Rendiconti di Matematica, Serie VII
Volume 11 , Roma (1991), 761-775

Some Results for a Generalised Integral Transform

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RIASSUNTO – *Si estende ad una classe di funzioni generalizzate una generalizzazione della trasformata di Laplace coinvolgente, nel nucleo, le funzioni di Weber del cilindro parabolico; si studiano inoltre le proprietà di uno spazio di funzioni di prova ed il suo duale. Si definiscono funzioni generalizzate trasformabili e si dimostra un teorema di analiticità per la trasformata integrale generalizzata. Infine viene mostrato come diverse classi di equazioni differenziali possano essere risolte con l'aiuto della trasformata integrale.*

ABSTRACT – *In this paper a generalisation of Laplace transform involving Weber's parabolic cylinder function in the kernel, is extended to a class of generalized functions and the properties of a testing function space and its dual are studied. Transformable generalized functions are defined and an analyticity theorem is proved for the generalised integral transform. It is shown that several class of differential equations can be solved with the help of the integral transform.*

KEY WORDS – *Testing function space - Generalized function - Weber transform.*

A.M.S. CLASSIFICATION: 46F12

1 – Introduction

The integral transform

$$F(s) = 2^{-\nu/2} \int_0^\infty (st)^\lambda e^{-\frac{1}{2}st} \mathcal{D}_\nu(\sqrt{2st}) f(t)\,dt$$

studied by B.M.L. TIWARI [5], where $\mathcal{D}_\nu$ is the Weber's parabolic cylinder function, has recently been extended by us to generalized functions

[3] and inversion and uniqueness theorems have been given for the transform of generalized functions. In this paper we extend the transform to a class of generalized functions other than that given in the earlier paper and study the properties of a testing function space and its dual. First we define a differential operator and examine the behaviour of the result of its nth operation on the kernal of our transform as this will be needed in our study and then given an analyticity theorem for the generalized transform. In [3] we have not shown the application of the transform. But in this paper we have shown in section 6 that several class of differential equations can be solved with the help of the weber transform.

2 – Differential operator

Let

$$u = x^{\lambda} e^{-\frac{1}{2}x} \mathcal{D}_{\nu}(\sqrt{2x})$$

where

$$\mathcal{D}_n(z) = e^{-\frac{1}{4}z^2} z^n \left\{1 - \frac{n(n-1)}{2z^2} + \frac{n(n-1)(n-2)(n-3)}{2.4z^4} \dots\right\}.$$

Then, $x^{-\lambda} u = e^{-\frac{1}{4}(\sqrt{2x})^2} \mathcal{D}_{\nu(\sqrt{2x})}$

$$\therefore \quad \frac{d}{dx}[x^{-\lambda} u] = \frac{d}{dx}\left[e^{-\frac{1}{4}(\sqrt{2x})^2} \mathcal{D}_{\nu(\sqrt{2x})}\right]$$

$$= (-1) e^{-\frac{1}{4}(\sqrt{2x})^2} \mathcal{D}_{\nu+1}(\sqrt{2x}) 2^{-\frac{1}{2}} x^{-\frac{1}{2}}$$

since

$$\frac{d^m}{dz^m}\left[e^{-\frac{1}{4}z^2} \mathcal{D}_{\nu}(z)\right] = (-1)^m e^{-\frac{1}{4}z^2} \mathcal{D}_{\nu+m}(z) \qquad m = 1, 2, 3 \dots,$$

(ERDÉLYI [2], p.119).

$$\therefore \quad \frac{d}{dx}[x^{-\lambda} u] = (-1) 2^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-\frac{1}{2}x} \mathcal{D}_{\nu+1}(\sqrt{2x})$$

or

$$x^{\frac{1}{2}} \frac{d}{dx}[x^{-\lambda} u] = (-1) 2^{-\frac{1}{2}} e^{-\frac{1}{2}x} \mathcal{D}_{\nu+1}(\sqrt{2x}).$$

$$\therefore \quad \frac{d}{dx}\left[x^{\frac{1}{2}}\frac{d}{dx}\{x^{-\lambda}u\} = \frac{d}{dx}\left[(-1)2^{-\frac{1}{2}}e^{-\frac{1}{2}x}\mathcal{D}_{\nu+1}(\sqrt{2x})\right] = \right.$$

$$\left. = (-1)^2 2^{-1} x^{-\frac{1}{2}x}\mathcal{D}_{\nu+2}(\sqrt{2x})\right].$$

$$\therefore \quad x^{\frac{1}{2}+\lambda}\frac{d}{dx}\left[x^{\frac{1}{2}}\frac{d}{dx}\{x^{-\lambda}u\}\right] = \frac{1}{2}x^{\lambda}e^{-\frac{1}{2}x}\mathcal{D}_{\nu+2}(\sqrt{2x}).$$

Let us now define an operator A_λ by

$$A_{\lambda,x}\phi(x) = x^{\frac{1}{2}+\lambda}\left[D_x x^{\frac{1}{2}} D_x\{x^{-\lambda}\phi(x)\}\right]$$

where $D_x = \frac{d}{dx}$.

Fron the rule for differentiation of products, we have

$$A_{\lambda,x}\phi(x) = \left[\lambda\left(\lambda+\frac{1}{2}\right)x^{-1} - \left(2\lambda-\frac{1}{2}\right)D_x + xD_x^2\right]\phi\ldots \tag{2.1}$$

Thus

$$A_{\lambda,x}\left\{(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu}(\sqrt{2st})\right\} = \frac{s}{2}(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu+2}(\sqrt{2st})$$

and

$$A_{\lambda,t}^2\left\{(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu}(\sqrt{2st})\right\} = \left(\frac{s}{2}\right)^2(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu+4}(\sqrt{2st})$$

$$\therefore \quad A_{\lambda,t}^n\left\{(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu}(\sqrt{2st})\right\} = \left(\frac{s}{2}\right)^n(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu+2n}(\sqrt{2st}) \quad \text{for} \quad n = 0,1,2,\ldots \tag{2.2}$$

For large t,

$$A_{\lambda,t}^n\left\{(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu}(\sqrt{2st})\right\} = \left(\frac{s}{2}\right)^n(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu+2n}(\sqrt{2st})$$

$$\sim \left(\frac{s}{2}\right)^n(st)^{\lambda}e^{-\frac{1}{2}st}e^{-\frac{1}{4}.2st}(\sqrt{2st})^{\nu+2n}$$

as $D_\nu(z) \sim z^\nu e^{-\frac{z^2}{4}}$ for large z and fixed ν (ERDĒLYI [1], p. 122)

$$= s^n (st)^{\lambda+\frac{\nu}{2}+n} e^{-st} 2^{\nu/2} < \infty \quad \text{provided Re} \quad s > 0\,.$$

Also for small t,

$$A^n_{\lambda,t}\left\{(st)^\lambda e^{-\frac{1}{2}st}\mathcal{D}_\nu(\sqrt{2st})\right\} = \left(\frac{s}{2}\right)^n (st)^\lambda e^{-\frac{1}{2}st}\mathcal{D}_{\nu+2n}(\sqrt{2st}) =$$

$$= \left(\frac{s}{2}\right)^n (st)^\lambda 2^{(\nu+2n)/2} e^{-st}\psi\left(-\frac{\nu+2n}{2}, \frac{1}{2}; st\right)$$

since $\mathcal{D}_\nu(x) = 2^{\nu/2}e^{-\frac{1}{4}x^2}\psi\left(-\frac{\nu}{2}, \frac{1}{2}; \frac{1}{2}x^2\right)$ where (ERDĒLYI [1], p. 267) $\psi(a,b;x)$ is a confluent hypergeometric function.

$$\sim \left(\frac{s}{2}\right)^n (st)^\lambda 2^{\frac{\nu}{2}+n} e^{-st}\left\{\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-n\right)} + 0(|st|)^{\frac{1}{2}}\right\}$$

since $\psi(a,c;x) = \dfrac{\Gamma(1-c)}{\Gamma(a-c+1)} + 0|x|^{1-\mathrm{Re}\, c}$ (ERDĒLYI [1], p. 262) $< \infty$ as $t \longrightarrow 0$ for $\mathrm{Re}\, s > 0$.

3 – Testing function space and its dual

Let us define functionals $\partial^\lambda_{\alpha,\beta,n}$; $n = 0,1,2,\ldots$ on certain smooth functions $\phi(t)(0 < t < \infty)$ by

$$\partial^\lambda_{\alpha,\beta,n}(\phi) = \sup_{0<t<\infty}\left|e^{\alpha t}t^{\beta+n}A^n_{\lambda,t}\phi(t)\right|.$$

Let us define $K_{\alpha,\beta}(I)$ to be space of all those complex-valued smooth functions $\phi(t)$ defined on $I(0,\infty)$ for which $\partial^\lambda_{\alpha,\beta,n}(\phi)$ is finite for all $n = 0,1,2,\ldots$ where α,β are suitably fixed real numbers and λ, a complex number with $\mathrm{Re}\,\lambda > 0$.

For any complex number γ, we have

$$\partial^\lambda_{\alpha,\beta,n}(\gamma\phi) = |\gamma|\partial^\lambda_{\alpha,\beta,n}(\phi)\,;$$

$$\partial^\lambda_{\alpha,\beta,n}(\phi+\psi) \le \partial^\lambda_{\alpha,\beta,n}(\phi) + \partial^\lambda_{\alpha,\beta,n}(\psi);\, \phi,\psi \in K_{\alpha,\beta}(I)$$

$$\therefore \quad \partial^{\lambda}_{\alpha,\beta,n} \quad \text{is a seminorm on} \quad K_{\alpha,\beta}(I).$$

Again

$$\partial^{\lambda}_{\alpha,\beta,0}(\phi) = 0$$

$\Longrightarrow$ $\phi(t)$ is the zero element in $K_{\alpha,\beta}(I)$. Hence $\partial^{\lambda}_{\alpha,\beta,0}$ is a norm. So the collection

$$M = \left\{\partial^{\lambda}_{\alpha,\beta,n}\right\}_{n=0}^{\infty}$$

is a countable multinorm on $K_{\alpha,\beta}(I)$ and equipped with the topology generated by M, $K_{\alpha,\beta}(I)$ is a countably multinormed space.

LEMMA 3.1. *For every fixed s such that* $\operatorname{Re} s > \alpha$ *and* $\beta + \operatorname{Re}\lambda > 0$, $\omega(st) \in K_{\alpha,\beta}(I)$ *where* $\omega(st) = 2^{\nu/2}(st)^{\lambda}e^{-1/2st}\mathcal{D}_{\nu}(\sqrt{2st})$.

PROOF. We have from (2.2)

$$A^{n}_{\lambda,t}\omega(st) = 2^{-\nu/2}\left(\frac{s}{2}\right)^{n}(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu+2n}(\sqrt{2st})$$

for $n = 0, 1, 2, \ldots$

Hence

$$\partial^{\lambda}_{\alpha,\beta,n}[\omega(st)] = \sup_{0<t<\infty}\left|e^{\alpha t}t^{\beta+n}A^{n}_{\lambda,t}\omega(st)\right|$$

$$= \sup_{0<t<\infty}\left|e^{\alpha t}t^{\beta+n}2^{-\nu/2}\left(\frac{s}{2}\right)^{n}(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu+2t}(\sqrt{2st})\right|.$$

Now, for large t and fixed s i.e. $st| \longrightarrow \infty$,

$$\left|2^{-\nu/2}e^{\alpha t}\left(\frac{s}{2}\right)^{n}t^{\beta+n}(st)^{\lambda}e^{-\frac{1}{2}st}\mathcal{D}_{\nu+2n}(\sqrt{2st})\right|$$

$$\sim \left|2^{-\nu/2}e^{\alpha t}t^{\beta+n}\left(\frac{s}{2}\right)^{n}(st)^{\lambda}e^{-\frac{1}{2}st}e^{-\frac{1}{4}(2st)}(\sqrt{2st})^{\nu+2n}\right|$$

$$= \left|(st)^{\lambda+\frac{\nu}{2}+2n}t^{\beta}e^{(\alpha-s)t}\right|$$

which tends to zero as $t \to \infty$, since s is fixed with $\operatorname{Re} s > \alpha$. For small t

$$\left|2^{-\nu/2} e^{\alpha t} t^{\beta+n} \left(\frac{s}{n}\right)^n (st)^\lambda e^{-\frac{1}{2}st} D_{\nu+2n}(\sqrt{2st})\right|$$

$$\left|2^{-\nu/2} \left(\frac{s}{2}\right)^n e^{\alpha t} t^{\beta+n} (st)^\lambda e^{-\frac{1}{2}st} 2^{\frac{\nu}{2}+n} e^{-\frac{1}{2}st} \psi\left(-\frac{\nu+2n}{2}, \frac{1}{2}; st\right)\right|$$

$$= \left|t^\beta (st)^{\lambda+n} e^{-(s-\alpha)t} \left\{\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}-n\right)} + 0(|st|)^{1/2}\right\}\right|$$

$$= \text{ a finite number as } \quad t \to 0,\ \operatorname{Re} s > \alpha \quad \text{and} \quad \beta + \operatorname{Re}\lambda > 0\,.$$

Hence $\partial^\lambda_{\alpha,\beta,n}[\omega(st)] < \infty$, $n = 0, 1, 2, \ldots$

This shows that $\omega(st) \in K_{\alpha,\beta}(I)$.

THEOREM 3.1. *$K_{\alpha,\beta}(I)$ is a complete countably multinormed space i.e., a Frechet space.*

PROOF. Let $\{\phi_v\}_{v=1}^{\infty}$ be a Cauchy sequence in $K_{\alpha,\beta}(I)$. Let Ω denote any arbitrary compact subset of $I(0,\infty)$.

Let us define an operator D^{-1} by

$$D^{-1} = \int_\tau^t dx$$

where τ is the fixed point in I. Thus for any smooth function $\psi(t)$ on $(0,\infty)$

$$D^{-1} D\psi(t) = \psi(t) - \psi(\tau)\,.$$

By the definition of $A_{\lambda,t}$; we have

$$A_{\lambda,t}\phi_v(t) = t^{1/2+\lambda} D t^{1/2} D t^{-\lambda} \phi_v(t)\,.$$

In view of seminorm $\partial^\lambda_{\alpha,\beta,n}$ we see that $A_{\lambda,t}\phi_v(t)$ converges uniformly on Ω as $v \to \infty$. Moreover, we have

$$t^{-\frac{1}{2}} D^{-1} t^{-\frac{1}{2}-\lambda} A_{\lambda,t}\phi(t) = t^{-\frac{1}{2}} D^{-1} t^{-\frac{1}{2}-\lambda} t^{\frac{1}{2}+\lambda} D t^{\frac{1}{2}} D t^{-\lambda} \phi_v(t)$$

(3.1)

$$= D_t t^{-\lambda}\phi_v(t) - \left(\frac{\tau}{t}\right)^{\frac{1}{2}} D_\tau \tau^{-\lambda}\phi_v(\tau)$$

and

$$tD^{-1}t^{-\frac{1}{2}}D^{-1}t^{-\frac{1}{2}-\lambda}A_{\lambda,t}\phi_v(t)$$

(3.2)
$$= \phi_v(t) - \left(\frac{t}{\tau}\right)^{\lambda}\phi_v(\tau) - l_\tau(t)D_\tau\tau^{-\lambda}\phi_v(\tau)$$

where $l_\tau(t) = 2t^\lambda\tau(t^{\frac{1}{2}}\tau^{-\frac{1}{2}} - 1)$.

Since multiplication by a power of t or multiplication by D^{-1} preserves the property of convergence of a converging function, the left hand side of (3.1) and (3.2) also converges uniformly on Ω as $v \to \infty$. Thus we see that the left hand side and the first two terms of the right hand side in (3.2) converges uniformly on Ω. Hence as $l_\tau(t) \neq 0$, $D_\tau\tau^{-\lambda}\phi(\tau)$ must converge as $v \to \infty$. This with (3.1) implies that $D_t t^{-\lambda}\phi_v(t)$ also converges uniformly on every Ω which, in turn, implies that $D_t\phi_v(t)$ does the same. Next, by virtue of (2.1), it follows that $D^2\phi(t)$ also converges uniformly on every compact subset of I.

We repeat this argument with ϕ_v replaced $A_\lambda^n\phi_v$ and $A_\lambda\phi_v$ replaced by $A_\lambda^{n+1}\phi_v$. This shows that for every non-negative integer n, $D^n\phi_v(t)$ converges uniformly on every Ω. Consequently there exists a smooth function $\phi(t)$ on I such that for each n and t, $D^n\phi_v(t) \to D^n\phi(t)$ as $v \to \infty$. It now follows easily that

(3.3) $$\partial^\lambda_{\alpha,\beta,n}(\phi_v - \phi) \to 0 \quad \text{as} \quad v \to \infty \qquad n = 0, 1, 2, \ldots$$

Finally, there exists a constant C_n, not depending on v such that

$$\partial^\lambda_{\alpha,\beta,n}(\phi_v) < C_n \qquad (\text{since } \phi \in K_{\alpha,\beta}(I)).$$

therefore, from (3.3),

$$\partial^\lambda_{\alpha,\beta,n}\phi < \partial^\lambda_{\alpha,\beta,n}\phi_v + \partial^\lambda_{\alpha,\beta,n}(\phi - \phi_v) < C_n + \epsilon.$$

This implies that $\phi \in K_{\alpha,\beta}(I)$ and is the limit in $K_{\alpha,\beta}(I)$ of $\left\{\phi_v\right\}_{v=1}^{\infty}$.

Thus $K_{\alpha,\beta}(I)$ is a sequentially complete countably multinormed space or a Frechet space.

Thus (i) members of $K_{\alpha,\beta}(I)$ are complex valued smooth function defined on I, (ii) $K_{\alpha,\beta}(I)$ is a complete countably multinormed space,

(ii) if $\left\{\phi_v\right\}_{v=1}^{\infty}$ converges in $K_{\alpha,\beta}(I)$ to zero, then for every non-negative integer $m\left\{D^m\phi_v\right\}_{v=1}^{\infty}$ converges to zero function uniformly on every compact subset of I, and so $K_{\alpha,\beta}(I)$ is a testing function space satisfying all the necessary conditions for it to be such a space. The collection of all continuous linear functionals on $K_{\alpha,\beta}(I)$ is called the dual of $K_{\alpha,\beta}(I)$ and is denoted by $K'_{\alpha,\beta}(I)$. Members of $K'_{\alpha,\beta}(I)$ are generalized functions. Since $K_{\alpha,\beta}(I)$ is complete, $K'_{\alpha,\beta}(I)$ is also complete.

4 – Properties of $K_{\alpha,\beta}(I)$

As in (Zemanian [6], pp. 32-36) $D(I)$ is the space which contains those complex-valued smooth functions $\phi(t)$ defined on $0 < t < \infty$ which have compact supports and $E(I)$ is the space of all complex-valued smooth functions on I. We now compare $K_{\alpha,\beta}(I)$ and $K'_{\alpha,\beta}(I)$ with $D(I)$, $E(I)$ and their duals and list some properties.

Property 4.1. *Fron the definition of the spaces $D(I)$ and $E(I)$ we see that $D(I) \subset K_{\alpha,\beta}(I) \subset E(I)$. Since $D(I)$ is dense in $E(I)$, it follows that $K_{\alpha,\beta}(I)$ is dense in $E(I)$.*

Let $\left\{\phi_v\right\}_{v=1}^{\infty}$ converges to ϕ in $D(I)$. Let the supports of ϕ_v and ϕ be containdes in the closed interval $[a,b]$, $0 < a < b < \infty$, we have

$$\begin{aligned}
\partial^{\lambda}_{\alpha,\beta,n}(\phi_v - \phi) &= \sup_{0<t<\infty} \left|e^{\alpha t} t^{\beta+n} A^n_{\lambda,t}(\phi_v - \phi)\right| \\
&= \sup_{a<t<b} \left|e^{\alpha t} t^{\beta} \sum_{r=0}^{2n} B_r t^r D^r(\phi_v - \phi)\right| \\
&\quad (B_r\text{s being some constants}) \\
&\leq \sum_{r=0}^{2n} \sup_{a<t<b} \left|e^{\alpha t} t^{\beta} B_r t^r D^r(\phi_v - \phi)\right| \\
&= \sum_{r=0}^{2n} \sup_{a<t<b} \left|e^{\alpha t} t^{\beta+r} B_r D^r(\phi_v - \phi)\right| .
\end{aligned}$$

If we take $C_r = \max_{a<t<b} \left| e^{\alpha t} t^{\beta+r} B_r \right|$, we see that

$$\partial^{\lambda}_{\alpha,\beta,n}(\phi_v - \phi) \leq \sum_{r=0}^{2n} C_r \sup_{a<t<b} \left| D(\phi_v - \phi) \right|$$

$< \epsilon$ for $v > N$, where N is a large positive integer. This is true from the property of the convergence of $\left\{\phi_v\right\}_{v=1}^{\infty}$ in $D(I)$.

We see that convergence in $D(I)$ implies convergence in $K_{\alpha,\beta}(I)$. Consequently, the restriction of $f \in K'_{\alpha,\beta}(I)$ to $D(I)$ is in $D'(I)$.

Property 4.2. *If $\alpha_1 < \alpha_2$ then $K_{\alpha_2,\beta}(I) \subset K_{\alpha_1,\beta}(I)$ and the topology of $K_{\alpha_2,\beta}(I)$ is stronger than the topology induced on it by the topology of $K_{\alpha_1,\beta}(I)$. Hence the restriction of any $f \in K'_{\alpha_1,\beta}(I)$ to $K_{\alpha_2,\beta}(I)$ is in $K'_{\alpha_2,\beta}(I)$.*

Property 4.3. *$K_{\alpha,\beta}(I)$ is a dense subspace of $E(I)$, whatever be the choicees of α and β. Indeed $D(I) \subseteq K_{\alpha,\beta}(I) \subseteq E(I)$, and since $D(I)$ is dense in $E(I)$ so in $K_{\alpha,\beta}(I)$. Moreover, in the proof of Theorem 3.1, we have seen that convergence of any sequence in $K_{\alpha,\beta}(I)$ implies its convergence in $E(I)$. Consequently, by corollary 1.8 2a* (Zemanian [6], p.21), *$E'(I)$ is a subspace of $K'_{\alpha,\beta}(I)$ for any permissible values of α and β.*

Property 4.4. *The differential operator $t^r A^r_{\lambda,t}$ $(r = 1, 2, \dots)$ are continuous linear mappings of $K_{\alpha,\beta}(I)$ into itself. For, we have*

$$\left| e^{\alpha t} t^{\beta+n} A^n_{\lambda,t} [t^r A^r_{\lambda,t}] \phi \right|$$

$$= \left| e^{\alpha t} t^{\beta+n} t^{-n} [a_0 + a_1 tD + \dots + a_{2n} t^{2n} D^{2n}] t^r A^r_{\lambda,t}(\phi) \right|$$

$$= \left| e^{\alpha t} t^{\beta} [\{a_0 + a_1 tD + \dots + a_{2n} t^{2n} D^{2n} big\} t^r] A^r_{\lambda,t}(\phi) + e^{\alpha t} t^{\beta+n+r} A^{r+n}_{\lambda,t}(\phi) \right.$$

$$\leq \left| e^{\alpha t} t^{\beta} [a_0 + r a_1 + r(r-1) a_2 + \dots + r a_r] t^r A^r_{\lambda,t}(\phi) \right| + \left| e^{\alpha t} t^{\beta+n+r} A^{n+r}_{\lambda,t} \phi \right|.$$

Hence for every $\phi \in K_{\alpha,\beta}(I)$, we have

$$\sup_{0<t<\infty} \left| e^{\alpha t} t^{\beta+n} A^{n}_{\lambda,t}[t^{r} A^{r}_{\lambda,t}(\phi)] \right|$$

$$\leq \sup_{0<t<\infty} \left| e^{\alpha t} t^{\beta+r} B_r A^{r}_{\lambda,t}(\phi) \right| + \sup_{0<t<\infty} \left| e^{\alpha t} t^{\beta+n+r} A^{n+r}_{\lambda,t}(\phi) \right|$$

where B_r is a constant, $< \infty$, for all $n = 0,1,2,\ldots$; $r = 0,1,2,\ldots$

The adjoint operator B^{r}_{λ} of $t^{r} A^{r}_{\lambda}$ is a generalized differential operator on $K'_{\alpha,\beta}(I)$ into $K'_{\alpha,\beta}(I)$ and is defined by

$$\langle B^{r}_{\lambda,t} f, \phi \rangle = \langle f, t^{r} A^{r}_{\lambda,t}(\phi) \rangle$$

5 - The generalized Weber transformation

All elements in $K'_{\alpha,\beta}(I)$ for some real numbers α and β are called Weber transformable generalized function. From property 4.2 we see that if f is a member of $K'_{\alpha,\beta}(I)$ for some real α, f is then a member of $K'_{\alpha',\beta}(I)$ for all $\alpha' > \alpha$. This implies that there exists a real number σ_f (possibly $\sigma_f = -\infty$) such that $f \in K'_{\alpha,\beta}(I)$ for every $\alpha > \sigma_f$ and $f \notin K'_{\alpha,\beta}(I)$ for $\alpha < \sigma_f$.

DEFINITION 5.1. *Let $f \in K'_{\alpha,\beta}(I)$ for some fixed real numbers α and β with $\mathrm{Re}\, s > \alpha$ and $(\lambda + \beta) \geq 0$. The Weber transformation $F(s)$ of f denoted by $\mathcal{D}_{\lambda,\nu}(f)$, is defined by*

$$F(s) = (\mathcal{D}_{\lambda,\nu} f)(s) = \langle f(t), \omega(st) \rangle \tag{5.1}$$

where

$$\omega(st) = 2^{-\nu/2}(st)^{\lambda} e^{-\frac{1}{2}st} \mathcal{D}_{\nu}(\sqrt{2st}) \quad \text{and} \quad s \in \Omega_f .$$

The region Ω_f is defined by

$$\Omega_f = \left\{ s \middle| \mathrm{Re}\, s > \sigma_f \,,\, s \neq 0 \,,\, -\frac{3}{4}\pi < \arg . s < \frac{3\pi}{4} \right\} . \tag{5.2}$$

If $\sigma_f < 0$, Ω_f is a cut half plane obtained by deleting all real non-negative values of s.

LEMMA 5.1. *Let α, α' be real numbers with $\alpha < \alpha'$, then for* $\operatorname{Re} z \geq \alpha'$; *$z \neq 0$, $-\frac{3}{4}\pi < \arg z < \frac{3}{4}\pi$ and $0 < t < \infty$, we have*

$$\left| e^{\alpha t}(zt)^{\lambda+\beta+n} e^{-\frac{1}{2}zt} \mathcal{D}_{\nu+2n}(\sqrt{2zt}) \right| \leq C(1 + Z^{\operatorname{Re}\mu})$$

where $C_{\lambda,\nu}$ is a constant with respect to z and t and

$$\mu = \lambda + \beta + \frac{\nu}{2} + 2n.$$

PROOF. Since $z \neq 0$ and $-\dfrac{3\pi}{4} < \arg z < \dfrac{3\pi}{4}$, from the series representation and asymptotic properties of Weber's parabolic cylinder function, we see that for $|z| \leq 1$, there exists a constant $M_{\lambda,\nu}$ independent of z, such that

$$\left| z^{\lambda+\beta+n} e^{-\frac{1}{2}z} \mathcal{D}_{\nu+2n}(\sqrt{2z}) \right| < M_{\lambda,\nu}$$

and another constant $N_{\lambda,\nu}$ independent of z, such that for $|z| > 1$,

$$\begin{aligned} \left| z^{\lambda+\beta+n} e^{-\frac{1}{2}z} \mathcal{D}_{\nu+2n}(\sqrt{2z}) \right| &< N_{\lambda,\nu} z^{\operatorname{Re}(\lambda+\beta+\frac{\nu}{2}+2n)} e^{-\operatorname{Re} z} \\ &= N_{\lambda,\nu} z^{\operatorname{Re}\mu} e^{-\operatorname{Re} z} \end{aligned}$$

where $\mu = \lambda + \beta + \nu/2 + 2n$.

Consequently for $\operatorname{Re} z > \alpha'$ and $0 < t < \infty$, there exist constants $B_{\lambda,\nu}$, independent of z and t, such that

$$\left| e^{\alpha t}(zt)^{\lambda+\beta+n} e^{-\frac{1}{2}zt} \mathcal{D}_{\nu+2n}(\sqrt{2zt}) \right| < B_{\lambda,\nu}(1 + |z|^{\operatorname{Re}\mu})(1 + t^{\operatorname{Re}\mu}) e^{(\alpha - \operatorname{Re} z)}.$$

Also for $\operatorname{Re} z > \alpha' > \alpha$, $(1 + t^{\operatorname{Re}\mu})e^{(\alpha-\operatorname{Re} z)}$ is uniformly bounded on $0 < t < \infty$ by another constant and so

$$\left| e^{\alpha t}(zt)^{\lambda+\beta+n} e^{-\frac{1}{2}zt} \mathcal{D}_{\nu+2n}(\sqrt{2zt}) \right| \leq C_{\lambda,\nu}(1 + |z|^{\operatorname{Re}\mu})$$

where $C_{\lambda,\nu}$ is a constant.

Hence the lemma.

THEOREM 5.1. Analiticity Theorem: *Let* $F(s) = (\mathcal{D}_{\lambda,\nu}f)(s)$ *for* $s \in \Omega_f$. *Then* $F(s)$ *is an analytic function on* Ω_f *and*

$$D_s F(s) = \langle f(t), D_s \omega(st) \rangle \tag{5.3}$$

where

$$D_s = \frac{\partial}{\partial s}$$

PROOF. Let s be an arbitrary but fixed point in Ω_f. Let us choose real numbers α, α' and positive r and r_1, such that

$$\alpha < \alpha' = \text{Re}\, s - r_1 < \text{Re}\, s - r < \text{Re}\, s .$$

Let C be a circle with centre at s and redius equal to r_1. We restriet r_1 and hence α' and r, in such a way that C lies entirely within Ω_f. Let Δs be a non-zero complex increment such that $|\Delta s| < r$ and let us consider the expression

$$\frac{F(s+\Delta s) - F(s)}{\Delta s} - \langle f(t), D_s\omega(st) \rangle = \langle f(t), \psi_{\Delta s}(t) \rangle \tag{5.4}$$

where $\psi_{\Delta s}(t) = \dfrac{\omega(\overline{s+\Delta st}) - \omega(st)}{\Delta s} - D_s\omega(st)$

The differentiation formula (SLATER [4], p. 25), the series expansion and the asymptotic behaviour of $\mathcal{D}_\nu(z)$ show that $D_s\omega(st)$ is a member of $K_{\alpha,\beta}(I)$ and hence equations (5.3) and (5.4) are meaningful.

To prove the theorem, we will have to show that $\psi_{\Delta s}(t) \to 0$ in $K_{\alpha,\beta}(I)$ as $|\Delta s| \to 0$. Using Canchy's integral formula we can write $A^n_{\lambda,t}\psi_{\Delta s}(t)$ in the form of a closed integral on C as

$$A^n_{\lambda,t}\psi_{\Delta s}(t) = \frac{1}{2\pi i}\int_C 2^{-\nu/2}\left(\frac{z}{2}\right)^n (zt)^\lambda e^{-\frac{1}{2}zt}\mathcal{D}_{\nu+2n}(\sqrt{2zt})\cdot$$

$$\left[\frac{1}{\Delta s}\left(\frac{1}{z-s-\Delta s} - \frac{1}{z-s}\right) - \frac{1}{(z-s)^2}\right] dz$$

$$= \frac{\Delta s}{2\pi i}\int_C \frac{2^{-\nu/2}\left(\frac{z}{2}\right)^n (zt)^\lambda e^{-\frac{1}{2}zt}\mathcal{D}_{\nu+2n}(\sqrt{2zt})}{(z-s)^2(z-s-\Delta s)} dz .$$

Hence

$$\left|e^{\alpha t}t^{\beta+n}A^n_{\lambda,t}\psi_{\Delta s}(t)\right|$$

$$= \left|\frac{\Delta s}{2\pi i}\int_C \frac{2^{-\nu/2}e^{\alpha t}t^{\beta}2^{-n}(zt)^{\lambda+n}e^{-\frac{1}{2}zt}\mathcal{D}_{\nu+2n}(\sqrt{2zt})}{(z-s)^2(z-s-\Delta s)}dz\right|$$

$$< \frac{|\Delta s|}{2\pi}\int_C \left|\frac{2^{-\nu/2-n}z^{-\beta}e^{\alpha t}(zt)^{\beta+\lambda+n}e^{-\frac{1}{2}zt}\mathcal{D}_{\nu+2n}(\sqrt{2zt})}{(z-s)^2(z-s-\Delta s)}dz\right|.$$

Let $Q_{\lambda,\nu}$ be the constant bound on

$$e^{\alpha t}(zt)^{\lambda+\beta+n}e^{-\frac{1}{2}zt}D_{\nu+zn}(\sqrt{2zt})$$

for $0 < t < \infty$ and $z \in C$ [Lemma 5.1]. Then we may write

$$\left|e^{\alpha t}t^{\beta+n}A^n_{\lambda,t}\psi_{\Delta s}(t)\right| \leq \frac{|\Delta s|}{2\pi}Q_{\lambda,\nu}\int_C \left|\frac{2^{-\nu/2-n}z^{-\beta}}{(z-s)^2(z-s-\Delta s)}dz\right|$$

$$\leq \frac{|\Delta s|Q_{\lambda,\nu}}{2\pi r_1^2(r_1-r)}2\pi r_1 \sup_{z\in C}\left|2^{-\nu/2-n}z^{-\beta}\right|$$

which $\to 0$ as $|\Delta s| \to 0$.

This proves the theorem.

6 – Solution of a class of differential equations

The Weber transform can be used to solve certain coundary value problems. From property 4.4, we see that the operator $t^r A^r_{\lambda,t}$ ($r = 0, 1, 2, \ldots$) is a continuous linear mapping of $K_{\alpha,\beta}(I)$ into itself. Its adjoint operator B^r_λ is a continuous linear mapping $K'_{\alpha,\beta}(I)$ into itself and is defined by

$$(6.1.) \qquad \langle B^r_\lambda f, \phi\rangle = \langle f, t^r A^r_{\lambda,t}\phi(t)\rangle$$

Se we see that

$$\begin{aligned}\mathcal{D}_{\lambda,\nu}\left\{B_\lambda^r f\right\} &= \langle B_\lambda^r f, 2^{-\nu/2}(st)^\lambda e^{-\frac{1}{2}st}\mathcal{D}_\nu(\sqrt{2st})\rangle \quad \text{by (5.1)}\\ &= \langle f, t^r A_{\lambda,t}^r 2^{-\nu/2}(st)^\lambda e^{-\frac{1}{2}st}\mathcal{D}_\nu(\sqrt{2st})\rangle \quad \text{by (6.1)}\\ &= \langle f, t^r 2^{-\nu/2}\left(\frac{s}{2}\right)^r (st)^\lambda e^{-\frac{1}{2}st}\mathcal{D}_{\nu+2r}(\sqrt{2st})\rangle\\ &= \langle f, 2^{-\frac{\nu+2r}{2}}(st)^{\lambda+r}e^{-\frac{1}{2}st}\mathcal{D}_{\nu+2r}(\sqrt{2st})\rangle\\ &= \mathcal{D}_{\lambda+r,\nu+2r}(f)\end{aligned}$$

Thus

$$\mathcal{D}_{\lambda,\nu}\{B^r f\} = \mathcal{D}_{\lambda+r,\nu+2r}(f) \tag{6.2}$$

We can exploit the relation (6.2) to solve a differential equation, with certain boundary conditions, of course, of the type

$$B_\lambda^r(f) = g \tag{6.3}$$

where g is a known generalized function belonging to $K'_{\alpha,\beta}(I)$ and is to be determined.

On applying Weber transform to (6.3) and using (6.2), we get

$$\begin{aligned}\mathcal{D}_{\lambda+r,\nu+2r}(f) &= \mathcal{D}_{\lambda,\nu}(g)\\ &= G(s), \quad \text{say}.\end{aligned}$$

Hence

$$f = D_{\lambda+r,\nu+2r}^{-1}[G(s)]$$

which gives a solution of (6.3); where

$$\mathcal{D}_{\lambda+r,\nu+2r}^{-1}[G(s)] =$$

$$= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{\Gamma\left(\lambda+\frac{3}{2}-\frac{\nu}{2}-s\right)}{\Gamma(\lambda+r-s+1)\Gamma\left(\lambda+r-s+\frac{3}{2}\right)}\Phi(s)ds\,,$$

in which $\Phi(s) = \int_0^\infty x^{-s} F(x)dx$, $s = C + iT$; C be a real with $\sigma_f < C < \infty$ and

$$F(x) = \langle f(u), \omega(x,u) \rangle .$$

Here we have not given the form B_λ^r. However, these can be calculated by the method of integration by parts. For $r = 1$, B_λ^r is given by

$$B_\lambda^r(f) = \left[t^2D^2 + \left(2\lambda + \frac{7}{2}\right) tD + \left(\lambda + \frac{3}{2}\right)(\lambda + 1)\right] f .$$

Acknowledgements

The authors are thankful to the referee for his valuable constructive suggestions.

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Lavoro pervenuto alla redazione il 21 febbraio 1990
ed accettato per la pubblicazione il 12 marzo 1991
su parere favorevole di A. Avantaggiati e di B. Lisena

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