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Second-order Differential Equations and Degenerate Lagrangians

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RIASSUNTO – *Si studiano le equazioni differenziali del secondo ordine ed i sistemi lagrangiani degeneri utilizzando le nozioni di struttura quasi tangente e quasi prodotto sulle varietà.*

ABSTRACT – *We give a continuation to an article of Sarlet et al. using results previously presented by us. Second order differential equations and degenerate Lagrangian systems are studied in the framework of almost tangent and almost product geometries.*

KEY WORDS – *Lagrangian mechanics - Degenerate - Almost tangent geometry.*

A.M.S. CLASSIFICATION: 58F05, 70H35

1 – Introduction

It is known that it is possible to develop in a geometric manner the Lagrangian formalism with the help of a special structure — called **almost tangent** — which is an intrinsical property of every tangent bundle TM of a manifold M (for further details see [13]). This kind of formalism shows that if $L : TM \longrightarrow R$ is a **regular** Lagrangian, that is, the Hessian matrix of L with respect to the velocities variables has maximal rank, then there is a symplectic form w_L on TM such that there

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is a unique vector field ξ on TM solving the equation

$$(*) \qquad i_\xi w_L = dE_L$$

where E_L is the energy associated with L. Moreover, ξ is a 2^{nd} order differential equation, called **Lagrange vector field**.

Sarlet et al. ([16]) proposed a different approach to the study of regular Lagrangian systems. Their basic idea was to start the study directly with a 2^{nd} order differential equation ξ on TM and then giving some conditions to find a regular Lagrangian such that ξ is precisely a Lagrange vector field. For this it was considered a special set of one-forms, denoted here by Δ^1_ξ and defined by

$$\Delta^1_\xi = \{\alpha \in \Delta^1(TM);\ (\mathcal{L}_\xi \circ J^*)(\alpha) = \alpha\},$$

where ξ is a 2^{nd} order differential equation, $\Delta^1(TM)$ is the set of all one-forms on TM, $\mathcal{L}_\xi$ is the Lie derivative and J^* is the adjoint endomorphism on $\Delta^1(TM)$ induced by the almost tangent structure $J : T(TM) \longrightarrow T(TM)$ on the double tangent bundle $T(TM)$ of M.

If we associate with each ξ an appropriate one-form α belonging to Δ^1_ξ then we obtain the Euler-Lagrange equations of motion. For this α must be exact, $\alpha = dL$, with L being regular. The interest on Δ^1_ξ led the quoted authors to obtain some geometric results like the invariance of Δ^1_ξ under the action of some tensors. The results were extended for the non-conservative case in de León & Rodrigues [7].

More recently the authors of the present article ([8,9]) proposed a geometrical study of **degenerate** Lagrangian systems suggested by the theory of lifts of tensor fields on manifolds, which may be summarized as follows. As we are dealing with degenerate Lagrangians, the form w_L is presymplectic and so we may decompose the tangent bundle of TM into a direct sum of Ker S_{w_L} and a complementary distribution. This distribution may be characterized by a couple of projectors on $T(TM)$ in such a way that one of them projects onto the Ker s_{w_L}. This gives the possibility of solving (*) along a distribution defined by the other projector. Now, this solution is not of 2^{nd} order type. So we consider a couple of projectors (P_0, Q_0) on the base manifold M and then we lift (P_0, Q_0) to TM, using the theory of lifting vector fields, forms and tensors. We suppose that the lifted projector of Q_0 projects onto Ker s_{w_L}. We consider an integral constraint submanifold S_0 of Im P_0 in M in such a way that w_L restricted to the tangent bundle TS_0 is symplectic. As in this case when the restriction of the original Lagrangian to the bundle TS_0 is regular, we may solve the corresponding equation of motion for this restricted Lagrangian.

The purpose of the present article is to go on with our procedure now using the method suggested by Sarlet et al. We may study the relation between 2^{nd} order differential equations and degenerate Lagrangians defining sets analogous to the above Δ^1_ξ. We will use the projectors to examine the Sarlet et al. results for the degenerate situation. Finally, we remark to the reader that throughout the text we will keep in mind some results of our previous article as well as the notation and terminology.

2 – Degenerate Lagrangian vector fields

Let (P, Q) be an almost product structure on TM which commutes with the almost tangent structure J, that is,

$$JP = PJ\,,$$

(also we have $JQ = QJ$, since $Q = Id - P$).

DEFINITION 2.1. *A vector field $\xi \in$ Im P is called a P-***semispray** *(or P-***second order differential equation***) if $J\xi = PV$, where V is the Liouville vector field on TM.*

With each P-semispray ξ on TM we now associate two R-linear operators on one-forms E_ξ and $\tilde{E}_\xi$ defined by

$$E_\xi = Id - \mathcal{L}_\xi \circ J^*,$$

$$\tilde{E}_\xi = P^* \circ E_\xi,$$

respectively.

REMARK. If $P = Id$, then ξ is a semispray (or second order differential equation) on TM and $\tilde{E}_\xi = E_\xi$ is the Euler-Lagrange operator introduced by Sarlet, Cantrijn and Crampin [16].

We can associate with ξ two sets of 1-forms Δ^1_ξ and $\tilde{\Delta}^1_\xi$ defined by

$$\Delta^1_\xi = \text{Ker } E_\xi,$$
$$\tilde{\Delta}^1_\xi = \text{Ker } \tilde{E}_\xi,$$

respectively.

Δ^1_ξ and $\tilde{\Delta}^1_\xi$ are in fact, vector spaces over R, by the linearity of E_ξ and $\tilde{E}_\xi$. We also have $\Delta^1_\xi \subset \tilde{\Delta}^1_\xi$.

Let us remark that Δ^1_ξ and $\tilde{\Delta}^1_\xi$ are not modules over the ring of functions; however, they are modules over the ring of constants of motion, that is, functions f satisfying $\xi f = 0$. In fact

$$E_\xi(f\alpha) = (\xi f)J^*\alpha + f(E_\xi\alpha)$$

and

$$\tilde{E}_\xi(f\alpha) = (\xi f)(PJ)^*\alpha + f(\tilde{E}_\xi\alpha).$$

Next, we shall compute the local expressions of E_ξ and $\tilde{E}_\xi$. First, since $PJ = JP$, we deduce that

$$P\left(\frac{\partial}{\partial q^i}\right) = P^j_i \frac{\partial}{\partial q^i} + \bar{P}^j_i \frac{\partial}{\partial v^i},$$

$$P\left(\frac{\partial}{\partial v^i}\right) = P^j_i \frac{\partial}{\partial v^j} \tag{2.1}$$

where

$$P^i_j = P^i_k P^k_j;\; \bar{P}^i_j = \bar{P}^i_k P^k_j + P^i_k \bar{P}^k_j, \tag{2.2}$$

because of $P^2 = P$, and (q^i, v^i) are the natural bundle coordinates on TM. From (2.1), we easily obtain

$$P^*(dq^i) = P^i_j dq^j,\; P^*(dv^i) = \bar{P}^i_j dq^j + P^i_j dv^j. \tag{2.3}$$

Then, if $\xi = PX$, where

$$X = \chi^i \frac{\partial}{\partial q^i} + \bar{\chi}^i \frac{\partial}{\partial v^i},$$

we have

$$\xi = \chi^i P_i^j \frac{\partial}{\partial q^j} + (\chi^i \bar{P}_i^j + \bar{\chi}^i P_i^j)\frac{\partial}{\partial v^j}, \tag{2.4}$$

$$PV = v^i P_i^j \frac{\partial}{\partial v^j}. \tag{2.5}$$

From (2.4) and (2.5), we deduce that

$$\chi^i P_i^j = v^i P_i^j, \; 1 \le j \le m \tag{2.6}$$

since $J\xi = PV$.

Hence, if $\alpha = \alpha_i dq^i + \bar{\alpha}_i dv^i$, we have

$$E_\xi \alpha = (\alpha_i - \xi(\bar{\alpha}_i) - \frac{\partial(\chi^j P_j^k)}{\partial q^i}\alpha_k)dq^i + (\bar{\alpha}_i - \frac{\partial(\chi^j P_j^k)}{\partial v^i}\bar{\alpha}_k)dv^i \tag{2.7}$$

$$\bar{E}_\xi \alpha = \left[(\alpha_i - \xi(\bar{\alpha}_i) - \frac{\partial(\chi^j P_j^k)}{\partial q^i}\alpha_k)P_\ell^i + (\bar{\alpha}_i - \frac{\partial(\chi^j P_j^k)}{\partial v^i}\bar{\alpha}_k)\bar{P}_\ell^i\right] dq^\ell$$

$$+ (\bar{\alpha}_i - \frac{\partial(\chi^j P_j^k)}{\partial v^i}\bar{\alpha}_k)P_\ell^i dv^\ell. \tag{2.8}$$

If $P = P_0^c$ is the complete lift of an almost product structure P_0 on M, then (2.7) and (2.8) become

$$E_\xi \alpha = (\alpha_i - \xi(\bar{\alpha}_i) - \frac{\partial(\chi^j P_j^k)}{\partial q^i}\alpha_k)dq^i + (\bar{\alpha}_i - \frac{\partial \chi^j}{\partial v^i}P_j^k \bar{\alpha}_k)dv^i, \tag{2.9}$$

$$\bar{E}_\xi \alpha = \left[(\alpha_i - \xi(\bar{\alpha}_i) - \frac{\partial(\chi^j P_j^k)}{\partial q^i}\alpha_k)P_\ell^i + (\bar{\alpha}_i - \frac{\partial \chi^j}{\partial v^i}P_j^k \bar{\alpha}_k)v^r \frac{\partial P_\ell^i}{\partial q^r}\right] dq^\ell$$

$$+ (\bar{\alpha}_i - \frac{\partial \chi^j}{\partial v^i}P_j^k \bar{\alpha}_k)P_\ell^i dv^\ell, \tag{2.10}$$

because $P_j^i = P_j^i(q^r)$ and $\bar{P}_j^i = v^r \frac{\partial P_j^i}{\partial q^r}$.

From (2.7) and (2.8), we deduce that α belongs to Δ^1_ξ (resp. $\tilde{\Delta}^1_\xi$) if and only if

$$\xi(\bar{\alpha}_i) = \alpha_i - \frac{\partial(\chi^j P^k_j)}{\partial q^i}\alpha_k,$$

$$\bar{\alpha}_i = \frac{\partial(\chi^j P^k_j)}{v^i}\bar{\alpha}_k \tag{2.11}$$

(resp.,

$$\xi(\bar{\alpha}_i)P^i_l = (\alpha_i - \frac{\partial(\chi^j P^k_j)}{\partial q^i}\alpha_k)P^i_l + (\bar{\alpha}_i - \frac{\partial(\chi^j P^k_j)}{\partial v^i}\bar{\alpha}_k)\bar{P}^i_l,$$

$$\bar{\alpha}_i P^i_l = \frac{\partial(\chi^j P^k_j)}{\partial v^i}\bar{\alpha}_k P^i_l). \tag{2.12}$$

DEFINITION 2.2.

(1) $\alpha \in \tilde{\Delta}^1_\xi$ *is called* P-**regular** *if* $w_\alpha = -d(J^*\alpha)$ *is a presymplectic form and* P *is adapted to* w_α.

(2) ξ *is called a* P-**regular Lagrangian vector field** *if there exists a* P*-regular* $\alpha \in \bar{\Delta}^1_\xi$ *which is exact, that is,* $\alpha = dL$, *for some function* $L : TM \longrightarrow R$.

REMARK. It is clear that the Lagrangian involved in the definition must be degenerate, otherwise we will be in the presence of a regular situation and in such situation we will take $P = Id$, obtaining the results of Sarlet et al. So throughout the text L, such that $dL = \alpha$, is assumed to be degenerate.

Let us suppose that ξ is a P-regular Lagrangian vector field and $\alpha = dL$. Then we have $w_\alpha = w_L$, where $w_L = -dd_J L$ is the usual Poincaré-Cartan form. Furthermore, the condition

$$P^*(\mathcal{L}_\xi(J^*(dL))) = P^*(dL),$$

which characterizes $\alpha = dL$, can be rewritten in the form

$$i_\xi w_L = P^*(dE_L), \tag{2.13}$$

where $E_L = (PV)L - L$ is the energy function. So, (2.13) is the motion equation corresponding to the degenerate Lagrangian L (see [6,9]).

Moreover, if we suppose that P is the complete lift to TM of an integrable almost product structure P_0 on M, then we can choose local coordinates (q^i) on M such that

$$I_m P_0 = < \frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^r} >$$

where rank $P_0 = r$. Therefore,

$$I_m P_0^c = < \frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^r}, \frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^r} >$$

and $P_0^c V = \sum_{i=1}^r v^i \frac{\partial}{\partial v^i}$. So, from (2.4) and (2.6), ξ is locally given by

$$\xi = \sum_{i=1}^{r} v^i \frac{\partial}{\partial q^i} + \sum_{i=1}^{r} \bar{\chi}^i \frac{\partial}{\partial v^i} \tag{2.14}$$

where $\bar{\chi}^i = \bar{\chi}^i(q, v)$, $1 \leq i \leq r$.

Hence, (2.9) and (2.10) becomes

$$E_\xi \alpha = \sum_{i=1}^{m} (\alpha_i - \xi(\bar{\alpha}_i)) dq^i + \sum_{i=r+1}^{m} \bar{\alpha}_i dv^i \tag{2.15}$$

and

$$\tilde{E}_\xi \alpha = \sum_{i=1}^{r} (\alpha_i - \xi(\bar{\alpha}_i)) dq^i. \tag{2.16}$$

From (2.15) and (2.16) (or from (2.11) and (2.12)) we deduce that $\alpha \in \Delta_\xi^1$ (resp., $\alpha \in \tilde{\Delta}_\xi^1$) if and only if

$$\alpha_i = \xi(\bar{\alpha}_i),\ 1 \leq i \leq r,\ \alpha_i = \bar{\alpha}_i = 0,\ r+1 \leq 1 \leq m$$

(resp., $\alpha_i = \xi(\bar{\alpha}_i)$, $1 \leq i \leq r$).

Now, let us suppose that $\phi : S_0 \longrightarrow M$ is an integral manifold of ImP_0. Then $T\phi : TS_0 \longrightarrow TM$ is an integral manifold of ImP_0^c. So J

and V restrict to TS_0 and $J/TS_0 = J_0$, $V/TS_0 = V_0$, where J_0 and V_0 are the almost tangent structure and Liouville vector field, respectively, corresponding to TS_0. Furthermore, $(T\phi)^*w_L$ is a symplectic form on TS_0, $\xi_0 = \xi/TS_0$ is a semispray on TS_0 and $L_0 = L \circ (T\phi)$ is a regular Lagrangian on TS_0. Hence (2.13) becomes

$$i_{\xi_0}(T\phi)^* w_L = d(E_L \circ T\phi), \tag{2.17}$$

on TS_0, where $(T\phi)^*w_L = -dd_{J_0}L_0 = w_{L_0}$ and $E_L \circ T\phi = V_0 L_0 - L_0 = E_{L_0}$.

Obviously, (2.16) is equivalent to (2.17) when $\alpha = dL$; then we have

$$\xi\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial q^i},\ 1 \le i \le r,$$

and, consequently,

$$\xi_0\left(\frac{\partial L_0}{\partial v^i}\right) = \frac{\partial L_0}{\partial q^i},\ 1 \le i \le r. \tag{2.18}$$

Thus, if $q(t)$ is a curve in S_0 which is a path of ξ_0, then (2.18) becomes the Euler-Lagrange equations corresponding to L_0:

$$\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{q}^i}\right) - \frac{\partial L_0}{\partial q^i} = 0,\ 1 \le i \le r$$

(see [8]).

3 – Actions preserving Δ^1_ξ and $\tilde{\Delta}^1_\xi$

In this section, we find a kind of tensor fields which preserves Δ^1_ξ and $\tilde{\Delta}^1_\xi$.

PROPOSITION 3.1. *Let R be a tensor field of type (1,1) on TM which commutes with J and satisfies $J \circ \mathcal{L}_\xi R = 0$. Then*

$$R^* \circ E_\xi = E_\xi \circ R^*. \tag{3.1}$$

If, moreover R commutes with P, then

$$R^* \circ \tilde{E}_\xi = \tilde{E}_\xi \circ R^*. \tag{3.2}$$

PROOF. Let $\alpha \in \Delta^1(TM)$. Then

$$R^*(E_\xi \alpha) = R^*(\alpha - \mathcal{L}_\xi(J^*\alpha)) = \alpha \circ R - \mathcal{L}_\xi(J^*\alpha) \circ R.$$

On the other hand, we have

$$E_\xi(R^*\alpha) = R^*\alpha - \mathcal{L}_\xi(J^*(R^*\alpha)) = \alpha \circ R - \mathcal{L}_\xi((J^* \circ R^*)\alpha).$$

But

$$(\mathcal{L}_\xi(J^*\alpha) \circ R)(X) = \mathcal{L}_\xi(J^*\alpha)(RX) = \xi\alpha(JRX) - \alpha J[\xi, RX] =$$
$$= \xi\alpha(RJX) - \alpha(RJ[\xi, X])$$

(because $JR = RJ$ and $J((\mathcal{L}_\xi R)X) = J[\xi, RX] - JR[\xi, X] = 0$)

$$= (\mathcal{L}_\xi((RJ)^*\alpha))(X).$$

Therefore (3.1) follows. Finally, (3.2) is a direct consequence of (3.1). □

From Proposition 3.1, we easily obtain the following.

COROLLARY 3.2. . *Let R be a tensor field of type (1,1) on TM which commutes with J and P and satisfies $J(\mathcal{L}_\xi R) = 0$. Then R^* preserves Λ^1_ξ and $\tilde{\Lambda}^1_\xi$.*

Next we shall find tensor fields of type (1,1) on TM satisfying these conditions. We only consider the most interesting case in which P is the complete lift of an integrable almost product structure P_0 on M.

PROPOSITION 3.3. . *Let A_0 be a tensor field of type (1,1) on M which commutes with P_0. Then the complete lift A_0^c of A_0 to TM commutes with J and P_0^c and satisfies $J(\mathcal{L}_\xi A_0^c) = 0$. Hence $(A_0^c)^*$ preserves Δ^1_ξ and $\tilde{\Delta}^1_\xi$.*

PROOF. In fact,

$$A_0^c J = J A_0^c = A_0^v,$$

and

$$A_0^c P_0^c = (A_0 P_0)^c = (P_0 A_0)^c = P_0^c A_0^c.$$

Moreover, we have

$$(J(\mathcal{L}_\xi A_0^c))X^c = J[\xi, (A_0 X)^c] - J A_0^c[\xi, X^c].$$

But $[\xi, Z^c]$ is vertical for any $Z \in \chi(M)$, (see [4]). Then we deduce that $J[\xi, (A_0 X)^c] = 0$. Furthermore, $J A_0^c = A_0^v$ and $A_0^v[\xi, X^c] = 0$. □

4 – Conservation of energy, point symmetries and Noether's theorem

Let ξ a P-semispray on TM. We have the following:

PROPOSITION 4.1. *Let α be a P-regular element of $\tilde{\Delta}_\xi^1$. Then*

$$i_\xi P^*(\alpha - d < PV, \alpha >) = 0. \tag{4.1}$$

PROOF. In fact, we have

$$P^*(\mathcal{L}_\xi(J^*\alpha)) = P^*(i_\xi d(J^*\alpha) + d i_\xi(J^*\alpha)) = P^*\alpha.$$

Hence,

$$i_\xi d(J^*\alpha) = P^*(\alpha - d < \xi, J^*\alpha >) = P^*(\alpha - d < PV, \alpha >),$$

since

$$P^* i_\xi d(J^*\alpha) = i_\xi d(J^*\alpha) \text{ and } J\xi = PV.$$

Therefore, we deduce

$$i_\xi P^*(\alpha - d < PV, \alpha >) = 0.$$ □

If ξ is a P-regular Lagrangian vector field and $\alpha = dL$, then $dL - d(PV, dL) = -dE_L$. Thus (4.1) becomes $i_\xi dE_L = 0$, and, hence, $\mathcal{L}_\xi E_L = 0$, that is, E_L is constant along the integral curves of ξ.

If P is the complete lift of an integrable almost product structure P_0 on M and S_0 is an integral manifold of ImP_0, then (4.1) becomes $\mathcal{L}_\xi E_{L_0} = 0$. Thus (4.1) represents a generalization of the usual law of conservation of energy.

PROPOSITION 4.2. *Suppose that P is the complete lift to TM of an integrable almost product structure P_0 on M and let $f \in C^\infty(M)$. Then if $\alpha \in \tilde{\Delta}^1_\xi$ we have $\alpha' = \alpha + d(\xi f) \in \tilde{\Delta}^1_\xi$.*

PROOF. We can choose local coordinates (q^i) on M such that $I_m P_0 = \left(\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^r}\right)$. Then ξ is given by (2.14) and we have $(P_0^c)^*(J^* d(\xi f)) = (P_0^c)^*(df)$.

Hence, we deduce

$$(P_0^c)^* \mathcal{L}_\xi (J^* \alpha') = (P_0^c)^* \mathcal{L}_\xi (J^* \alpha) + (P_0^c)^* \mathcal{L}_\xi J^* d(\xi f) = (P_0^c)^* \alpha +$$

$$+ (P_0^c)^* \mathcal{L}_\xi (df) = (P_0^c)^* (\alpha + d(\xi f)) = (P_0^c)^* \alpha'. \qquad \square$$

Now, let $\phi : S_0 \longrightarrow M$ be an integral manifold of $I_m P_0$. Proposition 4.2 shows that if $f \in C^\infty(M)$, then $L_0 = L \circ T\phi$ and $L_0 + \xi_0(f \circ \phi)$ are equivalent Lagrangians on TS_0. Thus Proposition 4.2 may be considered as a generalization of the usual gauge freedom.

The next result concerns with point symmetries of ξ.

DEFINITION 4.3. *A vector field Y on M is called a* **point symmetry** *of ξ if the complete lift Y^c of Y to TM commutes with ξ, that is, $[Y^c, \xi] = 0$.*

PROPOSITION 4.4. *Let us suppose that P is the complete lift P_0^c to TM of an almost product structure P_0 on M and let Y be a point symmetry of ξ which is an infinitesimal automorphism of P_0, that is, $\mathcal{L}_Y P_0 = 0$. Then $\mathcal{L}_{Y^c}$ preserves $\tilde{\Delta}^1_\xi$.*

PROOF. Let $\alpha \in \tilde{\Delta}^1_\xi$. Then

$$((P_0^c)^*(\mathcal{L}_{Y^c}\alpha))Z^c = ((\mathcal{L}_{Y^c}\alpha)(P_0 Z)^c = Y^c \alpha(P_0 Z)^c - \alpha([Y, P_0 Z]^c))$$

$$= Y^c \alpha(P_0 Z)^c - \alpha(P_0[Y, Z])^c = (\mathcal{L}_{Y^c}((P_0^c)^* \alpha))Z^c,$$

since $\mathcal{L}_Y P_0 = 0$.

Hence, we have

$$\begin{aligned}(P_0^c)^*(\mathcal{L}_{Y^c}\alpha) &= \mathcal{L}_{Y^c}((P_0^c)^*\alpha) = \mathcal{L}_{Y^c}((P_0^c)^*\mathcal{L}_\xi(J^*\alpha))\\ &= (P_0^c)^*(\mathcal{L}_{Y^c}(\mathcal{L}_\xi(J^*\alpha)))\\ &= (P_0^c)*(\mathcal{L}_\xi(\mathcal{L}_{Y^c}(J^*\alpha))) \quad (\text{since } [Y^c,\xi]=0)\\ &= (P_0^c)^*(\mathcal{L}_\xi((\mathcal{L}_{Y^c}J)^*\alpha)) + (P_0^c)^*(\mathcal{L}_\xi(J^*(\mathcal{L}_{Y^c}(\alpha)))\\ &\qquad (\text{since } (\mathcal{L}_{Y^c}J)^* = \mathcal{L}_{Y^c}\circ J^* - J^*\circ\mathcal{L}_{Y^c})\\ &= (P_0^c)^*\mathcal{L}_\xi(J^*(\mathcal{L}_{Y^c}\alpha)) \quad (\text{since } \mathcal{L}_{Y^c}J = 0).\end{aligned}$$

Therefore,

$$\mathcal{L}_{Y^c}\alpha \in \tilde{\Delta}_\xi^1.$$

□

COROLLARY 4.5. *Let Y be a vector field on M such that P_0Y is an infinitesimal automorphism of P_0 and a point symmetry of ξ, simultaneously. Then $\mathcal{L}_{(P_0Y)^c}(\tilde{\Delta}^1\xi) \subset \tilde{\Delta}^1\xi$.*

Now, let us suppose that ξ is a P_0^c-regular Lagrangian vector field with $\alpha = dL$ and P_0 is integrable. Let S_0 be an integral manifold of ImP_0 and $L_0, E_{L_0}, J_0, V_0, \xi_0$ as above. Then Corollary 4.5 tell us that the restriction of P_0Y to S_0 lifts to TS_0 and produces a Lagrangian equivalent to L_0.

Finally, we now prove a generalization of Noether's theorem.

PROPOSITION 4.6. *Let α be a P-regular element of $\tilde{\Delta}_\xi^1$. If $Y \in \chi(TM)$ satisfies*

(4.2) (1) $P^*\mathcal{L}_Y(J^*\alpha) = P^*(df)$, for $f \in C^\infty(TM)$

(4.3) (2) $i_Y(P^*(\alpha - d < PV, \alpha >)) = 0$,

then $F = f - < Y, J^*\alpha >$ is a first integral of ξ. Conversely, to each first integral F of ξ there corresponds a vector field satisfying (4.2) and (4.3).

PROOF. From (4.2) we have

$$i_Y d(J^*\alpha) = P^*(df - d < Y, J^*\alpha >) = P^*(df),$$

since $P^*i_Y d(J^*\alpha) = i_Y d(J^*\alpha)$.

Now, we have

$$\begin{aligned}\mathcal{L}_\xi F &= i_\xi(dF) = i_\xi P^*(dF)\\ &= i_\xi i_Y d(J^*\alpha) = -i_Y i_\xi d(J^*\alpha)\\ &= -i_Y(P^*(\alpha - d < PV, \alpha >)) = 0,\end{aligned}$$

since $i_\xi d(J^*\alpha) = P^*(\alpha - d < PV, \alpha >)$ (see Proposition 4.1).

Conversely, let F be a first integral of ξ. Then, there exists a unique vector field $Y \in \operatorname{Im} P$ such that

$$i_Y d(J^*\alpha) = P^*(dF).$$

Hence, we have

$$\begin{aligned}P^*\mathcal{L}_Y(J^*\alpha) &= P^*(i_Y d(J^*\alpha)) + P^*(d i_Y(J^*\alpha))\\ &= i_Y d(J^*\alpha) + P^*(d < Y, J^*\alpha >)\\ &= P^*(d(F - < Y, J^*\alpha >)).\end{aligned}$$

Therefore, if we put $f = F - < Y, J^*\alpha >$, we obtain (4.2). Furthermore, since $\mathcal{L}_\xi F = 0$, we have

$$\begin{aligned}0 &= i_\xi(dF) = i_\xi P^*(dF) = i_\xi i_Y d(J^*\alpha)\\ &= -i_Y i_\xi d(J^*\alpha) = -i_Y(P^*(\alpha - d < PV, \alpha >)),\end{aligned}$$

from which (4.3) follows. □

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