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On a Conjecture of Multiplicative Partitions

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RIASSUNTO – *Si indichi con $f(n)$ il numero dei fattori essenzialmente diversi del numero intero $n > 1$. In questo lavoro si dimostra che se $P_1(n) > 3$, allora $f(n) \leq n/\log n$, dove $P_1(n)$ indica il più piccolo fattore primo di n.*

ABSTRACT – *Let $f(n)$ denote the number of essentially different factorizations of the integer $n > 1$. In this paper, we prove that if $P_1(n) > 3$, then $f(n) \leq n/\log n$, where $P_1(n)$ denotes the smallest prime factor of n.*

KEY WORDS – *Multiplicative partitions - Prime factorization.*

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1 – Introduction

In this paper, n, m_i denote integers > 1; $P(n)$ denotes the largest prime factor of n, $P_1(n)$ the smallest prime factor of n and $w(n)$ the number of different prime factors of n. Let logarithm be with the base e. Consider the set $T(n) = \{(m_1, m_2, \ldots, m_s) \,|\, n = m_1 m_2 \ldots m_s\}$, where we identify those partitions which differ only by the order of the factors. We define $f(n) = |T(n)|$ and $f(1) = 1$. For example, $f(12) = 4$, since $12 = 6 \cdot 2 = 4 \cdot 3 = 3 \cdot 2 \cdot 2$ are the four multiplicative partitions of 12.

In 1983 HUGHES and SHALLIT [2] have proved that $f(n) \leq 2n^{\sqrt{2}}$ and made two conjectures:

Conjecture 1. $f(n) \leq n$.

Conjecture 2. $f(n) \leq \frac{n}{\log n}$ for $n \neq 144$.

In [2], they point out that the second is more doubtful. In 1986 and a year later, MATTICS and DODD [3] and CHEN XIAO-XIA [1] have proved the conjecture 1 independently. Six years have elapsed, but the conjecture 2 has not been proved.

In this paper, we shall prove the following

THEOREM. *If $P_1(n) > 3$, then $f(n) \leq \frac{n}{\log n}$.*

Note that $f(144) = f(2^4 3^2) > 144/\log 144$.

2 – Some Lemmas

To prove the theorem, we need the following lemmas.

LEMMA 1. *If $P_1(n) > 3$ and $w(n) \geq 2$, then* $\sum_{d|\frac{n}{p(n)}} 1 \leq \frac{16}{25} n^{\frac{1}{3}}$.

PROOF. Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, $p_1 < p_2 < \ldots < p_r$ be the prime factorization of n. If $r = 2$, we have

$$\frac{\sum_{d|\frac{n}{p(n)}} 1}{n^{\frac{1}{3}}} = \frac{\alpha_2}{p_2^{\frac{1}{3}\alpha_2}} \frac{\alpha_1 + 1}{p_1^{\frac{1}{3}\alpha_1}} \leq \frac{\alpha_2}{7^{\frac{1}{3}\alpha_2}} \frac{\alpha_1 + 1}{5^{\frac{1}{3}\alpha_1}}. \tag{1}$$

If $r \geq 3$, since $(\alpha + 1)/11^{\frac{1}{3}\alpha} < (\alpha + 1)/2^{\alpha} \leq 1$ and $2^{\alpha} \geq 2\alpha$, where α is any natural number, we have

$$\frac{\sum_{d|\frac{n}{p(n)}} 1}{n^{\frac{1}{3}}} = \frac{\alpha_r}{p_r^{\frac{1}{3}\alpha_r}} \frac{\alpha_1 + 1}{p_1^{\frac{1}{3}\alpha_1}} \prod_{i=2}^{r-1} \frac{\alpha_i + 1}{p_i^{\frac{1}{3}\alpha_i}} \leq \frac{\alpha_r}{11^{\frac{1}{3}\alpha_r}} \frac{\alpha_1 + 1}{5^{\frac{1}{3}\alpha_1}} \frac{\alpha_2 + 1}{7^{\frac{1}{3}\alpha_2}} \leq \tag{2}$$

$$\leq \frac{\frac{\alpha_2+1}{2}}{7^{\frac{1}{3}\alpha_2}} \frac{\alpha_1 + 1}{5^{\frac{1}{3}\alpha_1}} \leq \frac{\alpha_2}{7^{\frac{1}{3}\alpha_2}} \frac{\alpha_1 + 1}{5^{\frac{1}{3}\alpha_1}}.$$

By (1), (2), we get

$$\frac{\sum_{d|\frac{n}{p(n)}} 1}{n^{\frac{1}{3}}} \le \frac{\alpha_2}{7^{\frac{1}{3}\alpha_2}} \frac{\alpha_1+1}{5^{\frac{1}{3}\alpha_1}}.$$

Consider $g(x) = x/7^{\frac{1}{3}x}$ for $x > 0$. We have

$$g'(x) = \frac{7^{\frac{1}{3}x}\left(1 - \frac{\log 7}{3}x\right)}{7^{\frac{2}{3}x}}.$$

If $g'(x) = 0$, we get $x = 3/\log 7$, $1.5 < 3/\log 7 < 2$. Obviously, $g'(x) > 0$ for $0 < x < 3/\log 7$; $g'(x) < 0$ for $x > 3/\log 7$. Hence we get $g(\alpha) \le \max\{g(1), g(2)\}$ for α which is any natural number. Since $\frac{1}{7^{\frac{1}{3}}} < \frac{1}{7^{\frac{1}{3}}}\frac{2}{7^{\frac{1}{3}}} = \frac{2}{7^{\frac{2}{3}}}$, we get $\frac{\alpha_2}{7^{\frac{1}{3}\alpha_2}} \le \frac{2}{7^{\frac{1}{3}\cdot 2}}$. By similar method, we easily get $\frac{\alpha_1+1}{5^{\frac{1}{3}\alpha_1}} \le \frac{2}{5^{\frac{1}{3}\cdot 1}}$. So, we get

$$\frac{\sum_{d|\frac{n}{p(n)}} 1}{n^{\frac{1}{3}}} \le \frac{2}{5^{\frac{1}{3}}}\frac{2}{7^{\frac{2}{3}}} = \frac{4}{\sqrt[3]{245}} < \frac{4}{6.25} = \frac{16}{25}.$$

□

Lemma 2. *If $P_1(n) > 3$, then* $\sum_{\alpha|\frac{n}{p(n)}} \alpha \le \frac{1}{4}n$.

Proof. Let $n = \prod_{i=1}^{t} p_i^{\beta_i}$, $p_1 < p_2 < \ldots < p_t$ be the prime factorization of n, we have

$$\sum_{\alpha|\frac{n}{p(n)}} \alpha \le \frac{n}{p_t} \sum_{d|\frac{n}{p(n)}} \frac{1}{\alpha} \le \frac{n}{p_t} \prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right)^{-1}.$$

If $t = 1$, we have

$$\sum_{\alpha|\frac{n}{p(n)}} \alpha \le \frac{n}{p_1}\left(1 - \frac{1}{p_1}\right)^{-1} = \frac{1}{p_1 - 1}n \le \frac{1}{4}n.$$

If $t \geq 2$, we have

$$\sum_{\alpha | \frac{n}{p(n)}} \alpha \leq \frac{n}{p_t} \prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right)^{-1} = \frac{n}{p_t} \prod_{i=1}^{t} \frac{p_i}{p_i - 1} = \frac{n}{p_t} \frac{p_t}{p_1 - 1} \prod_{i=1}^{t-1} \frac{p_i}{p_{i+1} - 1} \leq$$

$$\leq \frac{n}{p_1 - 1} \prod_{i=1}^{t-1} \frac{p_i}{p_i} = \frac{1}{p_1 - 1} n \leq \frac{1}{4} n .$$

□

LEMMA 3. *If $n > 1$, then $f(n) \leq \sum_{d | \frac{n}{p(n)}} f(d)$.*

PROOF. Let $n = \prod_{j=1}^{q} p_j^{c_j}$, $p_1 < p_2 < \ldots < p_q$, be the prime factorization of n. Consider the sets: $T_{j_1 j_2 \ldots j_q}(n) = \{(p_q p_1^{c_1 - j_1} p_2^{c_2 - j_2} \ldots p_q^{(c_q - 1) - j_q}, m_2, \ldots, m_r) \mid n = p_q p_1^{c_1 - j_1} p_2^{c_2 - j_2} \ldots p_q^{(c_q - 1) - j_q} m_2 \ldots m_r\}$, $0 \leq j_i \leq c_i$, $1 \leq i \leq q - 1$; $0 \leq j_q \leq c_q - 1$, where we also identify those partitions which differ only by the order of factors.

We easily see that

$$|T_{j_1 j_2 \ldots j_q}(n)| = f(p_1^{j_1} p_2^{j_2} \ldots p_q^{j_q}) \quad \text{and}$$

$$T(n) = \bigcup_{j_1=0}^{c_1} \bigcup_{j_2=0}^{c_2} \cdots \bigcup_{j_q=0}^{c_q - 1} T_{j_1 j_2 \ldots j_q}(n) .$$

So we have

$$f(n) = |T(n)| \leq \sum_{j_1=0}^{c_1} \sum_{j_2=0}^{c_2} \cdots \sum_{j_q=0}^{c_q - 1} |T_{j_1 j_2 \ldots j_q}(n)| =$$

$$= \sum_{j_1=0}^{c_1} \sum_{j_2=0}^{c_2} \cdots \sum_{j_q=0}^{c_q - 1} f(p_1^{j_1} p_2^{j_2} \ldots p_q^{j_q}) = \sum_{d | \frac{n}{p(n)}} f(d) .$$

□

Lemma 4. *$f(n) \leq \frac{n}{15}$ for n with $P_1(n) > 3$ except $n = 5, 7, 11, 13, 25$.*

Proof. We first find out all n with $P_1(n) > 3$ not exceeding 60 and satisfying $n < 15f(n)$. Let n, with $P_1(n) > 3$ and $n < 60$, be of the form $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, $p_1 < p_2 < \ldots < p_s$. Since $5 \cdot 7 \cdot 11 > 60$, we have $s \leq 2$. If $s = 1$, since $5^3 > 60$, then n can only be expressed in one of the forms p_1, p_1^2. If $s = 2$, since $5^2 \cdot 7 > 60$, then n can only be expressed in the form $p_1 p_2$. If $n = p_1$, then $f(n) = 1$. All prime numbers n with $3 < n < 15$ are 5,7,11,13. If $n = p_1^2$ or $p_1 p_2$, then $f(n) = 2$. The only value of n with $P_1(n) > 3$ and $n = p_1^2$ or $p_1 p_2$ not exceeding 30 is 25. So, all n with $P_1(n) > 3$ not exceeding 60 and satisfying $n < 15f(n)$ are 5,7,11,13,25.

We shall prove that $f(n) \leq \frac{n}{15}$ for n with $P_1(n) > 3$ and $n > 60$.

We arrange all n with $P_1(n) > 3$ and $n > 60$ in increasing order to obtain the sequence $\{n_i\}$, $i = 1, 2, \ldots$. When $i = 1$, we have $f(n_1) = f(61) = 1$.

Suppose that $f(n_j) \leq \frac{n_j}{15}$ for j with $1 \leq j \leq i-1$, where $i - 1 \geq 1$. We shall prove that $f(n_i) \leq \frac{n_i}{15}$.

By Lemma 3 and Lemma 2, we have

$$f(n_i) \leq \sum_{d \mid \frac{n_i}{p(n_i)}} f(d) \leq f(1) - \frac{1}{15} + f(5) - \frac{5}{15} + f(7) - \frac{7}{15} + f(11) - \frac{11}{15} +$$

$$+ f(13) - \frac{13}{15} + f(25) - \frac{25}{11} + \frac{1}{15} \sum_{d \mid \frac{n_i}{p(n_i)}} d \leq$$

$$\leq 7 - \frac{62}{15} + \frac{1}{15}\frac{1}{4} n_i = \frac{43}{15} + \frac{1}{60} n_i < \frac{1}{15} n_i,$$

since $\frac{43}{15} + \frac{1}{60} n < \frac{n}{15}$ for $n > 60$. □

Corollary. *$f(n) \leq \frac{n}{\log n}$ for n with $P_1(n) > 3$ and $n \leq e^{15}$.*

PROOF. It is easy to verify that $f(n) \le \frac{n}{\log n}$ for $n = 5, 7, 11, 13, 25$ (or see [2]). If $n \ne 5, 7, 11, 13, 25$, $P_1(n) > 3$ and $n \le e^{15}$, by Lemma 4, we have

$$f(n) \le \frac{n}{15} \le \frac{n}{\log n} \qquad \text{for} \quad n \le e^{15}.$$

So, our assertion has been proved. □

3 – Proof of the theorem

We arrange all n with $P_1(n) > 3$ in increasing order to obtain the sequence $\{n_k\}$, $k = 1, 2, \ldots$. Let n_{k_0} be the largest number which belongs to the sequence and does not exceed e^{15}. By the Corollary, we know that $f(n_i) \le \frac{n_i}{\log n_i}$ for $1 \le i \le k_0$. Suppose that $f(n_i) \le \frac{n_i}{\log n_i}$ for $1 \le i \le k-1$, where $k - 1 \ge k_0$. We shall prove that $f(n_k) \le \frac{n_k}{\log n_k}$.

Since $f(n) \le \frac{n}{\log n}$ holds for n with $w(n) = 1$ (see [1]), we may suppose that $w(n) \ge 2$.

By Lemma 3, we have

$$f(n_k) \le \sum_{d \mid \frac{n_k}{p(n_k)}} f(d) = \sum_{\substack{d \mid \frac{n_k}{p(n_k)} \\ d \le n_k^{\frac{1}{3}}}} f(d) + \sum_{\substack{d \mid \frac{n_k}{p(n_k)} \\ d > n_k^{\frac{1}{3}}}} f(d) = S_1 + S_2 . \tag{3}$$

By $f(d) \le d$ and Lemma 1, we get

$$S_1 \le n_k^{\frac{1}{3}} \sum_{d \mid \frac{n_k}{p(n_k)}} 1 \le \frac{16}{25} n_k^{\frac{1}{3} \cdot 2} . \tag{4}$$

Since $n_k > e^{15}$, we have

$$\frac{n_k^{\frac{1}{3}}}{\log n_k} > \frac{e^5}{15} > \frac{64}{25} \quad \text{or} \quad \frac{16}{25} n_k^{\frac{2}{3}} < \frac{1}{4} \frac{n_k}{\log n_k} . \tag{5}$$

By the induction hypothesis and Lemma 2, we get

$$(6) \qquad S_2 \le \sum_{\substack{d \mid \frac{n_k}{p(n_k)} \\ d > n_k^{\frac{1}{3}}}} \frac{d}{\log d} < \frac{3}{\log n_k} \sum_{d \mid \frac{n_k}{p(n_k)}} d \le \frac{3}{4} \frac{n_k}{\log n_k}.$$

By (3)–(6), we get

$$f(n_k) \le \frac{n_k}{\log n_k}.$$

Our theorem is now proved by induction. □

Editor's Note

Conjecture 2 has been proved in general by F.W. DODD and L.E. MATTICS, *Extimating the number of multiplicative partitions*, Rocky Mountain Journal of Mathematics 17 (1987), 797-813. However, the direct proof for the special case $P_1(n) > 3$, presented here by the Author, is more accessible.

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