,\beta)$ *(*$b \neq 0$*, complesso,* $0 < \beta \leq 1$*), la classe delle funzioni* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$*, analitiche in* $U = \{z\colon |z| < 1\}$ *che, per* $z = re^{i\theta} \in U$*, soddisfano la condizione:*

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - 1}{2\beta\left(\dfrac{zf'(z)}{f(z)} - 1 + b\right) - \left(\dfrac{zf'(z)}{f(z)} - 1\right)} \right| < 1.$$

Inoltre, si dice che $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *appartiene alla classe* $C(b,\beta)$ *(*$b \neq 0$*, complesso,* $0 < \beta \leq 1$*) se e solo se* $zg'(z) \in S(1-b,\beta)$*. Questo articolo tratta lo studio di alcune proprietà di tale classe di funzioni.*

ABSTRACT – *Let* $S(1-b,\beta)$ *(*$b \neq 0$*, complex,* $0 < \beta \leq 1$*), denote the class of functions* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *analytic in* $U = \{z\colon |z| < 1\}$ *which satisfy for* $z = re^{i\theta} \in U$*,*

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - 1}{2\beta\left(\dfrac{zf'(z)}{f(z)} - 1 + b\right) - \left(\dfrac{zf'(z)}{f(z)} - 1\right)} \right| < 1.$$

Further $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *is said to belong to the class* $C(b,\beta)$ *(*$b \neq 0$*, complex,* $0 < \beta \leq 1$*) if and only if* $zg'(z) \in S(1-b,\beta)$*. This paper investigates certain properties*

of the above mentioned classes.

KEY WORDS – *Analytic - Starlike - Convex - Complex order.*

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1 – Introduction

Let A denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in $U = \{z: |z| < 1\}$. In [24] WIATROWSKI introduced the class of complex functions of order $b(b \neq 0$, complex) defined as follows:

DEFINITION 1. *A function $g(z) \in A$ is said to be convex function of order b ($b \neq 0$, complex), that is $g(z) \in C(b)$ if and only if $g'(z) \neq 0$ in U and*

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{zg''(z)}{g'(z)}\right\} > 0, \quad z \in U. \tag{1.1}$$

In [16] NASR and AOUF introduced the class $S(1-b)$, $b \neq 0$, complex, of starlike functions of order $1 - b$, defined as follows:

DEFINITION 2. *A function $f(z) \in A$ is said to be starlike function of order $1 - b$ ($b \neq 0$, complex), that is $f(z) \in S(1 - b)$ if and only if $f(z)/z \neq 0$ in U and*

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \quad z \in U. \tag{1.2}$$

It follows from (1.1) and (1.2) that

$$g(z) \in C(b) \quad \text{if and only if} \quad zg'(z) \in S(1 - b). \tag{1.3}$$

The class $C(b)$ has been studied by NASR and AOUF [14, 15] and AOUF [2].

Motivated by [7, 1, 13], we in the present paper, introduce the concept of "type" for the classes $S(1 - b)$ and $C(b)$, $b \neq 0$, complex, as follows:

DEFINITION 3. *A function* $(z) \in A$ *is a starlike of order* $1-b$ *and type* β, $f(z) \in S(1-b,\beta)$, *if and only if for all* $z \in U$, *the inequality*

$$(1.4)\qquad \left| \frac{\frac{zf'(z)}{f(z)} - 1}{2\beta\left(\frac{zf'(z)}{f(z)} - 1 + b\right) - \left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < 1,$$

holds for $b \neq 0$, *complex and* $0 < \beta \leq 1$.

DEFINITION 4. *A function* $g(z) \in A$ *is a convex of order* b *and type* β, $g(z) \in C(b,\beta)$, *if and only if for all* $z \in U$, *the inequality*

$$(1.5)\qquad \left| \frac{\frac{zg''(z)}{g'(z)}}{2\beta\left(\frac{zg''(z)}{g'(z)} + b\right) - \left(\frac{zg''(z)}{g'(z)}\right)} \right| < 1,$$

holds for $b \neq 0$, *complex and* $0 < \beta \leq 1$.

It follows from (1.4) and (1.5) that

$$(1.6)\qquad g(z) \in C(b,\beta) \quad \text{if and only if} \quad zg'(z) \in S(1-b,\beta).$$

By specializing b and β we obtain several subclasses studied by various authors in earlier papers:

(1) $S(1-b,1) = S(b)$ and $C(b,1) = C(b)$, $S\left(1-b, \frac{2M-1}{2M}\right) = F(b,M)$, $M > \frac{1}{2}$ and $C\left(b, \frac{2M-1}{2M}\right) = G(b,M), M > \frac{1}{2}$, are, respectively, the class of bounded starlike functions of complex order, introduced by NASR and AOUF [17], and the class of bounded convex functions of complex order, introduced by NASR and AOUF [15].

(2) $S(1-(1-\alpha)\cos\lambda e^{-i\lambda}, 1) = S^{\lambda}(\alpha)$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, $S(1-(1-\alpha),\beta) = S^{*}(\alpha,\beta)$, $0 \leq \alpha < 1$, $S(1-(1-\alpha)\cos\lambda e^{-i\lambda},\beta) = S^{\lambda}(\alpha,\beta)$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, are, respectively, the class of λ-spirallike functions of order α, introduced by LIBERA [10], the class of starlike functions of order α and type β studied by JUNEJA and MOGRA [7] and the class

of λ-spirallike functions of order α and type β, studied by MOGRA and AHUJA [13]. Also $C(\cos\lambda e^{-i\lambda}, 1) = C^{\lambda}$, $|\lambda| < \frac{\pi}{2}$, $C((1-\alpha)\cos\lambda e^{-i\lambda}, 1) = C^{\lambda}(\alpha)$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, $C((1-\alpha)\cos\lambda e^{-i\lambda}, \beta) = C^{\lambda}(\alpha,\beta)$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, are respectively, the class of λ-Robertson functions, studied by ROBERTSON [19], LIBERA and ZIEGLER [11], BAJPAI and MEHROK [4], the class of functions for which $zg'(z)$ is λ-spiral-shaped of order α, introduced and studied by CHICHRA [5] and SIZUK [23], the class of λ-Robertson functions of order α and type β, studied by AHUJA [1].

(3) $S\left(1-\cos\lambda e^{-i\lambda}, \frac{2M-1}{2M}\right) = F_{\lambda,M}$, $|\lambda| < \frac{\pi}{2}, M > \frac{1}{2}$, $S\left(1-(1-\alpha)\cos\lambda e^{-i\lambda}, \frac{2M-1}{2M}\right) = F_M(\lambda,\alpha)$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, $M > \frac{1}{2}$, and $S\left(1 - \cos\lambda e^{-i\lambda}, \frac{2-\cos\lambda}{2}\right) = H(\lambda)$, $|\lambda| < \frac{\pi}{2}$, are, respectively, the classes introduced and studied by KULSHRESTHA [9], AOUF [3] and GOEL [6]. Also $C\left(\cos\lambda e^{-i\lambda}, \frac{2M-1}{2M}\right) = G_{\lambda,M}$, $|\lambda| < \frac{\pi}{2}$, $M > \frac{1}{2}$, $C\left((1-\alpha)\cos\lambda e^{-i\lambda}, \frac{2M-1}{2M}\right) = G_M(\lambda,\alpha)$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1, M > \frac{1}{2}$, are the classes introduced and studied, respectively, by KULSHRESTHA [9] and AOUF [3].

(4) $S\left(1-(1-\alpha),\frac{1}{2}\right) = \overline{S}_{\alpha}$, $0 \leq \alpha < 1$, $S(0,\frac{1}{2})$ and $S\left(0, \frac{2M-1}{2M}\right)$, $M > \frac{1}{2}$, are the classes introduced and studied, respectively, by WRIGHT [25], MCCARTY [12], R. SINGH [21], and SINGH and SINGH [22]. Also $C(1-\alpha, 1) = C(\alpha)$, $0 \leq \alpha < 1$, is the class of convex functions of order α, introduced by ROBERTSON [20].

Since our classes includes various subclasses as noticed above, study of its various properties will lead to a unified study of these subclasses. In the present paper, we shall give at first a representation formula for the classes $S(1-b,\beta)$ and $C(b,\beta)$. A sufficient condition for a function to belong to $S(1-b,\beta)$ and $C(b,\beta)$ has been obtained. We maximize $|a_3 - \mu a_2^2|$ over the classes $S(1-b,\beta)$ and $C(b,\beta)$. Distortion theorems are obtained for the classes $S(1-b,\beta)$ and $C(b,\beta)$. Also we obtain the sharp radius of starlikeness for the class $S(1-b,\beta)$ and the sharp radious of convexity for the class $C(b,\beta)$.

2 – The representation formulas

Let Q denote the class of functions $\phi(z)$ which are analytic in U and which satisfy $|\phi(z)| \le 1$ for all z in U. We first give the following lemma.

LEMMA 1. *If a function* $H(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$, *analytic in* U, *satisfies the condition*

$$\left|\frac{H(z)-1}{2\beta(H(z)-1+b)-(H(z)-1)}\right| < 1, \tag{2.1}$$

for some $b \neq 0$, *complex*, $0 < \beta \le 1$ *and for all* $z \in U$, *then*

$$H(z) = \frac{1-((1-2\beta)+2\beta b)z\phi(z)}{1+(2\beta-1)z\phi(z)} \tag{2.2}$$

for some $\phi(z) \in Q$. *Conversely, a function* $H(z)$ *given by* (2.2) *for some* $\phi(z) \in Q$ *is analytic in* U *and satisfies* (2.1) *for all* z *in* U.

PROOF. The first half of the lemma is obtained immediately by an application of Schwarz's Lemma [18]; and the converse part follows from the observation that the function

$$w = \frac{1-((1-2\beta)+2\beta b)z}{1+(2\beta-1)z}$$

maps $|z| < 1$ onto the disc

$$\left|\frac{1-w}{2\beta(w-1+b)-(w-1)}\right| < 1$$

in the w-plane.

THEOREM 1. *A function* $f(z) \in A$, *is in the class* $S(1-b,\beta)$ *if and only if*

$$f(z) = z\exp\left\{-2\beta b\int_0^z \frac{\phi(t)}{1+(2\beta-1)t\phi(t)}dt\right\}, \tag{2.3}$$

for some $\phi(z) \in Q$.

PROOF. First suppose $f(z) \in S(1-b,\beta)$. Noting that $\frac{zf'(z)}{f(z)}$ satisfies the hypothesis of the first part of Lemma 1, we see that

$$\frac{zf'(z)}{f(z)} = \frac{1-((1-2\beta)+2\beta b)z\phi(z)}{1+(2\beta-1)z\phi(z)}$$

for some $\phi(z) \in Q$. Thus we have

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{-2\beta b\phi(z)}{1+(2\beta-1)z\phi(z)}. \tag{2.4}$$

An integration from 0 to z in (2.4) followed by an exponentation leads to (2.3).

Conversely, if (2.3) holds, then

$$\frac{zf'(z)}{f(z)} = \frac{1-((1-2\beta)+2\beta b)z\phi(z)}{1+(2\beta-1)z\phi(z)}.$$

Now the theorem follows by the converse part of Lemma 1.

From Theorem 1 and using (1.6), we get:

COROLLARY 1. *A function $g(z) \in A$, is in the class $C(b,\beta)$ if and only if*

$$g'(z) = \exp\left\{-2\beta b\int_0^z \frac{\phi(t)}{1+(2\beta-1)t\phi(t)}dt\right\},$$

for some $\phi(z) \in Q$.

An immediate consequence of Theorem 1, and a representation theorem for functions in $S^*(0,\beta)$ given by JUNEJA and MOGRA [7] may be shown in the following corollary:

COROLLARY 2. *$f(z) \in S(1-b,\beta)$ if and only if there is a function $f_1(z) \in S^*(0,\beta)$ such that*

$$f(z) = z\left[\frac{f_1(z)}{z}\right]^b.$$

Also an immediate consequence of Corollary 1, and a representation theorem for functions in $C^0(0,\beta)$ given by AHUJA [1] may be shown in the following corollary:

COROLLARY 3. *$g(z) \in C(b,\beta)$ if and only if there is a function $g_1(z) \in C^0(0,\beta)$ such that*

$$g'(z) = [g_1'(z)]^b .$$

REMARK ON THEOREM 1 AND COROLLARY 1

(1) For $\beta = 1$ in Theorem 1 and Corollary 1, we obtain, respectively, a representation formulas for the classes $S(1-b)$ and $C(b)$.

(2) For $\beta = \frac{2M-1}{2M}$, $M > \frac{1}{2}$ (or $\beta = \frac{1+m}{2}$, $m = 1 - \frac{1}{M}$, $M > \frac{1}{2}$) in Theorem 1 and Corollary 1, we obtain, respectively, a representation formulas for the classes $F(b,M)$ an $G(b,M)$.

(3) For $b = (1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, and $\beta = \frac{2M-1}{2M}$, $M > \frac{1}{2}$ (or $\beta = \frac{1+m}{2}$, $m = 1 - \frac{1}{M}$, $M > \frac{1}{2}$) in Theorem 1 and Corollary 1, we obtain, respectively, a representation formulas for the classes $F_M(\lambda,\alpha)$ and $G_M(\lambda,\alpha)$.

(4) Putting $b = 1-\alpha$, $0 \leq \alpha < 1$, in Theorem 1, we obtain a representation formula for the class $S^*(\alpha,\beta)$ determined by JUNEJA and MOGRA [7].

(5) Putting $b = (1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, in Corollary 1, we obtain a representation formula for the class $C^\lambda(\alpha,\beta)$ determined by AHUJA [1].

3 – The sufficient conditions

We now establish a sufficient condition for a function to be in $S(1-b,\beta)$ and $C(b,\beta)$.

THEOREM 2. *Let $f(z) \in A$. Then $f(z) \in S(1-b,\beta)$, if for $b \neq 0$, complex,*

$$\sum_{n=2}^{\infty} \{2n(1-\beta) - 1 + |1 - 2\beta(1-b)|\} |a_n| \leq 2\beta|b|, \tag{3.1}$$

whenever $\beta \in (0, \frac{1}{2}]$,

$$\sum_{n=2}^{\infty} \{(n-1) + |(2\beta-1)(n-1) + 2\beta b|\} |a_n| \leq 2\beta|b|, \tag{3.2}$$

whenever $\beta \in [\frac{1}{2}, 1]$,

holds.

PROOF. Let $|z| = r < 1$. Noting that

$$|zf'(z) - f(z)| < \sum_{n=2}^{\infty}(n-1)|a_n| r, \tag{3.3}$$

and

$$\begin{aligned} &|2\beta(zf'(z) - (1-b)f(z)) - (zf'(z) - f(z))| \geq \\ &\geq \left\{2\beta|b| - \sum_{n=2}^{\infty}(1-2\beta)n|a_n| - \sum_{n=2}^{\infty}|1 - 2\beta(1-b)||a_n|\right\} r, \end{aligned} \tag{3.4}$$

we see that

$$|zf'(z) - f(z)| - |2\beta(zf'(z) - (1-b)f(z)) - (zf'(z) - f(z))| \leq$$

$$\left[\sum_{n=2}^{\infty} \{2n(1-\beta) - 1 + |1 - 2\beta(1-b)|\} |a_n| - 2\beta|b|\right] r,$$

provided $0 < \beta \le \frac{1}{2}$. The last quantity is ≤ 0 by (3.1), so that $f(z) \in S(1-b,\beta)$. For the second part, we assume that (3.2) holds for $\beta \in [\frac{1}{2},1]$. In this case,

$$(3.5)\qquad \begin{aligned} &|2\beta(zf'(z)-(1-b)f(z))-(zf'(z)-f(z))| \ge \\ &\ge \left\{2\beta|b| - \sum_{n=2}^{\infty}|(2\beta-1)(n-1)+2\beta b|\,|a_n|\right\} r . \end{aligned}$$

Now the theorem follows, as before, from (3.3), (3.5) and (3.2).

Corollary 4. *Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$. Then $g(z) \in C(b,\beta)$ if for $b \ne 0$, complex,*

$$(3.6)\qquad \sum_{n=2}^{\infty} n\left\{2n(1-\beta)-1+|1-2\beta(1-b)|\right\}|b_n| \le 2\beta|b|,$$

whenever $\beta \in (0,\frac{1}{2}]$,

$$(3.7)\qquad \sum_{n=2}^{\infty} n\left\{(n-1)+|(2\beta-1)(n-1)+2\beta b|\right\}|b_n| \le 2\beta|b|,$$

whenever $\beta \in [\frac{1}{2},1]$,

holds.

Proof. The function $g(z)$ is in $C(b,\beta)$ if and only if $zg'(z) \in S(1-b,\beta)$. Now, since

$$zg'(z) = z + \sum_{n=2}^{\infty} nb_n z^n ,$$

by replacing a_n by $\{nb_n\}$ in Theorem 2, we have the theorem.

Remarks on theorem 2 and Corollary 4

(1) For $\beta = 1$ in Theorem 2 and Corollary 4, we obtain, respectively, a sufficient condition for a function to be in $S(1-b)$ and $C(b)$.

(2) For $\beta = \frac{2M-1}{2M}$, $M > \frac{1}{2}$, in Theorem 2 and Corollary 4, we obtain, respectively, a sufficient condition for a function to be in $F(b,M)$ and $G(b,M)$.

(3) For $b = (1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$, and $\beta = \frac{2M-1}{2M}$, $M > \frac{1}{2}$, in Theorem 2 and Corollary 4, we obtain, respectively, a sufficient condition for a function to be in $F_M(\lambda,\alpha)$ and $G_M(\lambda,\alpha)$.

(4) Putting $b = 1-\alpha$, $0 \le \alpha < 1$, in Theorem 2, we obtain the sufficient condition determined by JUNEJA and MOGRA [7].

(5) Putting $b = 1-\alpha$, $0 \le \alpha < 1$, and $\beta = \frac{1}{2}$, in Theorem 2, we obtain the sufficient condition determined by MCCARTY [12].

(6) Putting $b = (1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$, in Theorem 2 and Corollary 4, we obtain, respectively, the sufficient condition determined by MOGRA and AHUJA [13] and AHUJA [1].

4 – Coefficient bounds

Let Ω denotes the class of bounded analytic functions $w(z)$ in U, satisfying the conditions $w(0) = 0$ and $|w(z)| \le |z|$ for $z \in U$. We need in our discussion the following lemma:

LEMMA 2 [8]. *Let $w(z) = \sum_{n=1}^{\infty} c_n z^n \in \Omega$, if ν is any complex number, then*

$$|c_2 - \nu c_1^2| \le \max\{1, |\nu|\} . \tag{4.1}$$

Equality may be attained with the functions $w(z) = z^2$ and $w(z) = z$.

THEOREM 3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(1-b,\beta)$, $\beta \ne \frac{1}{2}$, and μ is any complex number, then*

$$|a_3 - \mu a_2^2| \le \beta|b| \max\{1, |2\beta b(2\mu - 1) - (2\beta - 1)|\} . \tag{4.2}$$

This inequality is sharp for each μ.

PROOF. Since $f(z) \in S(1-b,\beta)$, (2.3) gives

$$\frac{zf'(z)}{f(z)} = \frac{1+[(2\beta-1)-2\beta b]w(z)}{1+(2\beta-1)w(z)}, \tag{4.3}$$

where $w(z) = z\phi(z) = \sum_{k=1}^{\infty} c_k z^k \in \Omega$.

We get from (4.3) after expanding and equating coefficients that

$$a_2 = -2\beta b c_1 \tag{4.4}$$

$$a_3 = -\beta b\left\{[1-2\beta-2\beta b]c_1^2 + c_2\right\}. \tag{4.5}$$

Using (4.1), (4.4) and (4.5) we get the result. Since (4.1) is sharp, (4.2) is also sharp.

COROLLARY 5. *If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in C(b,\beta)$, $\beta \neq \frac{1}{2}$, and μ is any complex number, then*

$$\left|b_3 - \mu b_2^2\right| \leq \frac{\beta|b|}{3}\max\left\{1, \left|3\beta b\mu - 2\beta b - 2\beta + 1\right|\right\}.$$

This inequality is sharp for each μ.

THEOREM 4. *Let $f(z) \in S(1-b,\beta)$, and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in U$.*

(a) *If $\beta\left[|b|^2 + (k-1)\mathrm{Re}\{b\}\right] > (1-\beta)(k-1)\left[k-1+\mathrm{Re}\{b\}\right]$, let*

$$N = \frac{\beta\left[|b|^2 + (k-1)\mathrm{Re}\{b\}\right]}{(1-\beta)(k-1)\left[k-1+\mathrm{Re}\{b\}\right]}, \quad k = 2,3,\ldots,(n-1).$$

Then

$$|a_n| \leq \frac{1}{(n-1)!}\prod_{k=2}^{n}\left|(2\beta-1)(k-2)+2\beta b\right|, \tag{4.6}$$

for $n = 2, 3, \ldots, N+2$*; and*

(4.7) $$|a_n| \leq \frac{1}{(N+1)!(n-1)} \prod_{k=2}^{N+3} \left|(2\beta - 1)(k-2) + 2\beta b\right|, \quad n > N+2.$$

(b) *If* $\beta\left[|b|^2 + (k-1)\mathrm{Re}\{b\}\right] \leq (1-\beta)(k-1)\left[k - 1 + \mathrm{Re}\{b\}\right]$, *then*

(4.8) $$|a_n| \leq \frac{2\beta|b|}{n-1}, \quad \text{for} \quad n \geq 2.$$

The bounds in (4.6) *and* (4.8) *are sharp for all admissible* β, $b \neq 0$, *complex, and for each* n.

PROOF. Since $f(n) \in S(1-b, \beta)$, (4.3) gives

(4.9) $$\begin{aligned} &[(2\beta - 1)zf'(z) - (2\beta - 1)f(z) + 2\beta b f(z)]\, w(z) = \\ &= f(z) - zf'(z), \quad w \in \Omega. \end{aligned}$$

Now (4.8) may be written as

$$\begin{aligned} &\left\{2\beta bz + \sum_{k=2}^{\infty} [(2\beta - 1)(k-1) + 2\beta b]\, a_k z^k \right\} w(z) = \\ &= \sum_{k=2}^{\infty} (1-k) a_k z^k, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\left\{2\beta bz + \sum_{k=2}^{n-1} [(2\beta - 1)(k-1) + 2\beta b]\, a_k z^k \right\} w(z) = \\ &= \sum_{k=2}^{n} (1-k) a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k, \end{aligned}$$

where $\sum_{k=n+1}^{\infty} b_k z^k$ converges in U. Then, since $|w(z)| < 1$,

$$(4.10)\qquad \begin{aligned} &\left|2\beta bz + \sum_{k=2}^{n-1}[(2\beta-1)(k-1)+2\beta b]\,a_k z^k\right| \geq \\ &\geq \left|\sum_{k=2}^{n}(1-k)a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k\right| . \end{aligned}$$

Writting $z = re^{i\theta}$, $r < 1$, squaring both sides of (4.10), and then integrating, we get

$$\begin{aligned} &4\beta^2|b|^2 r^2 + \sum_{k=2}^{n-1}\left|(2\beta-1)(k-1)+2\beta b\right|^2 |a_k|^2 r^{2k} \geq \\ &\geq \sum_{k=2}^{n}(k-1)^2|a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |b_k|^2 r^{2k} . \end{aligned}$$

Let $r \longrightarrow 1$, then on some simplification we obtain

$$(4.11)\qquad \begin{aligned} &(n-1)^2|a_n|^2 \leq 4\beta^2|b|^2 + \sum_{k=2}^{n-1}\Big\{\left|(2\beta-1)(k-1)+\right. \\ &\left. + 2\beta b\right|^2 - (k-1)^2\Big\}|a_k|^2 , \quad n \geq 2 . \end{aligned}$$

Now there may be following two cases:

Let $\beta\left[|b|^2 + (k-1)\mathrm{Re}\{b\}\right] > (1-\beta)(k-1)\left[k-1+\mathrm{Re}\{b\}\right]$. Suppose that $n \leq N+2$; then for $n = 2$, (4.11) gives

$$|a_2| \leq 2\beta|b| ,$$

which gives (4.6) for $n = 2$. We establish (4.6) for $n \leq N+2$, from (4.11), by mathematical induction.

Suppose (4.6) is valid for $k = 2, 3, \dots, (n-1)$. Then it follows from (4.11)

$$(n-1)^2|a_n|^2 \leq 4\beta^2|b|^2 + \sum_{k=2}^{n-1}\left\{\left[|(2\beta-1)(k-1)+\right.\right.$$

$$\left.\left. + 2\beta b|^2 - (k-1)^2\right]\frac{1}{((k-1)!)^2}\prod_{p=2}^{k}|(2\beta-1)(p-2)+2\beta b|^2\right\} =$$

$$= \frac{1}{((n-2)!)^2}\prod_{k=2}^{n}|(2\beta-1)(k-2)+2\beta b|^2 .$$

Thus, we get

$$|a_n| \leq \frac{1}{(n-1)!}\prod_{k=2}^{n}|(2\beta-1)(k-2)+2\beta b| ,$$

which completes the proof of (4.6).

Next, we suppose $n > N+2$. Then (4.11) gives

$$(n-1)^2|a_n|^2 \leq 4\beta^2|b|^2 + \sum_{k=2}^{N+2}\left\{\left|(2\beta-1)(k-1)+\right.\right.$$

$$\left.\left. 2\beta b\right|^2 - (k-1)^2\right\}|a_k|^2 + \sum_{k=N+3}^{n-1}\left\{\left|(2\beta-1)(k-1)+\right.\right.$$

$$\left.\left. + 2\beta b\right|^2 - (k-1)^2\right\}|a_k|^2 \leq$$

$$\leq 4\beta^2|b|^2 + \sum_{k=2}^{N+2}\left\{\left|(2\beta-1)(k-1)+2\beta b\right|^2 - (k-1)^2\right\}|a_k|^2 .$$

On substituting upper estimates for $a_2, a_3, \dots, a_{N+2}$ obtained above, and simplifying, we obtain (4.7).

(b) Let $\beta\left[|b|^2 + (k-1)\mathrm{Re}\{b\}\right] \leq (1-\beta)(k-1)\left[k-1+\mathrm{Re}\{b\}\right]$, then it follows from (4.11)

$$(n-1)^2|a_n|^2 \leq 4\beta^2|b|^2 , \quad (n \geq 2)$$

which proves (4.8).

The bounds in (4.6) are sharp for the functions given by

$$f(z) = z(1-(2\beta-1)z)^{(-2\beta b)/(2\beta-1)}, \quad \beta \neq \frac{1}{2}. \tag{4.12}$$

The bounds in (4.8) are sharp for the functions given by

$$f_n(z) = z(1-(2\beta-1)z^{n-1})^{(-2\beta b)/[(2\beta-1)(n-1)]}, \quad \beta \neq \frac{1}{2}; \tag{4.13}$$

whereas for $\beta = \frac{1}{2}$,

$$f_n(z) = z\exp\left(\frac{b}{n-1}z^{n-1}\right), \quad (n \geq 2). \tag{4.14}$$

COROLLARY 6. *If* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in C(b,\beta)$.

(a) *If* $\beta\left[|b|^2 + (k-1)\mathrm{Re}\{b\}\right] > (1-\beta)(k-1)\left[k-1+\mathrm{Re}\{b\}\right]$, *let*

$$N = \left[\frac{\beta\left[|b|^2+(k-1)\mathrm{Re}\{b\}\right]}{(1-\beta)(k-1)[k-1+\mathrm{Re}\{b\}]}\right], \quad k = 2,3,\ldots,(n-1),$$

where N *is the greatest integer of the expression within the square bracket. Then*

$$|b_n| \leq \frac{1}{n!}\prod_{k=2}^{n}\left|(2\beta-1)(k-2)+2\beta b\right|, \tag{4.15}$$

for $n = 2,3,\ldots,N+2$; *and*

$$|b_n| \leq \frac{1}{n(n-1)(N+1)!}\prod_{k=2}^{N+3}\left|(2\beta-1)(k-2)+2\beta b\right|, \quad n > N+2. \tag{4.16}$$

(b) *If* $\beta\left[|b|^2 + (k-1)\mathrm{Re}\{b\}\right] \leq (1-\beta)(k-1)\left[k-1+\mathrm{Re}\{b\}\right]$, *then*

$$|b_n| \leq \frac{2\beta|b|}{n(n-1)}, \quad n \geq 2. \tag{4.17}$$

The estimates in (4.15) *are sharp for the function given by*

$$(4.18) \qquad g'(z) = (1-(2\beta-1)z)^{(-2\beta b)/(2\beta-1)}, \quad \beta \neq \frac{1}{2},$$

where $\beta\left[|b|^2 + (k-1)\mathrm{Re}\{b\}\right] > (1-\beta)(k-1)\left[K-1+\mathrm{Re}\{b\}\right]$, *while the estimates in* (4.17) *are sharp for the functions given by*

$$(4.19) \qquad g'_n(z) = (1-(2\beta-1)z^{n-1})^{(-2\beta b)/[(2\beta-1)(n-1)]}, \quad \beta \neq \frac{1}{2};$$

whereas for $\beta = \frac{1}{2}$

$$(4.20) \qquad g'_n(z) = \exp\left(\frac{b}{n-1} z^{n-1}\right), \quad (n \geq 2).$$

REMARKS ON THEOREM 4 AND COROLLARY 6

(1) Putting $b = 1-\alpha$, $0 \leq \alpha < 1$, in Theorem 4, we obtain a Theorem of JUNEJA and MOGRA [7].

(2) Putting $b = (1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, in Theorem 4 and Corollary 6, we obtain, respectively, a Theorem of MOGRA and AHUJA [13] and AHUJA [1].

(3) By choosing $b = \cos\lambda e^{-i\lambda}$ and $\beta = \frac{2-\cos\lambda}{2}$, $|\lambda| < \frac{\pi}{2}$, in Theorem 4, we get the result of GOEL [6].

(4) By choosing $\beta = 1$ and $\beta = \frac{2M-1}{2M}$, $M > \frac{1}{2}$, in Theorem 4 and Corollary 6, we get the results obtained by NASR and AOUF [14, 15, 16, 17], respectively.

(5) The coefficient estimates determined by KULSHRESTHA [9], ZAMARSKI [26], WRIGHT [25], MCCARTY [12], R. SINGH [21], and many others can be obtained from Theorem 3 and Corollary 6 by taking different values of β and b.

5 – Distortion Theorems

THEOREM 5. *Let $f(z) \in A$. If $f(z) \in S(1-b,\beta)$, then for $|z| = r$, $0 < r < 1$, and for all $\beta \in (0,\frac{1}{2})U(\frac{1}{2},1]$, $b \neq 0$, complex,*

$$|f(z)| \leq r \left[\frac{(1+(2\beta-1)r)^{\left(1-\frac{\mathrm{Re}\{b\}}{|b|}\right)}}{(1-(2\beta-1)r)^{\left(1+\frac{\mathrm{Re}\{b\}}{|b|}\right)}} \right]^{\frac{\beta|b|}{2\beta-1}}, \tag{5.1}$$

and

$$|f(z)| \geq r \left[\frac{(1-(2\beta-1)r)^{\left(1-\frac{\mathrm{Re}\{b\}}{|b|}\right)}}{(1+(2\beta-1)r)^{\left(1+\frac{\mathrm{Re}\{b\}}{|b|}\right)}} \right]^{\frac{\beta|b|}{2\beta-1}}; \tag{5.2}$$

whereas for $\beta = \frac{1}{2}$, $b \neq 0$, complex,

$$|f(z)| \leq r \exp\left(|b|r\right), \tag{5.3}$$

and

$$|f(z)| \geq r \exp\left(-|b|r\right). \tag{5.4}$$

All these estimates are sharp for all admissible values of β, b.

PROOF. Since $f(z) \in S(1-b,\beta)$, the condition (1.4) coupled with an application of Schwarz's Lemma [18], implies

$$\left| \frac{zf'(z)}{f(z)} - \xi \right| < R, \quad \text{where}$$

$$\xi = \frac{1-(2\beta-1)\left[2\beta(1-\mathrm{Re}\{b\})-1\right]r^2 + i2\beta(2\beta-1)\mathrm{Im}\{b\}r^2}{1-(2\beta-1)^2r^2},$$

and

$$R = \frac{2\beta|b|r}{1-(2\beta-1)^2r^2}, \quad (|z| = r).$$

Hence we have

$$\frac{1-2\beta|b|r+(2\beta-1)\Big[2\beta(\mathrm{Re}\{b\}-1)+1\Big]r^2}{1-(2\beta-1)^2r^2}\leq$$

$$(5.5)\qquad \leq \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\leq$$

$$\frac{1+2\beta|b|r+(2\beta-1)\Big[2\beta(\mathrm{Re}\{b\}-1)+1\Big]r^2}{1-(2\beta-1)^2r^2}.$$

Noting

$$\log\left(\left|\frac{f(z)}{z}\right|\right)=\mathrm{Re}\left(\log\frac{f(z)}{z}\right)=\mathrm{Re}\int_0^z\left[\frac{f'(s)}{f(s)}-\frac{1}{s}\right]ds=$$

$$=\int_0^r\frac{1}{t}\mathrm{Re}\left[te^{i\theta}\frac{f'(te^{i\theta})}{f(te^{i\theta})}-1\right]dt,$$

and using (5.5), we see that

$$(5.6)\qquad \log\left(\left|\frac{f(z)}{z}\right|\right)\leq 2\beta|b|\int_0^r\frac{1+(2\beta-1)\frac{\mathrm{Re}\{b\}t}{|b|}}{1-(2\beta-1)^2t^2}dt.$$

Now suppose $\beta\in(0,\frac{1}{2})\cup(\frac{1}{2},1]$ and $b\neq 0$, complex. Then from (5.6), we get

$$\log\left(\left|\frac{f(z)}{z}\right|\right)\leq\frac{\beta|b|}{2\beta-1}\log\left\{\frac{(1+(2\beta-1)r)^{\left(1-\frac{\mathrm{Re}\{b\}}{|b|}\right)}}{(1-(2\beta-1)r)^{\left(1+\frac{\mathrm{Re}\{b\}}{|b|}\right)}}\right\},$$

which gives (5.1). For the case when $\beta=\frac{1}{2}$, and $b\neq 0$, complex, (5.6) immediately proves (5.3). In view of

$$\log\left(\left|\frac{f(z)}{z}\right|\right)=\mathrm{Re}\left(\log\frac{f(z)}{z}\right)=\int_0^r\mathrm{Re}\left(\frac{\partial}{\partial t}\left(\log\frac{f(t)}{t}\right)\right)dt=$$

$$=\int_0^r\frac{1}{r}\mathrm{Re}\left(\frac{tf'(t)}{f(t)}-1\right)dt,$$

and with the aid of (5.5) we may write

$$\log\left(\left|\frac{f(z)}{z}\right|\right) \geq -2\beta|b| \int_0^r \frac{1-(2\beta-1)\frac{Re\{b\}t}{|b|}}{1-(2\beta-1)^2t^2}dt. \tag{5.7}$$

If $\beta \neq \frac{1}{2}$, then carrying out the integration in (5.7), we obtain (5.2). Further, when $\beta = \frac{1}{2}$, then we immediately get (5.4) from (5.7). The extremal function for all the inequalities is given by

$$f(z) = \begin{cases} z(1-(2\beta-1)\varepsilon z)^{(-2\beta b)/(2\beta-1)}, |\varepsilon| = 1, & \beta \neq \frac{1}{2}, \\ z\exp(b \in z), \quad |\varepsilon| = 1, & \beta = \frac{1}{2}. \end{cases} \tag{5.8}$$

COROLLARY 7. *Let $g(z) \in A$. If $g(z)$ is in $C(b,\beta)$, then for $|z| = r < 1$ and for all $b \neq 0$, complex, $\beta \in (0,\frac{1}{2})U(\frac{1}{2},1]$,*

$$|g'(z)| \leq \left[\frac{(1+(2\beta-1)r)^{\left(1-\frac{Re\{b\}}{|b|}\right)}}{(1-(2\beta-1)r)^{\left(1+\frac{Re\{b\}}{|b|}\right)}}\right]^{\frac{\beta|b|}{2\beta-1}}, \tag{5.9}$$

and

$$|g'(z)| \geq \left[\frac{(1-(2\beta-1)r)^{\left(1-\frac{Re\{b\}}{|b|}\right)}}{(1+(2\beta-1)r)^{\left(1+\frac{Re\{b\}}{|b|}\right)}}\right]^{\frac{\beta|b|}{2\beta-1}}; \tag{5.10}$$

whereas for $\beta = \frac{1}{2}$, $b \neq 0$, complex,

$$|g'(z)| \leq \exp(|b|r), \tag{5.11}$$

and

$$|g'(z)| \geq \exp(-|b|r). \tag{5.12}$$

The extremal function for all the inequalities is given by

$$(5.13)\qquad g_0'(z)=\begin{cases}(1-(2\beta-1)\varepsilon z)^{(-2\beta b)/(2\beta-1)}\,,|\varepsilon|=1\,,\quad \beta\neq\dfrac{1}{2}\,,\\[2ex] \exp(b\in z)\,,\quad |\varepsilon|=1\,,\quad \beta=\dfrac{1}{2}\,.\end{cases}$$

REMARK ON THEOREM 5 AND COROLLARY 7

(1) For $\beta=1$ in Theorem 5 and Corollary 7, we obtain, respectively, the bounds for $|f(z)|$ and $|g'(z)|$, where $f(z)\in S(1-b)$ and $g(z)\in C(b)$.

(2) For $\beta=\frac{2M-1}{2M}$, $M>\frac{1}{2}$ (or $\beta=\frac{1+m}{2}$, $m=1-\frac{1}{M}$, $M>\frac{1}{2}$) in Theorem 5 and Corollary 7, we obtain, respectively, the bounds for $|f(z)|$ and $|g'(z)|$, where $f(z)\in F(b,M)$ and $g(z)\in G(b,M)$.

(3) For $b=(1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda|<\frac{\pi}{2}$, $0\le\alpha<1$ and $\beta=\frac{2M-1}{2M}$, $M>\frac{1}{2}$ (or $\beta=\frac{1+m}{2}$, $m=1-\frac{1}{M}$, $M>\frac{1}{2}$) in Theorem 5 and Corollary 7, we obtain, respectively the bounds for $|f(z)|$ and $|g'(z)|$, where $f(z)\in F_M(\lambda,\alpha)$ and $g(z)\in G_M(\lambda,\alpha)$.

(4) Putting $b=(1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda|<\frac{\pi}{2}$, $0\le\alpha<1$, in Theorem 5 and Corollary 7, we obtain, respectively, a Theorem of MOGRA and AHUJA [13] and AHUJA [1].

6 – The radious of starlikeness and convexity

THEOREM 6. *The sharp radius of starlikeness of the class $S(1-b,\beta)$, $\beta\neq\frac{1}{2}$, is given by*

$$(6.1)\qquad r_s=\left\{\beta|b|+\sqrt{\beta^2|b|^2-(2\beta-1)^2\left[\frac{2\beta}{2\beta-1}\mathrm{Re}\{b\}-1\right]}\right\}^{-1}.$$

The expression in (6.1) is real and finite (<1) only when $\beta\neq\frac{1}{2}$ and such that

$$(6.2)\qquad \beta^2|b|^2\ge(2\beta-1)^2\left[\frac{2\beta}{2\beta-1}\mathrm{Re}\{b\}-1\right].$$

PROOF. From (5.5), we have

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{1-2\beta|b|r+(2\beta-1)^2\left[\frac{2\beta}{2\beta-1}\operatorname{Re}\{b\}-1\right]r^2}{1-(2\beta-1)^2r^2}.$$

where $|z| = r$.

Thus $\operatorname{Re}\{\frac{zf'(z)}{f(z)}\} > 0$ for $|z| < r_s$, where r_s is given by (6.1) provided that the expression under the radical sign in (6.1) is non-negative, i.e.,

$$\beta^2|b|^2 \geq (2\beta-1)^2\left[\frac{2\beta}{2\beta-1}\operatorname{Re}\{b\}-1\right]$$

which gives (6.2).

To show that (6.1) is sharp, we let

$$f_*(z) = z(1-(2\beta-1)z)^{(-2\beta b)/(2\beta-1)}, \quad \beta \neq \frac{1}{2} \text{ and}$$

$$t = \frac{r\left[(2\beta-1)r-\sqrt{\frac{\bar{b}}{b}}\right]}{1-(2\beta-1)r\sqrt{\frac{\bar{b}}{b}}},$$

and obtain

$$\frac{tf_*'(t)}{f_*(t)} = \frac{1-2\beta|b|r+(2\beta-1)^2\left[\frac{2\beta}{2\beta-1}b-1\right]r^2}{1-(2\beta-1)^2r^2},$$

which has a zero real part at r given by (6.1). This completes the proof of the theorem.

REMARKS ON THEOREM 6

(1) $\beta = 1$ and $\beta = \frac{2M-1}{2M}$, $M > 1$, $m = 1 - \frac{1}{M}$, leads, respectively, to the results obtained by NASR and AOUF [16,17].

(2) $b = \cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$ and $\beta = \frac{2M-1}{2M}$, $M > 1$, $m = 1 - \frac{1}{M}$, leads to the result obtained by KULSHRESTHA [9].

COROLLARY 8. *Let $g(z) \in A$ and $g(z)$ is a member of $C(b,\beta)$. Then the sharp radius of convexity of the class $C(b,\beta)$, $\beta \neq \frac{1}{2}$, is given by (6.1). The radius is real and finite (< 1) if $\beta \neq \frac{1}{2}$ and (6.2) is satisfied. The result is sharp for the function*

$$g_0'(z) = (1-(2\beta-1)z)^{(-2\beta b)/(2\beta-1)}, \quad \beta \neq \frac{1}{2} \text{ and}$$

$$t = \frac{r\left[(2\beta-1)r - \sqrt{\frac{\bar{b}}{b}}\right]}{1-(2\beta-1)r\sqrt{\frac{\bar{b}}{b}}}.$$

REMARKS ON COROLLARY 8

(1) Putting $b = (1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$, in Corollary 8, we get a theorem of AHUJA [1].

(2) Putting $\beta = 1$, in Corollary 8, we get a theorem of NASR and AOUF [14].

(3) Putting $b = \cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$ and $\beta = 1$ in Corollary 8, we get a theorem of LIBERA and ZIEGLER [11].

(4) Putting $b = (1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$, $0 \leq \alpha < 1$ and $\beta = 1$ in Corollary 8, we get a theorem of CHICHRA [5].

(5) On taking the appropriate values to b and β the above corollary can give the corresponding radius of convexity for the functions in the classes $G(b,M)$, $G_{\lambda,M}$ and $G_M(\lambda,\alpha)$.

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