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The Jordan Canonical Form in some θ-groups

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RIASSUNTO – *Si espone un metodo di classificazione delle orbite di un gruppo lineare algebrico riduttivo dovuto a V. Gatti e E. Viniberghi. Lo si applica poi ad alcuni casi particolari ed, attraverso una serie di riduzioni, si mostra come tali casi speciali, alcuni esempi di θ-gruppi, coincidano con alcuni casi classici e si recuperano così alcuni risultati noti.*

ABSTRACT – *We explain a method for the classification of orbits of a reductive linear algebraic group due to V. Gatti and E. Viniberghi. We then apply such method to some special cases, some examples of θ-groups, and we show, via a series of reductions, that these cases are actually classical. Thus we recover some known results.*

KEY WORDS – *Gruppi algebrici - Orbite.*

A.M.S. CLASSIFICATION: 17C - 20G

– Introduction

In this work we compute and classify the nilpotent orbits in two examples of θ-groups using a method developed in [5].

These groups form a class of connected linear algebraic groups that generalize the notion of adjoint group of a semisimple Lie algebra.

If θ is a semisimple automorphism of finite order of a reductive connected algebraic group G, and G^θ is the set of fixed points of G, we may consider the connected component of the identity of G^θ, and denote it

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G_0. Then the automorphism $d\theta$ induces on $\underline{g}$, the Lie algebra of G, a $\mathbf{Z}_m$-gradation, $\underline{g} = \coprod_{i \in \mathbf{Z}_m} \underline{g}_i$, where m is the order of θ. Then the adjoint representation of G induces a linear representation of G_0 on each $\underline{g}_i$. The image of G_0 in its representation on $\underline{g}_1$ is called a θ-group.

For $m = 1$ we have the adjoint group $Ad\ G$, and for $m = 2$ the isotropy group of the symmetric space associated to G.

It turns out, see [15], [16], [8], that for such groups there exists a Jordan decomposition, a Cartan subspace and a Weyl group which is finite and generated by complex reflections. Therefore the classification of the nilpotent orbits, which are analogous to Hilbert's "nullforms", is a decisive step for the classification of all orbits.

The classification of nilpotent orbits for the adjoint group was obtained by E.B. DYNKIN in [4] and later by P. BALA and R.W. CARTER in [1], [2], with a method similar to Dynkin's.

In this work the method used to classify nilpotent orbits in two examples of θ-groups was developed by V. GATTI and E. VINIBERGHI in [5]. Such method allows one to reduce the classification of the orbits of the action of a linear algebraic group G on a vector space V to that of certain special orbits of its linear subgroups.

In the first section, we describe the method in general and state the main result of [5]. Then, in Section 2, we translate the problem to a combinatorial computation of certain roots and weights.

In Section 3, we introduce the notion of θ-groups and state the theorem that allows us to classify the nilpotent orbits in the particular case of θ-groups.

In the last two sections, we finally examine the case of two examples of θ-groups, and recover results of [11] and [13].

1 – Description of the method

We give here a general description of the method developed in [5] to classify the orbits of a linear algebraic group G. We shall assume the extra hypothesis that G is a reductive group.

In general we shall follow the custom of indicating the group with the capital letter and with the corresponding small underlined letter its Lie algebra.

DEFINITION 1.1. *A subgroup H of an algebraic group G is said to be* **complete regular** *if H is the centralizer of a semisimple element of the Lie algebra of G.*

DEFINITION 1.2. *Let H be an algebraic group and U an H-module. An element $x \in U$ is called* **semifree** *if all the semisimple elements of the Lie algebra of its normalizer $\underline{h}_{<x>} = \{\alpha \in \underline{h}/\alpha x = \lambda x, \lambda \in \mathbb{C}\}$ belong to the center of the Lie algebra $\underline{h}$ of H and if U is spanned by $H \cdot x$.*

PROPOSITION 1.3. *If G is a connected linear algebraic group, $G \subset GL(V)$, $\underline{g}$ its Lie algebra, T a fixed maximal torus of G, and $\underline{t}$ the corresponding Lie algebra, then every $x \in V$ is in the G-orbit of a semifree element of V with respect to a suitable complete regular subgroup H containing T and a suitable subspace $U \subset V$ that is H-stable.*

PROOF. Let $x \in V$. Consider $G_{<x>} = \{g \in G : gx = \lambda x, \lambda \in \mathbb{C}^*\}$. First, we shall prove that there exists an element in the G orbit of X, say gx, such that $T_{<gx>}$ is a maximal torus in $G_{<gx>}$.

This is clear because we can take a maximal torus S in $G_{<x>}$, which is not necessarily contained in T, then S is certainly conjugate to a torus contained in T, namely, there exists $g \in G$ such that $gSg^{-1} \subset T$. It then follows that $gSg^{-1} \subset T_{<gx>}$ (since if $s \in S$, $gsg^{-1}(gx) = \lambda gx$). Now, since S is maximal in $G_{<x>}$, $gSg^{-1} = T_{<gx>}$ is maximal in $G_{<gx>}$. Furthermore, $\underline{t}_{<gx>}$ is maximal in $\underline{g}_{<gx>}$. Set $y = gx$. Let H be the centralizer in G of $\underline{t}_{<y>}$ and $\underline{h}$ its Lie algebra. Take U to be the linear span of $H \cdot y$.

U is clearly H-invariant; also, all the semisimple elements of $\underline{h}_{<y>}$ are in the center of $\underline{h}$ which we denote $Z(\underline{h})$. Indeed: Certainly $\underline{t}_{<y>} \subset Z(\underline{h}) \subset \underline{h}$, because $\underline{h} = c_{\underline{g}}(\underline{t}_{<y>})$; on the other hand if a is a semisimple element of $\underline{h}$, it certainly commutes with $\underline{t}_{<y>}$; hence $\{a, \underline{t}_{<y>}\}$ generates a torus in $\underline{g}_{<y>}$, but, by hypothesis, $\underline{t}_{<y>}$ is maximal and so $a \in \underline{t}_{<y>} \subset Z(\underline{h})$. Therefore, y is semifree. $\square$

Notice that since the semisimple elements of $\underline{h}_{<y>}$ are in the center of $\underline{h}$ they act as scalar transformations on U and, finally, the set of such elements coincides with $\underline{t}_{<y>}$.

DEFINITION 1.4. *A pair $\{H, U\}$, where H is a subgroup of G and U is an H-invariant subspace of V, is called a* special pair *if it satisfies the following conditions:*
(1) H is a complete regular subgroup of G which contains T;
(2) H coincides with the centralizer in G of the subalgebra of $\underline{t}$ consisting of all elements that act as scalar transformations on U;
(3) The H-module U has a semifree orbit (i.e. an orbit made of semifree elements).

REMARK 1.5 In the proof of Proposition 1.3, we have constructed a special pair.

If W_G is the Weyl group of G, then W_G acts on the set C of special pairs of G as follows. If $\overline{w} \in W_G \simeq N_G(T)/C_G(T)$, take a representative $w \in N_G(T)$ and define an action

$$\alpha : N_G(T) \times C \longrightarrow C$$

by

$$w \cdot \{H, U\} = \{H^w = wHw^{-1}, wU\}.$$

We shall now check that we obtain in this way a new special pair and that α is an action. Let $n \in N_G(T)$, we shall see that $n.\{H, U\}$ is a special pair.

(1) If H centralizes $Y \in \underline{g}$, Y semisimple, then H^n centralizes $Ad\ n(Y)$ which is again semisimple:

$$Ad(ngn^{-1})Ad\ n\ (Y) = Ad\ n\ Ad\ h(Y) - Ad\ n(Y).$$

Observe that if $U =< H \cdot x >$, (the linear span of $H \cdot x$), then $nU =< H^n \cdot nx >$.

(2) We need to show that if H centralizes $\underline{t}_{<y>}$, then H^n centralizes $\underline{t}_{<ny>}$. Observe that since $n \in N_G(T)$, $n^{-1}T_{<nx>}n = T_{<x>}$, and so $\underline{t}_{<nx>} = Ad\ n(\underline{t}_{<x>})$. Hence, if $Y \in \underline{t}_{<nx>}$, then Y must be equal to $Ad\ n(X)$ for some X in $\underline{t}_{<x>}$ and so:

$$Ad(nhn^{-1})(Y) = Ad(nhn^{-1})Ad\ n(X) = Ad\ n\ Ad\ h(x) = Ad\ n(X).$$

Vice versa, in an analogous way we can show that if an element centralizes $\underline{t}_{<nx>}$ then it is the conjugate of an element that centralizes $\underline{t}_{<x>}$.

(3) If $\underline{\bar{h}}$ is the Lie algebra of H^n, let $s \in \underline{\bar{h}}_{<nx>}$ be a semisimple element. We shall show that s is in the center of $\underline{\bar{h}}$. Since $Ad\ n(\underline{h}) = \underline{\bar{h}}$, consider $Ad\ n^{-1}(s) \in \underline{h}_{<x>}$. By the hypothesis, $Ad\ n^{-1}(s) \in Z(\underline{h})$ and so $s = Ad\ n\ Ad\ n^{-1}(s) \in Z(\underline{\bar{h}})$.

It is easy to verify that α is an action.

Finally, notice that in this action $C_G(T)$ acts trivially hence α induces an action $\overline{\alpha}$ of W_G on C. Indeed: recall that in the case when G is reductive and T a maximal torus, we have $C_G(T) = T$. Therefore, since $T \subset H$, $H^T = H$, and $tU = U$ since U is T-stable.

Definition 1.6. *We shall denote by $\bar{H}$ the normalizer of the pair $\{H, U\}$, that is*

$$\bar{H} = \{n \in G : nHn^{-1} = H, nU = U\}.$$

Proposition 1.7. *Let $\{H, U\}$ be a fixed special pair of G. Let O be the set of semifree orbits. Then $\bar{H}$ acts on O as follows: if $Hx \subset U$ is a semifree orbit, $p \in \bar{H}$, then Hpx is still a semifree orbit in U.*

Proof. To see this, it is enough to show that the semisimple elements of $\underline{h}_{<px>}$ lie in the center of $\underline{h}$, but this is obvious if we think that $Ad\ p(\underline{h}_{<x>}) = \underline{h}_{<px>}$, that the center is mapped into itself by $Ad\ p$ and that, by hypothesis, the semisimple elements of $\underline{h}_{<x>}$ are in the center. □

We thus arrive at the fundamental theorem of the classification method of [5].

Theorem 1.8. *Let $G \subseteq GL(V)$ be a reductive linear algebraic group. Take a pair $\{H_\alpha, U_\alpha\}$ in each class of W_G-equivalent special pairs; let $\bar{H}_\alpha$ be the normalizer of the pair $\{H_\alpha, U_\alpha\}$ in G. Take an orbit in each class of $\bar{H}_\alpha$-equivalence of semifree orbits of H_α in U_α and an element in each such orbit. Let M_α be the set of the elements thus chosen.*

Then the set $M = \bigcup_\alpha M_\alpha$ is a minimal complete system of representatives of the orbits of the linear group G.

PROOF. It is clear that M is a complete system of representatives because of Proposition 1.3.

We shall show that M is minimal. Suppose $x_2 = gx_1$, for $g \in G$. Notice that we may assume that

$$(Ad\ g)\underline{t}_{<x_1>} = \underline{t}_{<x_2>}.$$

Indeed, if

$$(Ad\ g^{-1})\underline{t}_{<x_2>} = \underline{t}'_{<x_1>} \neq \underline{t}_{<x_1>}$$

then $\underline{t}'_{<x_1>}$ is a maximal torus and so it is conjugate to $\underline{t}_{<x_1>}$ i.e. there exists a $g_1 \in G$ such that

$$(Ad\ g_1)\underline{t}_{<x_1>} = \underline{t}'_{<x_1>}$$

hence

$$(Ad\ gg_1)\underline{t}_{<x_1>} = \underline{t}_{<x_2>}.$$

From this observation and from the property (2) of the special pairs, we obtain

$$H_1^g = H_2.$$

Since $g^{-1}Tg$ is a maximal torus in H_1, there must exist $h_1 \in H$ such that $h_1g^{-1}Tgh_1^{-1} = T$, hence $gh_1^{-1} \in N_G(T)$ and so $g \in N_G(T) \cdot H_1$. But then $\{H_1, U_1\}$ and $\{H_2, U_2\}$ are W_G-equivalent, indeed: $x_2 = gx_1$, $U_1 =< H_1 \cdot x_1 >$, $U_2 =< H_2 \cdot x_2 >$ imply $g\{H_1, U_1\} = \{H_1^g, gU_1\} = \{H_2, U_2\}$, where $g \in N_G(T)H_1$, i.e. $g = nh_1$. Therefore

$$H_2 = gH_1g^{-1} = (nh_1)H_1(h_1^{-1}n^{-1}) = nH_1n^{-1}$$

and analogously

$$gU_1 =< H_1^g \cdot gx_1 >=< H_1^n \cdot gx_1 > .$$

Now a typical element of $< H_1^n \cdot gx_1 >$ is of the form

$$\sum_k \alpha_k \cdot nh^{(k)}n^{-1} \cdot nh_1x_1 = \sum_k \alpha_k nh^{(k)}h_1x_1 = n\sum_k \alpha_k h^{(k)}h_1x_1$$

for some scalars α_k, and so

$$< H_1^n \cdot gx_1 >= nU_1.$$

Hence the special pairs in consideration are equivalent and we may identify them.

Next, we have to see that x_1 and x_2 give rise to two equivalent semifree orbits under the action of $\tilde{H}_1$. This is true since

$$Ad\ g(\underline{t}_{<x_1>}) = \underline{t}_{<x_1>}$$

implies $g \in N_G(H_1)$, by hypothesis $gx_1 = x_2$ is semifree and this suffices to conclude that $g \in \tilde{H}_1$ and that the semifree orbits are equivalent. □

2 – Weight System of special pairs

Let G be a reductive, connected, linear algebraic group and Δ the root system of its Lie algebra $\underline{g}$ with respect to a maximal torus $\underline{t}$. Let Π be a set of simple roots and Λ the weight system of the $\underline{g}$-module V.

We shall indicate by $\underline{g}_\alpha$ and V_λ respectively the root space corresponding to $\alpha \in \Delta$ and the weight space corresponding to $\lambda \in \Lambda$.

Let $\{H, U\}$ be a special pair of the group G. Then H, being a complete regular subgroup of G, is reductive. Let Δ_0 be the system of roots of $\underline{h}$ with respect to $\underline{t}$. A system of simple roots $\Pi_0 \subset \Delta_0$ is W_G-equivalent to a subsystem of Π. Consider the subspace

$$\underline{P} = \{x \in V \ s.t. \ \underline{g}_{-\alpha} x = 0, \alpha \in \Pi_0\}.$$

$\underline{P}$ is a $\underline{t}$-invariant subspace of V, and it consists of minimal weight vectors of V. Let Π_1 be the set of weight of $\underline{t}$ on $\underline{P}$.

DEFINITION 2.1. *The pair of subsets $\{\Pi_0, \Pi_1\}$, where $\Pi_0 \subset \Delta$, and $\Pi_1 \subset \Lambda$, constructed in the previous section for a special pair $\{H, U\}$ is called the weight system of that special pair.*

Our aim is now to study the properties of such weight systems so as to characterize the weight systems obtainable from special pairs.

Let's observe that, if we denote by Π_1' the set of differences of all the elements of Π_1, then the property (2) of the Definition 1.4 may be rewritten as

$$\text{(A)} \qquad < \Pi_0 \cup \Pi_1' > \cap \Delta \subset \Delta_0.$$

In fact, since $H = C_G(\underline{t}_{<x>})$, we have that $\alpha(t) = 0$, for all $t \in \underline{t}_{<x>}$, $\alpha \in \Pi_0$; furthermore, since $\underline{t}_{<x>}$ coincides with the scalar transformations on U, we have that $\underline{t}_{<x>}$ acts as zero on $U_{\lambda-\lambda'}$, if $\lambda, \lambda' \in \Pi_1$:

$$t(u) - (\lambda - \lambda')(t)(u) = (\lambda(t) - \lambda'(t))u = 0$$

being $\lambda(t) =$ constant for all $\lambda \in \Pi_1$. On the other hand, by definition, the only roots $\alpha \in \Delta$ such that $\alpha(t) = 0$, for all $t \in \underline{t}_{<x>}$, are those of H, i.e. Δ_0.

Other necessary properties of the weight systems are:

(B) The weight system of $\underline{t}$ on $\underline{P}$ is Π_1;

(C) $\underline{g}_{-\alpha}\underline{P} = 0$, for all $\alpha \in \Pi_0$;

(D) The H-module $< H\underline{P} >$ has a semifree orbit.

We claim that properties (A)-(D) are sufficient to reconstruct the special pairs.

H is the complete, regular subgroup that has Π_0 as single roots. Also, $U =< H\underline{P} >$: given a special pair $\{H, U\}$, U is a completely reducible H-module, say

$$U = \bigoplus_{\chi} U_\chi$$

where U_χ is an irreducible H-module. Then U_χ contains the minimal vector v_χ, hence $U_\chi \cap \underline{P} \neq (0)$, but then this intersection generates U_χ, because this is irreducible. Repeating the argument for each χ, we get $< H\underline{P} >= U$.

REMARK 2.2 In the case when all the weights of the G-module V are of multiplicity 1, given a special pair $\{H, U\}$, $U = \bigoplus_{\lambda} V_\lambda$, λ in a subset Γ of Λ, $\underline{P}$ is just the direct sum of the v_λ where λ is a minimal weight in Γ.

So, with the additional hypothesis that the weights have multiplicity 1, we have

PROPOSITION 2.3. *Let $G \subseteq GL(V)$ be a reductive linear algebraic group for which all weights have multiplicity one. Then the map that assigns to every special pair its system of weights induces a bijection between the set of W_G-equivalent classes of special pairs and the set of W_G-equivalent classes of pairs of subsets $\{\Pi_0, \Pi_1\}, \Pi_0 \subset \Pi, \Pi_1 \subset \Delta$, which satisfy the following properties:*
(i) $\{\Pi_0 \cup \Pi_1'\} \cap \Delta \subset \Delta_0$,
(ii) $\lambda - \alpha \notin \Delta$ *if* $\lambda \in \Pi_1$, $\alpha \in \Pi_1$,
(iii) The H-module U has a semifree orbit, where H is a complete regular subgroup of G that has Π_0 as simple root system and $U = \langle HV_\lambda, \lambda \in \Pi_1 \rangle$.

REMARK Observe that (i) and (ii) imply
(iv) $\lambda_1 - \lambda_2 \notin \Delta$ if $\lambda_1, \lambda_2 \in \Pi_1$.
Indeed, if $\lambda_1, \lambda_2 \in \pi_1$ and $\lambda_1 - \lambda_2 \in \Delta$ then (i) implies that $\lambda_1 - \lambda_2 = \alpha$ where $\alpha \in \Delta_0$, but this contradicts (ii).

3 – θ-groups

We shall introduce an important class of linear groups. Let $\underline{g}$ be a simple Lie algebra over $\mathbf{C}$. Let θ be an automorphism of $\underline{g}$ of period $m \in \mathbf{N}$. θ is semisimple and induces on $\underline{g}$ a $\mathbf{Z}_m$-gradation

$$\underline{g} = \bigoplus_{i \in \mathbf{Z}_m} \underline{g}_i.$$

Let G be a connected group that has $\underline{g}$ as its Lie algebra and G_0 the connected subgroup corresponding to the subalgebra $\underline{g}_0$. From the property $[\underline{g}_0, \underline{g}_k] \subset \underline{g}_k$ it follows that the adjoint representation of G induces, by restriction, a linear representation ρ_k of G_0 on $\underline{g}_k$, for each $k \in \mathbf{Z}_m$. In particular, we shall have ρ_1. We shall say that the linear group $\rho_1(G_0)$ is associated to the graded Lie algebra $\underline{g}$.

DEFINITION 3.1. *The linear groups obtained in this fashion are called θ-groups.*

REMARK 3.2 Suppose that θ is induced from an automorphism of G which we denote again by θ and consider

$$G^{\theta} = \{g \in G : \theta g = g\}.$$

If G is a semisimple simply connected group, then G^{θ} is connected and coincides with G_0.

REMARK 3.3 Let $x \in \underline{g}_1$ and consider its Jordan decomposition as an element of $\underline{g}$, $x = x_s + x_n$. It is not difficult to see that x_s, x_n lie in $\underline{g}_1$. Because of this fact, we can reduce the classification of orbits to the classifcation of the nilpotent orbits. We shall return to this fact later.

DEFINITION 3.4. *A special pair $\{H, U\}$ is said to be of conic type, if the Lie algebra of H contains a nonzero scalar transformation of U.*

PROPOSITION 3.5. *Let G be a θ-group. Then*

(a) The map that assigns to each special pair of conic type its weight system induces a bijection between the W_G-equivalence classes of special pairs of conic type and the W_G-equivalence classes of pairs satisfying the conditions (i)-(iv) of Proposition 2.3 and such that the set $\Pi_0 \cup \Pi_1$ consists of linearly independent vectors.

(b) If $\{H, U\}$ is a special pair of conic type, then all the semifree elements of U form an open subset $\Omega \subset U$.

(c) Taking a complete system of nonequivalent special pairs of conic type $\{H_\alpha, U_\alpha\}$, and taking an element in the open orbit of each group H_α acting on U_α, we obtain a minimal complete system of representatives of nilpotent orbits of G.

PROOF. Let $\{H, U\}$ be a special pair of conic type of G. Up to W_G-equivalence, we may assume that the simple roots Π_0 of $\underline{h}$ is in Π. By definition, there exists an element $t \in \underline{t}$ that acts as the identity on U and whose centralizer is $\underline{h}$. Let $\{\Pi_0, \Pi_1\}$ be the weight system of $\{H, U\}$.

(a) Since $\{H, U\}$ is conic, Π_1 does not contain the zero weight (because then any scalar transformation is zero). But then all the weight in Π_1 have multiplicity 1 (see [8]). Hence the pair $\{H, U\}$ can be reconstructed by the weight system.

Clearly the pair $\{\Pi_0, \Pi_1\}$ satisfies (i) and (ii) and hence (iv). From (ii) and (iv) it follows that $(\lambda, \alpha) \leq 0$ if $\lambda \in \Pi, \alpha \in \Pi_0$, and $(\lambda, \mu) \leq 0$ if $\lambda, \mu \in \Pi_1$ (see [9]). Also, we know that $(\alpha, \beta) \leq 0$ if $\alpha, \beta \in \Pi_0$. Therefore, $\Pi_0 \cup \Pi_1$ is a set of vectors in the Euclidean space $\underline{t}^*$ with pairwise nonpositive inner product, hence if they are dependent there must exist a linear combination with nonnegative coefficients equal to zero,

$$\sum_i a_i \alpha_i + \sum b_j \lambda_j = 0, \tag{3.1}$$

$\alpha_i \in \Pi_0$, $\lambda_j \in \Pi_1$, $a_i, b_j \geq 0$.

Applying both sides of this equality to t we obtain:

$$\sum_i a_i \alpha_i(t) + \sum_j b_j \lambda_j(t) = 0$$

but $\alpha_i(t) = 0$ for all i because $\underline{h}$ centralizes t and $\lambda_j(t) = 1$ by the choice of t and so $\sum_j b_j = 0$, therefore, all b_j are zero. Hence (3.1) can be rewritten as

$$\sum_i a_i \alpha_i = 0.$$

Since Π_0 is a set of linearly independent vectors, all the a_i are zero. Hence $\Pi_0 \cup \Pi_1$ is a linearly independent set of vectors.

(b) The linear subgroup H of G is observable since G is. (See [15], [5], [8]). (We recall that if G is a reductive group acting on a vector space V, the morphism

$$\mathbb{C}[V]^G \longrightarrow \mathbb{C}[V]$$

induces a morphism of affine algebraic varieties

$$\pi : V \longrightarrow V/G$$

where V/G is the quotient variety. G is said to be observable if each fiber of π contains only a finite number of orbits).

Since all the elements of U are nilpotent with respect to H, then H has only a finite number of orbits, in particular it has a dense open orbit $\Omega_1 \subset U$ (if $U = \bigcup_{i=1}^{n} U_i$, then $U = \bigcup_{i=1}^{n} \overline{U}_i$ and, since U is irreducible, $\overline{U}_i = U$ for some i). Obviously, Ω_1 has maximal dimension among the orbits, which must be equal to the dimension of U.

Let $x \in U \subset \underline{g}_1$, a semifree element of the H-module U. Set $A = \bigoplus_{s \in \mathbb{Z}} A_s$, where

$$A_s = \{x \in \underline{g} \text{ s.t. } [t,x] = sx\}.$$

A is a reductive subalgebra of the Lie algebra $\underline{g}$. For this, it is enough to see that the restriction to A of the Killing form K is nondegenerate; this is true, because t induces on $\underline{g}$ a decomposition $\underline{g} = \oplus \underline{g}_\lambda, \underline{g}$ is simple and so K is nondegenerate on $\underline{g}$. On the other hand, if $x \in \underline{g}_\lambda$, $y \in \underline{g}_\mu$, we have

$$([t,x],y) = -(x,[t,y])$$

and so

$$\lambda(x,y) = -\mu(x,y)$$

hence

$$(\lambda + \mu)(x,y) = 0.$$

If $\lambda \neq -\mu$, then $(x,y) = 0$. But then if there is the equivalence λ there must exist the eigenvalue $-\lambda$ and the restriction of K to $\underline{g}_\lambda \oplus \underline{g}_{-\lambda}$ is non-degenerate. Also, $A_0 = \underline{h}$, $A_1 \supset U$.

The element x is nilpotent and is contained in the reductive algebra A, hence, by the Morozov-Jacobson theorem (see [1], [8]), we may view x inside a simple three-dimensional Lie algebra $< x, h, y >$, where h is a semisimple element belonging to $A_0 = \underline{h}$ which normalizes x. Since x is semifree, h is in the center of $\underline{h}$ (see Def. 1.2). Hence the Lie algebra $\underline{h}_x$ = centralizer of x, centralizes the pair (h,x) and so $\underline{h}_x$ coincides with the centralizer in A of the reductive subalgebra $< x, h, y >$. Therefore, $\underline{h}_x$ is reductive.

On the other hand, x being semifree implies that $\underline{h}_x$ is nilpotent: if z is a semisimple element of $\underline{h}_x \subset \underline{h}_{<x>}$, we have $z \in Z(\underline{h})$ and so $z = 0$ on $U =< H \cdot x >$. This means that $\underline{h}_x$ has no nontrivial semisimple elements and so $\underline{h}_x$ is nilpotent. Hence $\underline{h}_x$ is zero on U. Therefore the orbit of x has maximal dimension, and since H has an open orbit Ω_1, we get $x \in \Omega_1$.

(c) Thanks to Theorem 1.8, it is enough to show that the conic pairs, up to W_G-equivalence, are in a bijective correspondence with nilpotent orbits; for this it is sufficient to show that if $x \in V$ is a nilpotent element, then it is, up to conjugation, a semifree element of a suitable conic pair. This is true since there are only a finite number of nilpotent orbits, therefore $\mathbf{C}^* x$ is contained in the orbit of x; and so there exists $g \in G$ such that $gx = \alpha x$, $\alpha \in \mathbf{C}^*$, but then the semisimple part g_s of g sends x in a nonzero scalar multiple. Take the torus S generated by g_s which, up to conjugation, we may think contained in T. Let H be the centralizer in G of the Lie algebra of S, and U the subspace $< Hx >$. Then $\{H, U\}$ is the desired conic pair. □

4 – A first example of θ-group

Consider the simple Lie algebra $\underline{g} = s\ell(n, \mathbf{C})$, and the matrix

$$P = \begin{pmatrix} 1_k & 0 \\ 0 & -1_{n-k} \end{pmatrix}$$

for a fixed integer k, $1 \leq k \leq n-1$, where 1_s is the identity on $\mathbf{C}^s$. We note that $P = P^{-1}$.

Consider the automorphism

$$\begin{aligned} \theta : \underline{s\ell}(n, \mathbf{C}) &\longrightarrow \underline{s\ell}(n, \mathbf{C}), \\ A &\longmapsto PAP^{-1} \end{aligned}$$

θ is clearly an automorphism and has finite order

$$\theta^2 = 1.$$

Hence θ is semisimple, and it decomposes the Lie algebra $\underline{s\ell}(n, \mathbf{C})$ as follows:

$$\underline{g} = \underline{g}_0 \oplus \underline{g}_1,$$

where, using block matrices,

$$\underline{g}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \underline{g} \right\},$$

and

$$\underline{g}_1 = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in \underline{g} \right\}.$$

This is a Z_2-gradation of $\underline{s\ell}(n, \mathbb{C})$, and $\underline{g}_0$ is a subalgebra of $\underline{g}$.

Consider the group G_0 obtained by exponentiating the matrices of $\underline{g}_0$. It is well-known that the connected component of the identity of a group G that has L as Lie algebra is generated by exp X, with $X \in L$, cf. [3]. Then

$$\begin{aligned}
\exp \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \frac{1}{2} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^2 + \frac{1}{3!} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^3 + \cdots = \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \frac{1}{2} \begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} a^3 & 0 \\ 0 & d^3 \end{pmatrix} + \cdots = \\
&= \begin{pmatrix} \exp a & 0 \\ 0 & \exp d \end{pmatrix}.
\end{aligned}$$

Set

$$z = \begin{pmatrix} \exp a & 0 \\ 0 & \exp d \end{pmatrix}.$$

Recalling that

$$\det(\exp a) = \exp(\operatorname{tr} a)$$

we have

$$\begin{aligned}
\det z &= \det(\exp a) \cdot \det(\exp d) = \exp(\operatorname{tr} a) \cdot \exp(\operatorname{tr} b) \\
&= \exp(\operatorname{tr} a + \operatorname{tr} b) = \exp(0) = 1.
\end{aligned}$$

So

$$G_0 = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : \det x \det y = 1 \right\}.$$

The representation that we want to study is the action of G_0 on $\underline{g}_1$ by conjugation

$$\begin{aligned}
G_0 \times \underline{g}_1 &\longrightarrow \underline{g}_1 \\
(Z, x) &\longmapsto Z x Z^{-1}.
\end{aligned}$$

We need to make some preliminary observations.

We may think of $\underline{g}_1$ as pairs of homomorphisms; more precisely if

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in \underline{g}_1$$

where b is a $k \times (n-k)$ block and c is a $(n-k) \times k$ block, we may view this matrix as the pair of linear maps

$$V \underset{c}{\overset{b}{\rightleftarrows}} W$$

where $\dim V = n-k$, $\dim W = k$.

Also, since the scalar matrices $\mathbf{C}I$ acts trivially, the G-orbits of $\underline{g}_1$ are the same as the $G \times \mathbf{C}$ orbits of $\underline{g}_1$.

We may consider the homomorphism

$$\varphi : G \times \mathbf{C}^* \longrightarrow GL_k \times GL_{n-k}$$

which maps the pair (g, α) into $(\alpha X, \alpha Y)$ where $g = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} = X \oplus Y$. Then

$$\operatorname{Ker} \varphi = \left\{ (\alpha 1_k \oplus \alpha 1_{n-k},\ \alpha^{-1}); \alpha^n = 1 \right\},$$

and φ is surjective, because if $(Z, W) \in GL_k \times GL_{n-k}$, $w = \det W \neq 0$, $z = \det Z \neq 0$, we may consider zw and one of its n-th roots α. Then the pair (X, Y) where $X = (\alpha^{-1} 1_k) Z, Y = (\alpha^{-1} 1_{n-k}) W$ maps onto (Z, W). Hence

$$\frac{G \times \mathbf{C}^*}{\operatorname{Ker} \varphi} \approx GL_k \times GL_{n-k}.$$

We can then reduce our problem to the study of the action of $GL(k, \mathbf{C}) \times GL(n-k, \mathbf{C})$ on the space $\hom(V^{n-k}, W^k) \times \hom(W^k, V^{n-k})$, given as follows

$$\left(\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right) \longmapsto \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \begin{pmatrix} X^{-1} & 0 \\ 0 & Y^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & XAY^{-1} \\ YBX^{-1} & 0 \end{pmatrix},$$

or

$$((X, Y), (A, B)) \longmapsto (XAY^{-1}, YBX^{-1}).$$

REMARK The above can be restated as the problem of studying pairs of linear maps between two fixed vector spaces V, W up to change of bases in V and W.

REMARK In general, if $\theta : \underline{g} \longrightarrow \underline{g}$ is an automorphism of order m of a semisimple Lie algebra $\underline{g}$ then θ induces a $\mathbb{Z}_m$-gradation of $\underline{g}$

$$\underline{g} = \underline{g}_0 \oplus \underline{g}_1 \oplus \cdots \oplus \underline{g}_{m-1}.$$

Observe that if $x \in \underline{g}$, then its semisimple and nilpotent parts x_s and x_n lie again in $\underline{g}_1$. Indeed, if

$$x = x_s + x_n$$

and if ζ is the fixed root of unity, then

$$\theta x = \zeta x = \zeta x_s + \zeta x_n$$

gives the Jordan decomposition of θx so by the uniqueness we must have

$$\theta x_s = \zeta x_s$$

$$\theta x_n = \zeta x_n.$$

The decomposition

$$x = x_s + x_n$$

can then be viewed as the Jordan decomposition in our example of θ-group.

Thanks to this and to the fact that we have a Cartan subspace and a finite Weyl group (see [15], [16]), we can reduce ourselves to the problem of classifying nilpotent orbits. Indeed, let $x = x_s + x_n$ and $y = y_s + y_n$ and suppose $gx = y$. Then $gx_s + gx_n = y_s + y_n$ and, by the uniqueness of the decomposition, $gx_s = y_s, gx_n = y_n$. Assuming to have classified the semisimple elements, we take two elements X and Y with the same semisimple part

$$X = X_s + X_n$$

$$Y = X_s + Y_n$$

and in the same G-orbit, then

$$X_s = gX_s$$
$$Y_n = gX_n.$$

Hence we see that it is sufficient to classify the nilpotent orbits with respect to the centralizer of a semisimple element.

For the classification of the semisimple elements, we shall recall here some well-known facts.

If x_s is semisimple, then it is in some maximal torus and two elements in a maximal torus are conjugate with respect to the adjoint group if and only if they are conjugate via the Weyl group. Indeed: if they are conjugate by the Weyl group W, they are obviously conjugate by G because, for example, $W = N_G(T)\big/C_G(T)$. Vice versa, if $x, y \in \underline{t}$ are conjugate by G, then $f(x) = f(y)$ for all $f \in \mathcal{P}(\underline{g})^G$ = the set of polynomial functions on $\underline{g}$ invariant under G, but

$$\mathcal{P}(\underline{g})^G \simeq \mathcal{P}(\underline{t})^W$$

(see [6], [3]), hence $f(x) = f(y)$ for all $f \in \mathcal{P}(\underline{t})^W$; but W is a finite group hence all the orbits are closed and so the invariants parameterize the orbits; therefore x and y are W-equivalent.

Back to our example. Observe that the elements of $\underline{g}_1$ can be thought of as endomorphism

$$X : V \oplus W \longrightarrow V \oplus W$$

with the property $X = -PXP = -\theta(x)$, i.e.

$$PX = -XP.$$

If λ is an eigenvalue of x, then $-\lambda$ is an eigenvalue as well, because

$$X(v_1 + w_1) = \lambda(v_1 + w_1)$$

so

$$PXP(v_1 - w_1) = PX(v_1 + w_1) = P\lambda(v_1 + w_1) = \lambda(v_1 - w_1)$$

i.e.

$$-X(v_1 - w_1) = -\lambda(v_1 - w_1).$$

Considering the two equalities

$$X(v_1 + w_1) = \lambda(v_1 + w_1),$$
$$X(v_1 - w_1) = -\lambda(v_1 - w_1),$$

adding and subtracting, we obtain

$$Xv_1 = \lambda w_1$$
$$Xw_1 = \lambda v_1.$$

Setting $U = V \oplus W$, and indicating with U_λ the eigenspace corresponding to λ, we have

(1) $U_0 = (V \cap U_0) \oplus (W \cap U_0)$
(2) $U_\lambda \oplus U_{-\lambda} = ((U_\lambda \oplus U_{-\lambda}) \cap V) \oplus ((U_\lambda \oplus U_{-\lambda}) \cap W)$.

Set

$$(U_\lambda \oplus U_{-\lambda}) \cap V = W_{\pm\lambda}$$
$$(U_\lambda \oplus U_{-\lambda}) \cap W = W_{\pm\lambda}.$$

Let X be semisimple. Then U is the direct sum of the spaces $U_\lambda \oplus U_{-\lambda}$, and the centralizer C_X of X in $GL(U)$ is the direct product

$$\prod_\lambda GL(U_\lambda).$$

Analogously to the classical theory of the Jordan canonical form, we can limit our consideration to the block relative to the eigenvalues $\pm\lambda$, and, as a first case, let us assume $\lambda \neq 0$. Let us consider the restriction of X to $U_\lambda \oplus U_{-\lambda}$, which we may write as

$$V_{\pm\lambda} \underset{B}{\overset{A}{\rightleftarrows}} W_{\pm\lambda}.$$

Choose a basis of $U_\lambda \oplus U_{-\lambda}$ of eigenvectors of X: $\{v_1 + w_1, \cdots, v_s + w_s\}$ in U_λ and $\{v_1 - w_1, \cdots, v_s - w_s\}$ in $U_{-\lambda}$. Then it is easy to see that

$\{v_1, \cdots, v_s\}$ is a basis of $V_{\pm\lambda}$ and $\{w_1, \cdots, w_s\}$ is a basis of $W_{\pm\lambda}$. With this choice of bases, we have $Av_i = \lambda w_i$ and $Bw_i = \lambda v_i$ so that

$$AB = \lambda^2$$

$$BA = \lambda^2,$$

and the matrix of X can be written as

$$\begin{pmatrix} 0 & \lambda I \\ \lambda I & 0. \end{pmatrix}$$

The problem we are reduced to study is that of the nilpotent orbits of the linear maps between the spaces $V_{\pm\lambda}, W_{\pm\lambda}$ under the action of the centralizer C_X of $X = (A, B)$. For this, recall that $(Z, Y) \in C_X$ if and only if $Z \in GL(V_{\pm\lambda}), Y \in GL(W_{\pm\lambda})$ and

$$YA = AZ$$

$$ZB = BY.$$

So

$$C_X = \{(Z, Y) \text{ s.t. } Y = AZA^{-1}\},$$

hence

$$C_X \simeq GL(s, \mathbf{C}),$$

where

$$s = \dim V_{\pm\lambda} = \dim W_{\pm\lambda}.$$

Let us take a nilpotent pair (A_n, B_n) which commutes with the semisimple pair (A, B). In the basis chosen above, $(A, B) = (\lambda I_s, \lambda I_s)$ hence we must have $A_n = B_n$ as matrices with respect to the chosen basis. Also, if $(Z, Y) \in C_X$, $Z = Y$ as matrices. Hence our action reduces to the action of $GL(s, \mathbf{C})$ by conjugation on the set of nilpotent matrices and we know that such a problem is solved by the theory of the Jordan canonical form, i.e. we know that each nilpotent orbit is represented by a partition of the integer s, that is to say a sequence of integers

$$(p_1, p_2, \cdots, p_l),$$

whose sum is s and such that $p_1 \geq p_2 \geq \cdots \geq p_t$. The parts p_i will be equal to the dimension of the i-th block in the Jordan canonical form. Let us consider now the case of the eigenvalue $\lambda = 0$. We shall classify the nilpotent pairs

$$V \underset{B}{\overset{A}{\rightleftarrows}} W$$

under the action of the group $GL(V) \times GL(W)$. Recalling the isomorphism

$$\hom(V, W) \approx V^* \otimes W$$

we can write

$$\hom(V, W) \times \hom(W, V) \simeq (V^* \otimes W) \oplus (W^* \otimes V).$$

Our aim is to compute the weights of the module

$$\underline{g}_1 \simeq (V^* \otimes W) \oplus (W^* \oplus V).$$

Let $\{v_1, \cdots, v_n\}$ be the basis of V and $\{v^1, \cdots, v^n\}$ a dual basis of V^*. Analogously, let $\{w_1, \cdots, w_m\}$ be a basis of W and $\{w^1, \cdots, w^m\}$ its dual. Then

$$\{v^i \otimes w_j, i = 1, \cdots, n; j = 1, \cdots m\}$$

and

$$\{w^j \otimes v_i, i = 1, \cdots, n; j = 1, \cdots, m\}$$

are basis of $V^* \otimes W$ and $W^* \otimes V$ respectively.

Take the torus $T \subset G, T = D_n \times D_m$, consisting of diagonal $(n + m) \times (n + m)$ matrices with nonzero determinant. For $\tau \in T$, we have

$$\tau(v^i \otimes w_j) = \beta_j \alpha_i^{-1}(v^i \otimes w_j)$$
$$\tau(w^j \otimes v_i) = \alpha_i \beta_j^{-1}(w^j \otimes v_i)$$

and so the weights are

$$\Lambda = \{\beta_j - \alpha_i, \alpha_i - \beta_j; i = 1, \cdots, n, j = 1, \cdots, m\}.$$

In this case, the Weyl group $\mathcal{W}$ is isomorphic to the direct product of the two symmetric groups S_n and S_m which act as permutations of the α's and the β's. Recall that the root system of $\underline{g}$ is

$$\Delta = \{\alpha_i - \alpha_j, i \neq j; \beta_h - \beta_k, h \neq k\}.$$

Proposition 3.5 allows us to obtain the desired classification by taking the classes of $\mathcal{W}$-equivalence of the pairs of subsets $\{\Pi_0, \Pi_1\}$ of $\{\Pi, \Lambda\}$ satisfying properties (i)-(iii) of Proposition 2.3, and such that $\Pi_0 \cup \Pi_1$ is a linearly independent set.

We can represent $\Pi_0 \cup \Pi_1$ with a Dynkin diagram as is done in the classification theory of semisimple Lie algebras. Comparing such diagrams with the list of their classifications contained in [1], [2], (see also [17], [4]) we can see that in the case in which $\underline{g}_0$ is of type A_n, as in the case we are considering, $\Pi_0 = \phi$ and so $H = T$ and $\underline{P} = U$.

So we can state the following.

PROPOSITION 4.1. *We can obtain a complete classification of the nilpotent orbits of our θ-group as follows: make a list of the subsets $\Pi_1 \subset \Lambda$ up to $\mathcal{W}$-equivalence such that*

[1] *the elements of Π_1 are linearly independent;*

[2] *the difference of two elements in Π_1 is not an element of Δ.*

In this way each set Π_1 determines a subspace $U \subset V$,

$$U = \bigoplus_{\chi \in \Pi_1} V_\chi.$$

If $x \in V$ is a vector such that x has a nonzero component in each V_χ, then x has a dense orbit in U and so this is a semifree orbit. The set of x thus obtained, is the required representative system.

We shall now sketch the explicit computation of such a set.

If Λ' is the set of those weights in Λ of the form $\alpha - \beta$, then

$$\Lambda = \Lambda' \cup (-\Lambda').$$

Let us write the weight of Λ' in a matrix form

$$\begin{array}{cccc} \alpha_1 - \beta_1 & \alpha_2 - \beta_1 & \dots & \alpha_n - \beta_1 \\ \alpha_1 - \beta_2 & \alpha_2 - \beta_2 & \dots & \alpha_n - \beta_2 \\ \alpha_1 - \beta_3 & \alpha_2 - \beta_3 & \dots & \alpha_n - \beta_3 \\ \vdots & & & \\ \alpha_1 - \beta_m & \alpha_2 - \beta_m & \dots & \alpha_n - \beta_m . \end{array}$$

Condition (2) of Proposition 4.1 is translated by saying that we must not choose two weights in the same row or column, and, because of (1), we must not choose twice the same weight. Also, because of the action of the Weyl group which permutes the α's among themselves and the β's among themselves, we can further restrict the number of choices. Arguing this way, we can see that every possible set Π_1 is obtained as union of disjoint parts of the following type

$$\begin{aligned} \Pi_1^{(1)} &= \{\alpha_1 - \beta_1, \beta_1 - \alpha_2, \alpha_2 - \beta_2, \cdots, \alpha_k - \beta_k, \beta_k - \alpha_{k+1}\}, \\ \Pi_1^{(2)} &= \{\beta_1 - \alpha_1, \alpha_1 - \beta_2, \beta_2 - \alpha_2, \cdots, \beta_s - \alpha_s, \alpha_s - \beta_{s+1}\}, \\ \Pi_1^{(3)} &= \{\alpha_1 - \beta_1, \beta_1 - \alpha_2, \cdots, \alpha_t - \beta_t\}, \\ \Pi_1^{(4)} &= \{\beta_1 - \alpha_1, \alpha_1 - \beta_2, \cdots, \beta_r - \alpha_r\}. \end{aligned}$$

To such weight systems correspond the following semifree elements:

$$\begin{aligned} x^{(1)} &= v^1 \otimes w_1 + w^1 \otimes v_2 + \cdots + v^k \otimes w_k + w^k \otimes v_{k+1}, \\ x^{(2)} &= w^1 \otimes v_1 + v^1 \otimes w_2 + \cdots w^s \otimes v_s + v^s \otimes w_{s+1}, \\ x^{(3)} &= v^1 \otimes w_1 + \cdots + v^t \otimes w_t, \\ x^{(4)} &= w^1 \otimes v_1 + \cdots + w^r \otimes v_r . \end{aligned}$$

PROPOSITION 4.2. *If V and W are finite dimensional complex vector spaces, then every nilpotent pair of endomorphisms (A, B)*

$$V \underset{B}{\overset{A}{\rightleftarrows}} W$$

is equivalent under the action of $G = GL(V) \times GL(W)$ to a direct sum of pairs of type $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}$.

REMARK These four possibilities are indicated in [11] by means of "ab-diagrams" as follows:

$$\begin{aligned} x^{(1)} &= abab\cdots aba \\ x^{(2)} &= baba\cdots bab \\ x^{(3)} &= abab\cdots ab \\ x^{(4)} &= baba\cdots ba \end{aligned}$$

and in [13] with respectively,

$$M_{n+1,n} = [\mathbf{C}^{n+1} \underset{s_n}{\overset{r_n}{\rightleftarrows}} \mathbf{C}^n]$$

$$M_{n,n+1} = [\mathbf{C}^{n} \underset{r_n}{\overset{s_n}{\rightleftarrows}} \mathbf{C}^{n+1}]$$

$$M_{n} = [\mathbf{C}^{n} \underset{j_n(0)}{\overset{\mathrm{id}}{\rightleftarrows}} \mathbf{C}^{n}]$$

$$M_{n}^1 = [\mathbf{C}^{n} \underset{\mathrm{id}}{\overset{j_n(0)}{\rightleftarrows}} \mathbf{C}^{n}]$$

where id $= 1_n$,

$$j_n(0) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix},$$

$$r_n = \begin{pmatrix} 0 & 1 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}, \quad s_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

5 – The orthosimplectic case

Let $E = \begin{pmatrix} 0 & 1_s \\ -1_s & 0 \end{pmatrix}$, $P = \begin{pmatrix} 1_k & 0 \\ 0 & E \end{pmatrix}$ and consider again $\underline{g} = s\ell(n, \mathbf{C})$, with $n = s + k$. Consider the automorphism

$$\begin{aligned} \theta : \underline{g} &\longrightarrow \underline{g} \\ A &\longmapsto -P\ A^t P^{-1}. \end{aligned}$$

Notice that $\theta^2 A = P^2 A P^2$ and $P^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so that θ^2 is the automorphism we studied in the previous section.

So $\theta^4 = 1$ and $\underline{g}$ decomposes into the sum of four eigenspaces

$$\underline{g_0} = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \underline{g} \text{ s.t. } a = -a^t, b = -Eb^t E^{-1}\},$$

$$\underline{g_1} = \{\begin{pmatrix} 0 & b \\ iEb^t & 0 \end{pmatrix}\},$$

$$\underline{g_2} = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in g, a + a^t, d = E\ d^t\ E^{-1}\},$$

$$\underline{g_3} = \{\begin{pmatrix} 0 & b \\ -iEb^t & 0 \end{pmatrix}\}.$$

Now we must observe that $\underline{g}$, can be interpreted as follows: Let $(W, <,>)$ be a vector space equipped with a symplectic form and $(V(,))$ a vector space with an orthogonal form. Given a linear map

$$b : W \longrightarrow V$$

we can define its adjoint b^* by

$$(bw, v) =< w, b^* v >,$$

$$b^* : V \longrightarrow W.$$

If the symplectic form has matrix $-E$ in some basis, we can write

$$(bw)^t v = w^t b^t v = w^t E^{-1} E b^t v =< w, Eb^t v > .$$

So

$$b^* = E\, b^t.$$

Hence we have

$$\underline{g_1} = \left\{\begin{pmatrix} 0 & b \\ ib^* & 0 \end{pmatrix}\right\},$$

and the map

$$\underline{g_1} \simeq \hom(W,V) \xrightarrow{\bullet} \hom(W,V) \times \hom(V,W)$$
$$b \longmapsto (b, b^*).$$

Similarly to what we saw in the previous case, the automorphism θ is the differential of the automorphism

$$\theta\colon SL(n,\mathbb{C}) \longrightarrow SL(n,\mathbb{C})$$
$$A \longmapsto P(A^{-1})^t P.$$

Then

$$G = G^\theta = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, X \text{ orthogonal}, Y \text{ symplectic} \right\}.$$

Hence our problem is to classify the pairs

$$V \underset{A}{\overset{A^*}{\rightleftarrows}} W$$

where V is orthogonal, W symplectic under the action of $SO(V) \times Sp(W)$.

Once again we can reduce to classify the nilpotent orbits.

Remark It is easy to see that

$$AA^* : V \longrightarrow V$$

and

$$A^*A : W \longrightarrow W$$

are antisymmetric endomorphisms i.e.

$$(AA^*)^* = -AA^*$$
$$(A^*A)^* = -A^*A.$$

Remark If U is a vector space with a form $(,)$ which may be orthogonal or symplectic and Y a semisimple antisymmetric endomorphism, then $U = \bigoplus_\mu U^\mu$; and, if $\mu \neq 0$, U^μ and $U^{-\mu}$ are dually paired.

If (X, X^*) is a semisimple pair, then XX^* is a semisimple element of End V and X^*X is a semisimple element of End W. By the analysis done in the previous section, we know that we may reduce to the case

$$V_{\pm\lambda} \underset{X_{\pm\lambda}}{\overset{X^*_{\pm\lambda}}{\rightleftarrows}} W_{\pm\lambda}$$

when $\lambda \neq 0$, and we know that XX^* and X^*X both have eigenvalue λ^2.

In the notation of the previous observation, we have

$$V_{\pm\lambda} = V^{\lambda^2} \qquad \text{and} \qquad W_{\pm\lambda} = W^{\lambda^2}.$$

Notice that $X^*_{\pm\lambda}$ is not the adjoint of $X_{\pm\lambda}$, since the form is degenerate on $V_{\pm\lambda}$, but if we consider V^{λ^2} i.e. $V_{\pm\lambda} \oplus V_{\pm i\lambda}$ and $W_{\pm\lambda} \oplus W_{\pm i\lambda}$ then $X^*_{\pm\lambda} \oplus X^*_{\pm i\lambda}$ is the adjoint of $X_{\pm\lambda} \oplus X_{\pm i\lambda}$.

To simplify the notation, let us set

$$V = V_{\pm\lambda} \oplus V_{\pm i\lambda}, W = W_{\pm\lambda} \oplus W_{\pm i\lambda},$$

$$X^*_{\pm\lambda} \oplus X^*_{\pm i\lambda} = A_\lambda \oplus A_{i\lambda} = A,$$

$$B = B_\lambda \oplus B_{i\lambda} = X_{\pm\lambda} \oplus X_{\pm i\lambda}.$$

Let $\{v_1, \cdots, v_n\}$ be a basis of $V_{\pm\lambda}$ and $\{w_1, \cdots, w_m\}$ be a basis of $W_{\pm\lambda}$ such that $Av_i = \lambda w_i$, $Bw_i = \lambda v_1$. Then in these bases the matrix of $A_\lambda \oplus B_\lambda$ is

$$\begin{pmatrix} O & \lambda I \\ \lambda I & O \end{pmatrix}.$$

Analogously, we can choose dual bases so that $A \oplus B$ restricted to $V_{\pm i\lambda} \oplus W_{\pm i\lambda}$ is

$$\begin{pmatrix} O & i\lambda I \\ i\lambda I & O \end{pmatrix}. \tag{4.1}$$

In these bases the matrix of the symplectic form is given by

$$\begin{pmatrix} O & -iI \\ iI & O \end{pmatrix}. \tag{4.2}$$

Indeed:

$$< w_h, w'_k > = < \frac{1}{\lambda} A v_h, w'_k > = (\frac{1}{\lambda} v_h, A^* w'_k)$$
$$= (\frac{1}{\lambda} v_h, -B w'_k) = (\frac{1}{\lambda} v_h, -i\lambda v'_k) = -i\delta_{hk}$$

where primes denote dual bases.

Normalizing $\overline{w}_k = i w_k$, we can assume that the matrix of the symplectic form is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

but then (4.1) changes into

$$\begin{pmatrix} O & i\lambda I \\ -i\lambda I & O \end{pmatrix}$$

while (4.2) stays the same.

Let $(X,Y) \in GL(V) \times GL(W)$ be an element of the centralizer of (A,B), then X centralizes BA and Y centralizes AB.

In the chosen bases we have

$$AB = \begin{pmatrix} \lambda^2 & & & & & \\ & \ddots & & & & \\ & & \lambda^2 & & & \\ & & & -\lambda^2 & & \\ & & & & \ddots & \\ & & & & & -\lambda^2 \end{pmatrix}.$$

So if $g \in GL(W)$ centralizes AB then

$$g = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$$

(with block notations). If g is a symplectic matrix then

$$g = \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^{-1})^t \end{pmatrix}.$$

Analogously, the centralizer of BA in $SO(V)$ is

$$\{h = \begin{pmatrix} \beta & 0 \\ 0 & (\beta^{-1})^t \end{pmatrix}\}.$$

Finally, if (g,h) is to centralize (A,B), then an easy calculation shows that $\alpha = \beta$. So the desired centralizer is isomorphic to $GL(n,\mathbb{C})$ where $n = \dim V_{\pm\lambda} = \dim W_{\pm\lambda} = \dim V_{\pm i\lambda} = \dim W_{\pm i\lambda}$. Next, we want to classify under the action of this group, the nilpotent orbits of elements that commute with the fixed semisimple pair (A_s, A_s^*). Such nilpotent elements must be of the form

$$\begin{pmatrix} & 0 & & A_1 & 0 \\ & & & 0 & -A_1^t \\ -A_1 & 0 & & & \\ 0 & -A_1^t & & 0 & \end{pmatrix}$$

which we may think of as $A_1 \in g\ell(n,\mathbb{C})$. Hence we are back to the classical case of the Jordan canonical form.

Let us examine now the case $\lambda = 0$.

We know that the restriction of the forms of V and W to V_0 and W_0 respectively give rise to new orthogonal and symplectic spaces.

We want to classify nilpotent orbits of homomorphisms $(X, X^*) \in \hom(V,W) \times \hom(W,V)$ under the action of $SO(V) \times Sp(W) = G$. Recall that there is a canonical isomorphism

$$\hom(V,W) \simeq V^* \otimes W \simeq V \otimes W,$$

where the last isomorphism is due to the existence of the orthogonal form on V.

Assume $\dim V = 2k$, $k \in \mathbb{N}$. We know that $\dim W$ must be even, say $\dim W = 2h$. Fix bases $\{v_1, .., v_k, v_1', .., v_k'\}$ and $\{w_1, .., w_h, w_1', .., w_h'\}$ of V and W respectively such that

$$(v_i, v_j) = \delta_{ij},$$

$$< w_i, w_j' > = \delta_{ij}.$$

Take T to be the direct sum of the tori of $SO(V)$ and $Sp(V)$:

$$T_1 = \left\{ \begin{pmatrix} \alpha_1 & & & & & \\ & \ddots & & & & \\ & & \alpha_k & & & \\ & & & \alpha_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \alpha_k^{-1} \end{pmatrix} \right\} \simeq \left\{ \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_k \end{pmatrix} = \alpha \right\},$$

$$T_2 = \left\{ \begin{pmatrix} \beta_1 & & & & & \\ & \ddots & & & & \\ & & \beta_h & & & \\ & & & \beta_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \beta_h^{-1} \end{pmatrix} \right\} \simeq \left\{ \begin{pmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_h \end{pmatrix} = \beta \right\}.$$

Then on an element of the basis of $V \otimes W$ we have

$$\begin{aligned} (\alpha,\beta)\cdot v_i \otimes w_j &= \alpha_i\beta_j(v_i \otimes w_j) \\ (\alpha,\beta)\cdot v_j' \otimes w_j &= \alpha_i^{-1}\beta_j(v_i' \otimes w_j) \\ (\alpha,\beta)\cdot v_i \otimes w_j^+ &= \alpha_i\beta_j^{-1}(v_i \otimes w_j') \\ (\alpha,\beta)\cdot v_i' \otimes w_j' &= (\alpha_i\beta_j)^{-1}(v_i' \otimes w_j^+). \end{aligned}$$

So the set of weights is

$$\Lambda = \{\pm(\alpha_i + \beta_j), \pm(\alpha_i - \beta_j)\}.$$

The Weyl group is given by the direct product of the Weyl group of $SO(V)$ and that of $Sp(W)$, namely,

$$\mathcal{W}_O \simeq (\mathbb{Z}_2)^{k-1} \times S_k,$$

$$\mathcal{W}_{Sp} \simeq (\mathbb{Z}_2)^h \times S_h.$$

The group $\mathcal{W}_O$ is generated by the transformation $\varepsilon_{ij}, \sigma_{ij}$ defined as follows

$$\varepsilon_{ij} : \begin{cases} v_i \longmapsto v_j' \\ v_j \longmapsto v_i' \\ \text{id elsewhere} \end{cases} \qquad \sigma_{ij} : \begin{cases} v_i \longmapsto v_j \\ v_j \longmapsto v_i \\ \text{id elsewhere} \end{cases}$$

while W_{Sp} is generated by τ_{ij} defined analogously to σ_{ij} and by

$$\eta_i : \begin{cases} v_i \longmapsto v_i' \\ \text{id elsewhere.} \end{cases}$$

Notice that if $\dim V$ is odd, we have the case of odd orthogonal space and the Weyl group is analogous to the one in the symplectic case. Finally, we want to apply Proposition 2.3. We find out that $\Pi_0 = \phi$ and so $\underline{P} = U$ (see [1], [2], [17], [4].)

PROPOSITION 5.1. *We list the subsets Π_1 of Λ, up to $\mathcal{W}$-equivalence, such that*
[1] *Π_1 is a linearly independent set,*
[2] *the difference of two elements of Π_1 is not in Δ.*
Each such set Π_1 determines a subspace $U \subset V$,

$$U = \bigoplus_{\chi \in \Pi_1} V_\chi.$$

Let $x \in U$ be a vector with nonzero component in each weight space of U. Then x is in the semifree orbit of U. The set of x thus obtained, is the desired set of representations of orthosymplectic orbits.

An algorithm for obtaining this list is sketched here. Let us represent each element of Λ with a dot and let us connect two dots if the corresponding weights have nonzero inner product. We obtain a graph. Π_1 will be a disjoint union of connected subgraphs. Let us be concerned only with the connected parts.

If $\Pi_1^{(1)}$ is a connected component of the graph and it satisfies (1) and (2) of Proposition 5.1, then the only possibilities for the dimensions of $\dim W$ and $\dim V$ are so that

$$|\dim V - \dim W| \leq 2.$$

The claim now is that each of the five possibilities exists, and gives a unique orbit. For example, if we start with $\alpha_1 + \beta_1$ we obtain the string

$$\Pi_1^{(1)} = \{\alpha_1+\beta_1, -(\beta_1+\alpha_2), \alpha_2+\beta_2, -(\beta_2+\alpha_3), \cdots, \alpha_n+\beta_n, -(\beta_n+\alpha_{n+1})\}$$

and any other is $\mathcal{W}$-equivalent to this.

REMARK In the notation of [11] the indecomposables with respect to $O(V) \times Sp(W)$ are

$$(1) \qquad \begin{array}{c} abab\cdots ab \\ bab\cdots ba \end{array}$$

$$(2) \qquad aba\cdots aba$$

$$(3) \qquad \begin{array}{c} aba\cdots aba \\ aba\cdots aba \end{array}$$

$$(4) \qquad bab\cdots bab$$

$$(5) \qquad \begin{array}{c} bab\cdots bab \\ bab\cdots bab \end{array}$$

The final result can then be stated as follows:

THEOREM 5.2. *If V is an orthogonal space, W a symplectic space, every nilpotent pair of homomorphisms is equivalent under the action of $G = O(V) \times Sp(W)$ to a direct sum of pairs of type (1)-(5). Also, a G-orbit splits into two orbits with respect to $SO(V) \times Sp(W)$ if and only if in every row of the ab-diagram the number of the a's is even and there are an even number of rows (see [11]).*

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