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Existence and Regularity Results for a Nonlinear Elliptic Equation

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RIASSUNTO – *In questo lavoro studiamo l'esistenza e la regolarità delle soluzioni del problema:*

$$u \in W_0^{1,p}(\Omega) : -(a_i(x,u,\nabla u))_{x_i} - (b_i(x)|u|^{p-2}u)_{x_i} + h(x,u) = (f_i)_{x_i}$$

dove $\Omega \subset \mathbb{R}^n$ *è un aperto limitato,* $A(u) = -(a_i(x,u,\nabla u))_{x_i}$ *è un operatore del tipo Leray-Lions su* $W_0^{1,p}(\Omega)$*, i coefficienti* $b_i(x)$ *appartengono ad un opportuno spazio di Lorentz e* $h(x,u)$ *è un termine non lineare che soddisfa la condizione di segno* $h(x,u)u \geq 0$ *e nessuna condizione di crescenza rispetto alla* u.

ABSTRACT – *In this paper we are concerned with the existence and regularity of the solutions of the problem:*

$$u \in W_0^{1,p}(\Omega) : -(a_i(x,u,\nabla u))_{x_i} - (b_i(x)|u|^{p-2}u)_{x_i} + h(x,u) = (f_i)_{x_i}$$

where $\Omega \subset \mathbb{R}^n$ *is a bounded open set,* $A(u) = -(a_i(x,u,\nabla u))_{x_i}$ *is a Leray - Lions operator on* $W_0^{1,p}(\Omega)$*,* $b_i(x)$ *are in a suitable Lorentz space and* $h(x,u)$ *is a nonlinear term satisfying the sign condition* $h(x,u)u \geq 0$ *but no growth condition with respect to* u.

KEY WORDS – *Nonlinear elliptic equations - Existence result - Regularity result - Rearrangements.*

A.M.S. CLASSIFICATION: 35J25 - 35J65 - 35D05 - 35D10

– Introduction

Let us consider the problem:

$$(*)\quad \begin{cases} Nu = -(a_i(x,u,\nabla u))_{x_i} - (b_i(x)|u|^{p-2}u)_{x_i} + h(x,u) = (f_i)_{x_i} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $p > 1$, Ω is a bounded open set of $\mathbb{R}^n$ $(n \geq 3)$, $A(u) = -(a_i(x,u,\nabla u))_{x_i}$ is a Leray-Lions operator on $W_0^{1,p}(\Omega)$, $h(x,u)$ is a strongly nonlinear term in that no growth restriction is imposed while it is assumed the sign condition $h(x,u)u \geq 0$ and the coefficients $b_i(x)$ are in a suitable Lorentz space.

In the linear case ($p = 2$, $h(x,u) = 0$) existence and regularity theorems are well known when the coefficents are in $L^k(\Omega)$ with $k \geq n$ (see e.g. [22]). Besides, when the coefficients b_i are in $L(n,\infty)$, existence and uniqueness results are proved in [4], while regularity theorems are in [2]. There exist similar results when the coefficients belong to the "intermediate" Lorentz spaces $L(n,p)$ (see [11]).

Existence theorems for strongly nonlinear elliptic problems are in [9], [10], [15], [23], where the theory of pseudomonotone operators is used. Similar problems are considered in [6], [8],[20], [21], where apriori estimates of the solutions allow to prove existence theorems.

In [7] and [13] regularity results are proved for the solutions of (*) when $b_i = 0$ and $f_i \in L^q(\Omega)$ with $q < n/(p-1)$ or $q > n/(p-1)$.

In this paper we firstly give an existence theorem for (*) when the coefficents b_i belong to $L(n/(p-1), r/(p-1))$ with $n \leq r \leq \infty$. For the proof we use a technique that can be found in [10]. Furthermore with the same hypotheses on the coefficients b_i, we prove a regularity theorem. Precisely, supposed $|f| \in L(q,p')$ and set $s = [q(p-1)]^*$, we show that the solution u of (*) belongs to the Lorentz space $L(s,p)$, provided that q is less than a critical value depending on the norm of b_i when $r = \infty$, while if $r < \infty$ there isn't any critical value. When $r = \infty$, the analogous result for the linear case is in [2].

1 – Notations and preliminar results

Let u be a real valued measurable function defined in a bounded open set Ω of $\mathbb{R}^n$ and $\mu(t)$ denote its distribution function, that is:

$$\mu(t) = \text{meas}\{x \in \Omega : |u(x)| > t\}, \qquad t \geq 0.$$

The decreasing rearrangement of u is defined by:

$$u^*(s) = \inf\{t \geq 0 : \mu(t) < s\}, \qquad s \in [0, \operatorname{meas}\Omega]$$

and, denoting by $\Omega^\#$ the ball centered in the origin with the same measure as Ω, the spherically symmetric rearrangement of u is defined by:

$$u^\#(x) = u^*(C_n|x|^n), \qquad x \in \Omega^\#$$

where C_n is the measure of the $n-$dimensional unit ball. We recall some properties of the rearrangements:

$$\int_\Omega |u(x)|^p dx = \int_0^\infty |u^*(s)|^p ds = \int_{\Omega^\#} |u^\#(x)|^p dx, \qquad p \geq 1$$

$$\int_{\Omega^\#} |\nabla u^\#|^p dx \leq \int_\Omega |\nabla u|^p dx,$$

$$\int_E |u| dx \leq \int_0^{\operatorname{meas} E} u^*(s) ds, \qquad E \subset \Omega.$$

Moreover we point out the Hardy's inequality:

$$\int_\Omega |uv| dx \leq \int_0^\infty u^*(s) v^*(s) ds = \int_{\Omega^\#} u^\# v^\# dx. \tag{1.1}$$

For more details on rearrangements see [5], [12], [14], [16].

If $1 < p < \infty$ and $1 \leq q \leq \infty$ the Lorentz space $L(p,q)$ is the class of functions u such that:

$$\|u\|_{p,q} = \left(\int_{\Omega^\#} [u^\#(x)|x|^{n/p}]^q \frac{dx}{|x|^n} \right)^{1/q} < \infty,$$

$$\|u\|_{p,\infty} = \sup u^\#(x)|x|^{n/p} < \infty.$$

For the theory of these spaces see for example [17], [18], [19]. We just recall that $L(p,p)$ is the classical L^p space and:

$$p > r \Rightarrow L(p,\infty) \subset L(r,1);$$

$$(1.2) \qquad 1 < q < r < \infty \Rightarrow L(p,1) \subset L(p,q) \subset L(p,r) \subset L(p,\infty) \quad \text{and}$$

$$\|u\|_{p,r} \leq \left(\frac{q}{p}\right)^{1/q-1/r} \|u\|_{p,q}\,.$$

Moreover the following decomposition lemma holds ([22]; see also [11]):

LEMMA 1.1. *If $1 < p < \infty$, $1 \leq q < \infty$ and $u \in L(p,q)$, then for any $\varepsilon > 0$ there are $k(\varepsilon) \geq 0$, $u' \in L^\infty(\Omega)$, $u'' \in L(p,q)$ such that:*

$$u = u' + u'', \qquad \|u'\|_\infty \leq k(\varepsilon), \qquad \|u''\|_{p,q} < \varepsilon\,.$$

Finally we remind two inequalities.
If $t \geq p$ it is easy to prove that (see [1], [3]):

$$(1.3) \qquad \int_{\Omega^\#} |u^\#|^p |x|^{n(p/t^*-1)} dx \leq \left(\frac{t}{n-t}\right)^p \int_{\Omega^\#} |\nabla u^\#|^p |x|^{n(p/t-1)} dx,$$

where $t^* = nt/(n-t)$.

The following inequality is a generalization of the well known interpolation inequality in L^p spaces:

$$(1.4) \qquad \|u\|_{p,q} \leq \|u\|_{t,q}^\alpha \|u\|_{r,q}^{1-\alpha}$$

where $t \leq p \leq r$ and $\dfrac{1}{p} = \dfrac{\alpha}{t} + \dfrac{1-\alpha}{r}$.

2 – Existence theorem

Let us consider the following Dirichlet problem:

$$(2.1)\quad \begin{cases} Nu = -(a_i(x,u,\nabla u))_{x_i} - (b_i(x)|u|^{p-2}u)_{x_i} + h(x,u) = (f_i)_{x_i} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $p > 1$ and:

(i) Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 3$;

(ii) $a_i(x,\eta,\xi)$ is a Carathéodory function for $\xi \in \mathbb{R}^n, \eta \in \mathbb{R}, x \in \Omega$ and

$$a_i(x,\eta,\xi)\xi_i \geq |\xi|^p, \qquad \xi \in \mathbb{R}, \quad \eta \in \mathbb{R}, \quad \text{a.e.} \quad x \in \Omega;$$

(iii) $$|a_i(x,\eta,\xi)| \leq c_1[|\xi|^{p-1} + |\eta|^{p-1} + k(x)]$$

where $c_1 > 0$ and $k(x) \in L^{p'}(\Omega)$ with $\dfrac{1}{p} + \dfrac{1}{p'} = 1$;

(iv) $$[a_i(x,\eta,\xi) - a_i(x,\eta,\xi^*)](\xi_i - \xi_i^*) > 0, \qquad \xi \neq \xi^*;$$

(v) $h(x,\eta)$ is a Carathéodory function for $x \in \Omega$, $\eta \in \mathbb{R}$ and

$$h(x,\eta)\eta \geq 0 \qquad \eta \in \mathbb{R}, \quad \text{a.e.} \quad x \in \Omega\,;$$

(vi) $$\sup_{|\eta|\leq t} |h(x,\eta)| \leq h_t(x) \in L^1(\Omega), \qquad t > 0;$$

(vii) $|f(x)| = (\sum_i |f_i(x)|^2)^{1/2} \in L^{p'}(\Omega)$;

(viii) $B(x) = (\sum_i |b_i(x)|^2)^{1/2} \in L\left(\dfrac{n}{p-1}, \dfrac{r}{p-1}\right)$, with $n \leq r \leq \infty$.

In this section we study the existence of weak solutions of problem (2.1), i.e. the existence of (at least) one function $u \in W_0^{1,p}(\Omega)$ such that:

$$(2.2)\quad \begin{cases} h(\cdot,u(\cdot)) \in L^1(\Omega), \qquad h(\cdot,u(\cdot))u(\cdot) \in L^1(\Omega) \\ \displaystyle\int_\Omega [a_i(x,u,\nabla u)\varphi_{x_i} + b_i(x)|u|^{p-2}u\varphi_{x_i} + h(x,u)\varphi]dx = -\int_\Omega f_i\varphi_{x_i}dx \\ \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \quad \text{and for} \quad \varphi = u. \end{cases}$$

We prove the following result:

THEOREM 2.1. *Let us assume, in addition to the hypotheses* (i)-(vii), *that one of the following conditions holds:*

(a) $B(x) \in L\left(\dfrac{n}{p-1}, \infty\right)$ *that is* $B^\#(x) \leq \dfrac{B^{p-1}}{|x|^{p-1}}$;

moreover:

$$B < \frac{n-p}{p};$$

(b) $B(x) \in L\left(\frac{n}{p-1}, \frac{r}{p-1}\right)$, *with* $n \leq r < \infty$;
moreover:

$$\|B\|_{n/(p-1), r/(p-1)} < \left(\frac{n-p}{p}\right)^{p-1} \left(\frac{n}{n-p}\right)^{(p-1)/r}.$$

Then the Dirichlet problem (2.2) *has a solution* $u \in W_0^{1,p}(\Omega)$.

PROOF. Let us consider the operator $A : W_0^{1,p} \longrightarrow W^{-1,p'}$ defined by:

$$Au = -(A_i(x, u, \nabla u))_{x_i}$$

where $A_i(x, u, \nabla u) = a_i(x, u, \nabla u) + b_i(x)|u|^{p-2}u$, $i = 1, .., n$.

We just have to prove that A is a pseudomonotone operator, that maps bounded sets into bounded sets and that is coercive (see [9], [23]).

We start proving that A is coercive, i.e.:

$$\frac{< Av, v >}{\|v\|} \longrightarrow \infty \qquad \text{for } \|v\| \to \infty$$

where $\|v\|$ denote the norm of v in $W_0^{1,p}(\Omega)$.
By (ii), Holder's and Hardy's inequalities, we have:

$$(2.3) \quad \frac{< Av, v >}{\|v\|} = \frac{1}{\|v\|}\left(\int_\Omega a_i(x, v, \nabla v) v_{x_i} dx + \int_\Omega b_i(x)|v|^{p-2} v v_{x_i} dx\right)$$

$$\geq \|v\|^{p-1} - \frac{1}{\|v\|}\int_\Omega B(x)|v|^{p-1}|\nabla v| dx$$

$$\geq \|v\|^{p-1} - \left(\int_\Omega B(x)^{p'}|v|^p dx\right)^{1/p'}$$

$$\geq \|v\|^{p-1} - \left(\int_{\Omega^\#} B^\#(x)^{p'}|v^\#|^p dx\right)^{1/p'}.$$

If $B \in L\left(\frac{n}{p-1},\infty\right)$, by (1.3) with $t = p$, we obtain:

$$(2.4)\qquad \left(\int_{\Omega^\#} B^\#(x)^{p'}|v^\#|^p dx\right)^{1/p'} \le B^{p-1}\left(\int_{\Omega^\#} \frac{|v^\#|^p}{|x|^p}dx\right)^{1/p'} \le \left(\frac{Bp}{n-p}\right)^{p-1}\|v\|^{p-1}.$$

Hence, from (2.3), and (2.4), we get:

$$(2.5)\qquad \frac{< Av, v >}{\|v\|} \ge \left[1 - \left(\frac{Bp}{n-p}\right)^{p-1}\right]\|v\|^{p-1}.$$

On the other hand, from hypothesy (a), we deduce that $pB/(n-p) < 1$, then A is coercive.

If $B \in L\left(\frac{n}{p-1},\frac{r}{p-1}\right)$, with $n \le r < \infty$, by Holder's inequality, (1.2) and (1.3), we obtain:

$$(2.6)\qquad \left(\int_{\Omega^\#} B^\#(x)^{p'}|v^\#|^p dx\right)^{1/p'} \le \|B\|_{n/(p-1),r/(p-1)}\|v\|^{p-1}_{p^*,rp/(r-p)}$$

$$\le \left(\frac{n-p}{n}\right)^{(p-1)/r}\|B\|_{n/(p-1),r/(p-1)}\|v\|^{p-1}_{p^*,p}$$

$$\le \left(\frac{n-p}{n}\right)^{(p-1)/r}\left(\frac{p}{n-p}\right)^{p-1}\|B\|_{n/(p-1),r/(p-1)}\|v\|^{p-1}.$$

Hence, from (2.3) and (2.6), we have:

$$(2.7)\qquad \frac{< Av, v >}{\|v\|} \ge$$

$$\ge \left[1 - \left(\frac{n-p}{n}\right)^{(p-1)/r}\left(\frac{p}{n-p}\right)^{p-1}\|B\|_{n/(p-1),r/(p-1)}\right]\|v\|^{p-1}.$$

Thus, using (b), A is coercive.

Now we prove that A maps bounded sets into bounded sets. We consider a function $u \in W_0^{1,p}(\Omega)$ such that $\|u\| \leq C$ with $C > 0$ arbitrary constant.

By (iii) and Hardy's inequality, we get:

$$\|a_i(x,u,\nabla u) + b_i(x)|u|^{p-2}u\|_{p'} \leq c[\|u\|^{p-1} + \|k\|_{p'}] + \\ + \Big(\int_{\Omega^\#} B^\#(x)^{p'}|u^\#|^p dx\Big)^{1/p'}$$

where $c > 0$ is a suitable constant. Therefore, from (2.4) and (2.6), we obtain the conclusion.

Finally we prove that A is a pseudomonotone operator, that is for any sequence $(u_j)_j \subset W_0^{1,p}(\Omega)$ such that $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{j\to\infty} < Au_j, u_j - u > \leq 0$, it follows that $< Au_j, u_j - u > \to 0$ and Au_j converges to Au in the weak*-topology of $W^{-1,p'}(\Omega)$.

Although the proof of pseudomonotonicity is similar to that of the result in [10], for completeness we give a sketch of it here.
By (iii), we get:

$$(2.8) \qquad |A_i(x,\eta,\xi)| \leq c_1[|\xi|^{p-1} + |\eta|^{p-1} + k(x)] + B(x)|\eta|^{p-1}.$$

Moreover, by (ii), we have:

$$(2.8)' \qquad A_i(x,\eta,\xi)\xi_i \geq |\xi|^p - B(x)|\eta|^{p-1}|\xi|.$$

Since $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, there exists a subsequence, still denoted by $(u_j)_j$, such that $u_j \to u$ in L^p and $u_j \to u$ a.e. in Ω. Moreover there exists a function $\rho \in L^p(\Omega)$ such that:

$$(2.9) \qquad |u_j(x)| \leq \rho(x), \qquad \forall j \in \mathbb{N},\ a.e.\ x \in \Omega.$$

If we set:

$$(2.10) \qquad p_j(x) = \\ = [A_i(x,u_j(x),\nabla u_j(x)) - A_i(x,u(x),\nabla u(x))][(u_j(x))_{x_i} - (u(x))_{x_i}],$$

then we have:

$$(2.11) \qquad \limsup_{j\to\infty} \int_\Omega p_j dx =$$
$$= \limsup_{j\to\infty} < Au_j, u_j - u > - \lim_{j\to\infty} < Au, u_j - u > \leq 0 .$$

Moreover we get:

$$(2.10)' \quad p_j = A_i(x, u_j, \nabla u_j)(u_j)_{x_i} - \{A_i(x, u, \nabla u)[(u_j)_{x_i} - u_{x_i}] +$$
$$+ A_i(x, u_j, \nabla u_j) u_{x_i}\} = A_i(x, u_j, \nabla u_j)(u_j)_{x_i} - g_j ,$$

with $(g_j)_j$ an equi-integrable sequence of functions in $L^1(\Omega)$. From (2.10), using (2.8), (2.8)′ and Young's inequality, we get:

$$(2.12) \qquad p_j \geq c_2 |\nabla u_j|^p - c_3 \{ |\nabla u|^p + |u|^p + B(x)^{p'} |u_j|^p +$$
$$+ |u_j|^p + B(x)^{p'} |u|^p + k_1(x) \} ,$$

where c_2 and c_3 are suitable constants and $k_1(x)$ is a function in $L^1(\Omega)$. If x is a point of Ω such that $p_j(x) < 0$, by (2.12) and (2.9), we have:

$$(2.13) \qquad |\nabla u_j(x)|^p \leq \alpha(x)$$

where $\alpha(x)$ is indipendent of j and finite *a.e.* in Ω.

Set $p_j = p_j^+ - p_j^-$, in which p_j^+ and p_j^- are the positive and negative parts of p_j respectively. On the other hands, we have:

$$p_j(x) =$$
$$= [A_i(x, u_j(x), \nabla u_j(x)) - A_i(x, u_j(x), \nabla u(x))][(u_j(x))_{x_i} - (u(x))_{x_i}] +$$
$$+ [A_i(x, u_j(x), \nabla u(x)) - A_i(x, u(x), \nabla u(x))][(u_j(x))_{x_i} - (u(x))_{x_i}] .$$

Hence, by (iv), for any $x \in \Omega$ such that $p_j^-(x) > 0$, we get:

$$- p_j^-(x) \geq$$
$$\geq [A_i(x, u_j(x), \nabla u(x)) - A_i(x, u(x), \nabla u(x))][(u_j(x))_{x_i} - (u(x))_{x_i}]$$
$$= r_j(x) .$$

Moreover by (2.13) and by the *a.e.* convergence of $(u_j)_j$ to u, $r_j \to 0$ *a.e.* in Ω. Hence $p_j^- \to 0$ *a.e.* in Ω.
By (2.10)′, we obtain:

$$\int_E p_j dx = \int_E A_i(x, u_j, \nabla u_j)(u_j)_{x_j} dx - \int_E g_j dx .$$

Since it is possible to prove, using hypothesy (a) or (b) as in (2.5) and (2.7), that:

$$\int_E A_i(x, u_j, \nabla u_j)(u_j)_{x_j} dx > 0 ,$$

it follows:

$$\int_E p_j^- dx \leq \int_E |g_j| dx .$$

From equi-integrability of $(g_j)_j$, we deduce the equi-integrability of $(p_j^-)_j$ and by Vitali's theorem we obtain that $p_j^- \to 0$ in $L^1(\Omega)$.
Besides, from (2.11), we get:

$$\limsup_{j\to\infty} \int_\Omega p_j^+ dx = \limsup_{j\to\infty} \int_\Omega p_j dx + \lim_{j\to\infty} \int_\Omega p_j^- dx \leq 0 ,$$

hence $p_j^+ \to 0$ in $L^1(\Omega)$.

Now we are able to prove that $< Au_j, u_j - u > \to 0$. Indeed we have:

$$| < Au_j, u_j - u > | \leq | < Au_j - Au, u_j - u > | + | < Au, u_j - u > |$$

$$\leq \int_\Omega |p_j| dx + | < Au, u_j - u > |$$

and the right side goes to 0, as $j \to \infty$.

To conclude the proof of pseudomonotonicity, we have to prove that Au_j converges to Au in the weak*– topology of $W^{-1,p'}(\Omega)$.
To this aim we observe that there exists a subsequence of $(p_j)_j$, still denoted by $(p_j)_j$, such that $p_j \to 0$ *a.e.* in Ω. Moreover from the previous inequalities, we have that $\sup_j |\nabla u_j(x)| < \infty$ *a.e.* $x \in \Omega$.

Now if we consider a subsequence of $\{|\nabla u_j(x)|\}$ converging to $\xi^{\#}$, then for the corresponding subsequence of indices, we have:

$$p_j(x) \to [A_i(x,u(x),\xi^{\#}) - A_i(x,u(x),\nabla u(x))][\xi^{\#} - u(x)_{x_i}] = 0 .$$

Hence $\xi^{\#} = \nabla u(x)$ and $\nabla u_j(x) \to \nabla u(x)$ *a.e.* in Ω. Finally by Vitali's theorem, we get:

$$< Au_j, v > = \int_\Omega a_i(x,u_j,\nabla u_j) v_{x_i} dx + \int_\Omega b_i(x)|u_j|^{p-2} u_j v_{x_i} dx \longrightarrow$$

$$\longrightarrow \int_\Omega a_i(x,u,\nabla u) v_{x_i} + \int_\Omega b_i(x)|u|^{p-2} u v_{x_i} = < Au, v > .$$

3 – Regularity theorem

The aim of this section is to obtain a regularity theorem for the solution u of (2.2). Precisely we want to study the behaviour of u when the index of sommability of $|f|$ increases. In the linear case ($p = 2$) the problem has been studied in [2].

THEOREM 3.1. *With the the assumption of theorem* 2.1, *let* $u \in W_0^{1,p}(\Omega)$ *be a solution of* (2.2) *and* $s = [q(p-1)]^*$. *The following results hold:*

(a)′ *if* $B(x) \in L\left(\frac{n}{p-1}, \infty\right)$ *and* $|f| \in L(q,p')$ *with*

$$p' \le q < \frac{n}{(B+1)(p-1)} ,$$

then u *belongs to* $L(s,p)$ *and*

$$\|u\|_{s,p} \le C \, \|f\|_{q,p'}^{p'/p}$$

$$\textit{where } C = \frac{q(p-1)}{\{[n-q(p-1)]^{p/p'} - B[q(p-1)]^{p/p'}\}^{p'/p}}$$

(b)′ *if* $B(x) \in L\left(\frac{n}{p-1}, \frac{r}{p-1}\right)$, *with* $n \le r < \infty$ *and* $|f| \in L(q, p')$ *with*

$$p' \le q < \frac{n}{p-1},$$

then u *belongs to* $L(s,p)$ *and*

$$\|u\|_{s,p}^{p-1} \le N\|u\|_p^{p-1} + K\|f\|_{q,p'}$$

with N *and* K *constants depending on* n, p *and* q.

Proof of (a)′. Let us consider, for $s \in [0, \text{meas}\Omega]$, the collection $E(s)$ of subsets of Ω such that:

$$\text{meas}\, E(s) = s,$$

$$s_1 < s_2 \Rightarrow E(s_1) \subset E(s_2),$$

$$s = \mu(t) \Rightarrow E(s) = \{x \in \Omega : |u(x)| > t\}.$$

Now we set:

$$s(x) = \inf\{\bar{s} \in [0, \text{meas}\Omega] : x \in E(\bar{s})\},$$

$$G(x) = \left[\frac{s(x)}{C_n}\right]^{p'/q-1},$$

$$g(t) = g(-t) = \left[\frac{\mu(t)}{C_n}\right]^{p'/q-1}, \qquad t \ge 0.$$

The following properties hold (for the proof see [2], [1]):

$$G^{\#}(x) = |x|^{n(p'/q-1)} \tag{3.1}$$

$$g(u(x)) = G(x) \quad \text{if} \quad \text{meas}\{y : |u(y)| = |u(x)|\} = 0. \tag{3.2}$$

Now we construct a suitable test function.

Set for $0 < k < \operatorname{ess\,sup}|u| \leq \infty$,

$$g_k(t) = \begin{cases} g(t) & \text{if } |t| < k \\ 0 & \text{if } |t| \geq k \end{cases}$$

we introduce the function

$$L(s) = \int_0^s g_k(t)dt$$

which is Lipschitz continuous. We observe that the function $L(u)$ considered in [2] can't be used in (2.2) as test function. Hence we have to use an approximation procedure. To this aim we introduce the following truncated function $\bar{u}_n$ of u:

$$\bar{u}_n(x) = \begin{cases} n & \text{if } u > n \\ u(x) & \text{if } |u| \leq n \\ -n & \text{if } u < -n\,. \end{cases}$$

The lemma 1.1 of [22] implies:

$$\varphi_n = L(\bar{u}_n) \in W_0^{1,p} \cap L^\infty \text{ and } (\varphi_n)_{x_i} = L'(\bar{u}_n)(\bar{u}_n)_{x_i}\,.$$

Then choosing the test function

$$\varphi_n(x) = \int_0^{\bar{u}_n} g_k(t)dt\,,$$

we have:

$$(3.3)\qquad \int_\Omega [a_i(x,u,\nabla u) + b_i(x)|u|^{p-2}u + f_i(x)]\,(\bar{u}_n)_{x_i}\, g_k(\bar{u}_n)\,dx + \\ + \int_\Omega h(x,u)\Big(\int_0^{\bar{u}_n} g_k(t)\,dt\Big)\,dx = 0\,.$$

Now we observe that, as $n \to \infty$:

$$h(x,u)\int_0^{\bar u_n} g_k(t)\,dt \longrightarrow h(x,u)\int_0^{u} g_k(t)dt \qquad \text{a.e. in } \Omega$$

and

$$\left|h(x,u)\int_0^{\bar u_n} g_k(t)\,dt\right| \leq C|h(x,u)||u| \in L^1(\Omega)\,.$$

Hence by Lebesgue's dominated convergence theorem, we get:

$$\int_\Omega h(x,u)\Big(\int_0^{\bar u_n} g_k(t)\,dt\Big)\,dx \longrightarrow \int_\Omega h(x,u)\Big(\int_0^{u} g_k(t)\,dt\Big)\,dx \qquad \text{in } L^1(\Omega)\,.$$

On the other hand, from the recalled lemma of [22], as $n \to \infty$:

$$(\bar u_n)_{x_i}\, g_k(\bar u_n) \to u_{x_i}\, g_k(u) \qquad \text{a.e. in } \Omega,$$

and besides, by Holder's inequality:

$$\left|\int_E [a_i(x,u,\nabla u) + b_i(x)|u|^{p-2}u + f_i(x)]\,(\bar u_n)_{x_i}\, g_k(\bar u_n)\,dx\right| \leq$$

$$\leq C\left(\int_\Omega |a_i(x,u,\nabla u) + b_i(x)|u|^{p-2}u + f_i(x)|^{p'}dx\right)^{1/p'}\left(\int_E (\bar u_n)^p_{x_i}\,dx\right)^{1/p}$$

for every measurable subset E of Ω. Hence, using Vitali's theorem, we obtain:

$$\int_\Omega [a_i(x,u,\nabla u) + b_i(x)|u|^{p-2}u + f_i(x)]\,(\bar u_n)_{x_i}\, g_k(\bar u_n)\,dx \longrightarrow$$

$$\longrightarrow \int_\Omega [a_i(x,u,\nabla u) + b_i(x)|u|^{p-2}u + f_i(x)]\,u_{x_i}\, g_k(u)\,dx \qquad \text{in } L^1(\Omega)\,.$$

Then from (3.3), as $n \to \infty$, we obtain:

$$(3.4) \qquad \int_\Omega [a_i(x,u,\nabla u) + b_i(x)|u|^{p-2}u + f_i(x)]\, u_{x_i}\, g_k(u)\, dx +$$

$$+ \int_\Omega h(x,u)\Big(\int_0^u g_k(t)\, dt\Big)\, dx = 0\,.$$

From (3.4), using (ii), the definition of g_k and (3.2), we get:

$$(3.5) \qquad \int_{|u|<k} |\nabla u|^p G(x)\, dx \le I_1 + I_2 + I_3,$$

where:

$$(3.5_a) \qquad I_1 = -\int_{|u|<k} b_i(x)|u|^{p-2}u u_{x_i} G(x)\, dx,$$

$$(3.5_b) \qquad I_2 = -\int_{|u|<k} f_i(x) u_{x_i} G(x)\, dx,$$

$$(3.5_c) \qquad I_3 = -\int_\Omega h(x,u)\left(\int_0^u g_k(t)\, dt\right) dx.$$

To estimate the terms in the right hand side of (3.5) we have to remind the following inequality (for the proof see [2], [1]):

$$\int_\Omega G(x)|\nabla u|^p\, dx \ge \int_{\Omega^\#} G^\#(x)|\nabla u^\#|^p\, dx \quad .$$

which gives, together with (1.3) and (3.1), the following "Sobolev type inequality":

$$(3.6) \qquad \int_{\Omega^\#} |u^\#|^p |x|^{n(p/s-1)}\, dx \le \left[\frac{q(p-1)}{n-q(p-1)}\right]^p \int_\Omega G(x)|\nabla u|^p\, dx.$$

Now we estimate I_1. Set $u_k(x) = \min\{|u(x)|, k\}$, the Holder's and Hardy's inequalities give:

$$|I_1| \leq \int_{|u|<k} B(x)|u|^{p-1}|\nabla u| G(x)\,dx$$

$$\leq \left(\int_{|u|<k} |\nabla u|^p G(x)\,dx\right)^{1/p} \left(\int_{\Omega} B(x)^{p'}|u_k|^p G(x)\,dx\right)^{1/p'}$$

$$\leq \left(\int_{|u|<k} |\nabla u|^p G(x)\,dx\right)^{1/p} \left(\int_{\Omega^\#} B^\#(x)^{p'}|u_k^\#|^p G^\#(x)\,dx\right)^{1/p'}.$$

On the other hand, taking into account that $B(x) \in L\left(\frac{n}{p-1}, \infty\right)$ and using (3.1) and (3.6), we get:

$$\int_{\Omega^\#} B^\#(x)^{p'}|u_k^\#|^p G^\#(x)\,dx \leq B^p \int_{\Omega^\#} |u_k^\#|^p |x|^{n(p'/q-1)-p}\,dx$$

$$\leq B^p \left[\frac{q(p-1)}{n-q(p-1)}\right]^p \int_{|u|<k} G(x)|\nabla u|^p\,dx.$$

Hence we have:

$$|I_1| \leq \left[\frac{Bq(p-1)}{n-q(p-1)}\right]^{p/p'} \int_{|u|<k} G(x)|\nabla u|^p\,dx. \tag{3.7}$$

Now, using the Holder's and Hardy's inequalities, we evaluate I_2:

$$|I_2| \le \int_{|u|<k} |f||\nabla u| G(x)\,dx$$

$$\le \left(\int_{\Omega} |f|^{p'} G(x)\,dx \right)^{1/p'} \left(\int_{|u|<k} |\nabla u|^p G(x)\,dx \right)^{1/p}$$

$$\le \left(\int_{\Omega^\#} |f^\#|^{p'} G^\#(x)\,dx \right)^{1/p'} \left(\int_{|u|<k} |\nabla u|^p G(x)\,dx \right)^{1/p}.$$

Then by (3.1):

$$|I_2| \le \|f\|_{q,p'} \left(\int_{|u|<k} |\nabla u|^p G(x)\,dx \right)^{1/p}. \tag{3.8}$$

Finally, taking into account (v) we deduce:

$$I_3 \le 0. \tag{3.9}$$

From (3.5), using (3.7), (3.8) and (3.9), we have:

$$\left(\int_{|u|<k} |\nabla u|^p G(x)\,dx \right)^{1/p'} \le$$

$$\le \left[\frac{Bq(p-1)}{n-q(p-1)} \right]^{p/p'} \left(\int_{|u|<k} |\nabla u|^p G(x)\,dx \right)^{1/p'} + \|f\|_{q,p'}.$$

Now if we suppose $q < \dfrac{n}{(p-1)(B+1)}$ then $\left[\dfrac{Bq(p-1)}{n-q(p-1)} \right]^{p/p'} < 1$, and so, as $k \to \infty$, we obtain:

$$(3.10) \qquad \left(\int_\Omega |\nabla u|^p G(x)\,dx\right)^{1/p'} \leq$$

$$\leq \frac{[n-q(p-1)]^{p/p'}}{[n-q(p-1)]^{p/p'} - [Bq(p-1)]^{p/p'}} \|f\|_{q,p'}.$$

By (3.10), using (3.6), part a)′ of the theorem is proved.

PROOF OF (b)′. As in the previous case, introduced the functions $s(x)$, $G(x)$ and $g(t)$, we get (3.5) with I_1, I_2 and I_3, defined by (3.5_a), (3.5_b) and (3.5_c) respectively.

Now we consider I_1. From the lemma 1.1, for any $\varepsilon > 0$ fixed, there exist two functions $B' \in L^\infty(\Omega)$ and $B'' \in L\left(\frac{n}{p-1}, \frac{r}{p-1}\right)$ and a constant $k(\varepsilon) > 0$ such that:

$$B = B' + B'', \qquad \|B'\| \leq k(\varepsilon), \qquad \|B''\|_{n/(p-1),r/(p-1)} \leq \varepsilon\,.$$

Hence (3.5_a) gives:

$$(3.11) \qquad |I_1| \leq \int_{|u|<k} B(x)|u|^{p-1}|\nabla u|G(x)dx$$

$$\leq \int_{|u|<k} |B'(x)||u|^{p-1}|\nabla u|G(x)dx + \int_{|u|<k} |B''(x)||u|^{p-1}|\nabla u|G(x)dx$$

$$= I_1' + I_1''$$

where:

$$I_1' = \int_{|u|<k} |B'(x)||u|^{p-1}|\nabla u|G(x)dx$$

$$I_1'' = \int_{|u|<k} |B''(x)||u|^{p-1}|\nabla u|G(x)dx\,.$$

By Holder's and Hardy's inequalities and (3.1), we get:

$$(3.12)\quad I_1' \le k(\varepsilon)\left(\int_{|u|<k} |\nabla u|^p G(x)dx\right)^{1/p}\left(\int_{\Omega} |u_k|^p G(x)dx\right)^{1/p'}$$

$$\le k(\varepsilon)\left(\int_{|u|<k} |\nabla u|^p G(x)dx\right)^{1/p}\left(\int_{\Omega^\#} |u_k^\#|^p G^\#(x)dx\right)^{1/p'}$$

$$= k(\varepsilon)\left(\int_{|u|<k} |\nabla u|^p G(x)dx\right)^{1/p} \|u_k\|^{p-1}_{q(p-1),p}\,.$$

On the other hand, (1.4) and Young's inequality imply:

$$(3.13)\qquad k(\varepsilon)\|u_k\|^{p-1}_{q(p-1),p} \le k(\varepsilon)\|u_k\|_p^{\alpha(p-1)}\|u_k\|_{s,p}^{(1-\alpha)(p-1)}$$

$$\le c(\varepsilon)\|u_k\|_p^{(p-1)} + \varepsilon\|u_k\|_{s,p}^{(p-1)}$$

where $c(\varepsilon)$ is a suitable constant depending on ε. Hence by (3.12) and (3.13) , we obtain:

$$I_1' \le c(\varepsilon)\|u_k\|_p^{(p-1)}\left(\int_{|u|<k} |\nabla u|^p G(x)dx\right)^{1/p} +$$

$$+\varepsilon\|u_k\|_{s,p}^{(p-1)}\left(\int_{|u|<k} |\nabla u|^p G(x)dx\right)^{1/p},$$

and, finally, by (3.6):

$$(3.14)\qquad I_1' \le c(\varepsilon)\|u_k\|_p^{(p-1)}\left(\int_{|u|<k} |\nabla u|^p G(x)dx\right)^{1/p} +$$

$$+\varepsilon\left[\frac{q(p-1)}{n-q(p-1)}\right]^{p-1}\int_{|u|<k} |\nabla u|^p G(x)dx\,.$$

Now we consider I_1''. By Holder's inequality:

$$(3.15)\qquad I_1'' \leq \left(\int_{|u|<k} |\nabla u|^p G(x)dx\right)^{1/p} \left(\int_{\Omega} |B''|^{p'} |u_k|^p G(x)dx\right)^{1/p'}$$

On the other hand, using Hardy's and Holder's inequalities, (1.2) and (3.6), we get:

$$\int_{\Omega} |B''|^{p'} |u_k|^p G(x)dx \leq \int_{\Omega^\#} |(B'')^\#|^{p'} |u_k^\#|^p G^\#(x)dx$$

$$\leq \|B''\|^{p'}_{n/(p-1),r/(p-1)} \|u_k\|^p_{s,rp/(r-p)}$$

$$\leq \varepsilon^{p'} \left(\frac{p}{s}\right)^{p/r} \|u_k\|^p_{s,p}$$

$$\leq \varepsilon^{p'} \left(\frac{p}{s}\right)^{p/r} \left[\frac{q(p-1)}{n-q(p-1)}\right]^p \int_{|u|<k} |\nabla u|^p G(x)dx.$$

Hence from (3.15) we get:

$$(3.16)\qquad I_1'' \leq \varepsilon \left(\frac{p}{s}\right)^{(p-1)/r} \left[\frac{q(p-1)}{n-q(p-1)}\right]^{p-1} \int_{|u|<k} |\nabla u|^p G(x)\, dx.$$

Taking into account (3.14) and (3.16), (3.11) gives:

$$(3.17)\qquad |I_1| \leq c(\varepsilon) \|u_k\|_p^{p-1} \left(\int_{|u|<k} |\nabla u|^p G(x)\, dx\right)^{1/p} +$$

$$+\varepsilon \left[\frac{q(p-1)}{n-q(p-1)}\right]^{p-1} \left[1 + \left(\frac{p}{s}\right)^{(p-1)/r}\right] \int_{|u|<k} |\nabla u|^p G(x)\, dx.$$

From (3.5), by (3.17), (3.8) and (3.9), we have:

$$\left(\int\limits_{|u|<k} |\nabla u|^p G(x)\,dx\right)^{1/p'} \leq \varepsilon \left[\frac{q(p-1)}{n-q(p-1)}\right]^{p-1} \left[1+\left(\frac{p}{s}\right)^{(p-1)/r}\right] \times$$

$$\times \left(\int\limits_{|u|<k} |\nabla u|^p G(x)\,dx\right)^{1/p'} + c(\varepsilon)\,\|u_k\|_p^{p-1} + \|f\|_{q,p'}\,.$$

Let $\varepsilon > 0$ such that:

$$\varepsilon \left[\frac{q(p-1)}{n-q(p-1)}\right]^{p-1} \left[1+\left(\frac{p}{s}\right)^{(p-1)/r}\right] < 1,$$

we get:

$$\left(\int\limits_{|u|<k} |\nabla u|^p G(x)\,dx\right)^{1/p'} \leq N'\|u_k\|_p^{p-1} + K'\|f\|_{q,p'}$$

with N' and K' suitable constants. As $k \to \infty$,we obtain:

$$\left(\int\limits_{\Omega} |\nabla u|^p G(x)\,dx\right)^{1/p'} \leq N'\|u\|_p^{p-1} + K'\|f\|_{q,p'}$$

which gives, together with (3.6), the theorem.

REFERENCES

[1] A. Alvino: *Sulla diseguaglianza di Sobolev in spazi di Lorentz*, Boll. Un. Mat. Ital., Vol. 5, 14-A (1977), 148-156.

[2] A. Alvino - P. Buonocore - G. Trombetti: *On Dirichlet problem for second order elliptic equations*, Nonlinear Analisys T.M.A., Vol. 14, No. 7 (1990), 559-570.

[3] A. ALVINO - P.L. LIONS - G. TROMBETTI: *On optimization problems with prescribed rearrangements*, Nonlinear Analysis T.M.A., Vol. 13, No. 2 (1989), 185-220.

[4] A. ALVINO - G. TROMBETTI: *Esistenza ed unicità per il teorema di Dirichlet*, Atti del Convegno "Methods of Functional Analysis and Theory of Elliptic Equations", Napoli, Liguori (1982), 269-280.

[5] C. BANDLE: *Isoperimetric Inequalities and Applications*, Monographs and Studies in Math., n. 7, Pitman, London, (1980).

[6] A. BENSOUSSAN - L. BOCCARDO - F. MURAT: *On a nonlinear partial differential equation having natural growth terms and unbounded solutions*, Ann. Inst. H.P. Analyse non lineaire, Vol. 5, No. 4 (1988), 347-364.

[7] L. BOCCARDO - D. GIACCHETTI: *Alcune osservazioni sulla regolarità delle soluzioni di problemi fortemente non lineari e applicazioni*, Ricerche di Mat., Vol. XXXIV, No. 2 (1985), 309-323.

[8] L. BOCCARDO - D. GIACCHETTI: *Existence results via regularity for some nonlinear elliptic problems*, Comm. in Part. Diff. Eq., Vol. 14, No. 5 (1989), 663-680.

[9] H. BREZIS - F.E. BROWDER: *Some properties of higher order Sobolev Spaces*, J. Math. Pures et Appl., Vol. 61 (1982), 245-259.

[10] F.E. BROWDER: *Pseudo-monotone operators and nonlinear elliptic boundary value problems on unbounded domains*, Proc. Nat. Acad. Sci. USA, Vol. 74,No. 7 (1977), 2659-2661.

[11] P. BUONOCORE: *Second order elliptic equations whose coefficients have their first derivatives in Lorentz spaces*, Ricerche di Mat., Vol. XXXII (1983), 187-202.

[12] K.M. CHONG - N.M. RICE: *Equimeasurable rearrangements of functions*, Queen's papers in pure and applied mathematics, No. 28, Queen's University, Ontario, (1971).

[13] E. GIARRUSSO - D. NUNZIANTE: *Regularity theorems in limit cases for solutions of linear and nonlinear elliptic equations*, Rend. Inst. Mat. Univ. Trieste, Vol. 20 (1988), 39-58.

[14] B. KAWHOL: *Rearrangements and convexity of level sets in P.D.E.*, Lecture Notes in Math., No. 1150, Springer, Berlin-New York, (1985).

[15] J. LERAY - J.L. LIONS: *Quelques résultats de Visik sur les problémes elliptique non linéaires par la méthode de Minty-Browder*, Bull. Soc. Math. France, Vol. 93 (1965), 97-107.

[16] J. MOSSINO: *Inégalités Isopérimériques et Applications en Physique*, Collection Travaux en Cours, Hermann Paris, (1984).

[17] E.T. OKLANDER: *Interpolacion, espacios de Lorentz y teorema de Marcinkiewicz*, Cursos y Seminarios Mat., fasc. 20, Univ. Buenos Aires, (1965).

[18] R. O'NEIL: *Convolution operators and $L(p,q)$ spaces*, Duke Math. J., Vol. 30 (1963), 129-142.

[19] R. O'Neil: *Integral transforms and tensor products on Orlicz spaces and $L(p,q)$ spaces*, J. Analyse Math., Vol. 21 (1968), 1-276.

[20] J.M. Rakotoson: *Réarrangement relatif dans les équations elliptiques quasilinéaires avec un second membre distribution: Application a un thórème d'existence et de régularité*, J. Diff. Eq., Vol. 66 (1987), 391-419.

[21] J.M. Rakotoson - R. Teman: *Relative Rearrangement in Quasilinear Elliptic Variational Inequalities*, Ind. Univ. Math. Journ., Vol. 36, No. 4 (1987), 757-810.

[22] G. Stampacchia: *Equations elliptiques du second ordre à coefficients discontinus*, Seminaire de Mathématique Supérieure n. 16. Les Presses de l'Université de Montréal. Montréal, (1966).

[23] J.R. Webb: *Boundary Value Problems for Strongly Nonlinear Elliptic Equations*, J. London Math. Soc., Vol. 21 (1980), 123-132.

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