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A Capacitability Theorem in Finitely Additive Gambling

A. MAITRA - R. PURVES - W. SUDDERTH(*)

RIASSUNTO – *Una casa da giuoco Γ assegna a ciascuno stato x, nello spazio discreto X, una collezione non vuota $\Gamma(x)$ di distribuzioni di probabilità finitamente additive su X. Un giocatore, partendo da un certo stato x, sceglie in $\Gamma(x)$ una distribuzione σ_0 per il successivo stato x_1. Quindi egli sceglie in $\Gamma(x_1)$ una distribuzione condizionale $\sigma_1(x_1)$ per lo stato x_2, e così via. Egli si propone di massimizzare la probabilità che la successione $x_1, x_2, \ldots$ appartenga ad un fissato insieme A boreliano (o anche soltanto sousliniano) dello spazio prodotto di una successione di copie di X. Se si denota con $\Gamma(A)(x)$ l'estremo superiore di questa probabilità al variare di tutte le possibili scelte di $\sigma_0, \sigma_1, \ldots$, allora la funzione $\Gamma(\cdot)(x)$ ha proprietà di regolarità analoghe a quelle di una capacità: in particolare, $\Gamma(A)(x)$, è eguale all'estremo inferiore dei numeri della forma $\Gamma(O)(x)$, con O insieme aperto contenente A. Ciò permette di approssimare un'ampia classe di problemi di scommessa mediante i classici problemi di Dubins e Savage.*

ABSTRACT – *A gambling house Γ assigns to each state x of the discrete space X a nonempty collection $\Gamma(x)$ of finitely additive probability distributions on X. A player in the house Γ starts at some state x. The player chooses the distribution σ_0 for the next state x_1 from $\Gamma(x)$ and then chooses the conditional distribution $\sigma_1(x_1)$ for x_2 from $\Gamma(x_1)$ and so on. Suppose the goal is to control the stochastic process $x_1, x_2, \ldots$ so that it will lie in a certain Borel (or even Souslin) subset A of the product space $X \times X \times \ldots$ and that $\Gamma(A)(x)$ is the supremum over all choices of $\sigma_0, \sigma_1, \ldots$ of the probability that the player attains this goal. Then the set function $\Gamma(\cdot)(x)$ has regularity properties like those of a capacity and, in particular, $\Gamma(A)(x) = \inf\{\Gamma(O)(x): O \text{ is open and } O \supseteq A\}$. Consequently, quite general gambling problems can be approximated by the classical problems of Dubins and Savage.*

KEY WORDS – *Finitely additive gambling - Stochastic control - Optimal reward operator - Regularity - Capacity - Souslin sets.*

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1 – Introduction

Suppose X is a nonempty set of possible states for a process and that to each $x \in X$ is associated a nonempty collection $\Gamma(x)$ of finitely additive probability measures defined on all subsets of X. Then, starting from any x, one can construct a random sequence $x_1, x_2, \ldots$, by selecting $\sigma_0 \in \Gamma(x)$ to be distribution of x_1, then selecting $\sigma_1(x_1) \in \Gamma(x_1)$ to be the conditional distribution of x_3 given x_1, x_2, and so on. The sequence $\sigma = \{\sigma_0, \sigma_1, \ldots\}$ is a *strategy at* x in the *gambling house* Γ. As is explained in DUBINS and SAVAGE [2, pp. 7-21], each strategy σ can be regarded as a finitely additive probability measure defined on the collection of closen subsets of the set $H = X \times X \times \ldots$, where X is given the discrete topology and H the product topology. In addition, there is a natural extension of each measure σ to the sigma-field generated by the clopen subsets of H and even further to the collection of *Souslin sets* A of the form

$$A = \bigcup_{\alpha} \bigcap_{k=1}^{\infty} B(\alpha_1, \ldots, \alpha_k)$$

where the union is all over the sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ of positive integers and $B(\alpha_1, \ldots, \alpha_k)$ is a closed subset of H for avery α and every k (PURVES and SUDDERTH [8, Theorem 5.3]). (A reader unfamiliar with finite additivity can assume X is countable and all measures are countably additive to get the gist of the result stated below).

For each x, let $S(x)$ be the collection of strategies σ available at x in Γ, and, for each Souslin set A, define

$$\Gamma(A)(x) = \sup\{\sigma(A) : \sigma \in S(x)\}. \tag{1.1}$$

Thus $\Gamma(A)(x)$ is the *optimal reward* for a player starting at x who seeks to control the process $x_1, x_2, \ldots$ so that it will lie in A.

Here is our main result.

THEOREM 1.1. *For every $x \in X$ and every Souslin set $A \subseteq H$,*

$$\Gamma(A)(x) = \inf\{\Gamma(O)(x) : O \text{ is open } O \supseteq A\}. \tag{1.2}$$

The fact that, for every x and A,

$$\Gamma(A)(x) = \sup\{\Gamma(C)(x) : C \text{ is closed}, \quad C \subseteq A\}$$

is immediate from the analogous fact about a single strategy σ (cf. PURVES and SUDDERTH [8, Theorem 5.3]).

The proof of Theorem 1.1. is based on the study of another operator Γ^* which is defined, for $x \in X$ and $E \subseteq H$, by

$$\Gamma^*(E)(x) = \inf\{\Gamma(O)(x) : O \text{ open}, O \supseteq E\}. \tag{1.3}$$

The operator $\Gamma^*(\cdot)(x)$ is a capacity in certain special case such as when X is countable and all measures under consideration are countably additive (MAITRA, PURVES, and SUDDERTH [4, Lemma 3.1]). In such cases, Theorem 1.1 follows from the capacitability theorem of CHOQUET [1]. In general, $\Gamma^*(\cdot)(x)$ fails to be a capacity although it will be shown to have certain properties akin to those of a capacity. For example, the usual "going up" property fails but there is an analogous result, Proposition 7.1, in which the natural numbers are replaced by the collection of stop rules.

Here is how the rest of the paper is organized. The next section introduces some terminology and notation. Section 3 shows that a certain functional equation is satisfied by each of the operators Γ and Γ^*. The basic technique for proving equality (1.2) is presented in section 4 and is used in section 5 to verify the equality for sets A which are G_δ's. The next step is to verify it for $G_{\delta\sigma}$'s in section 6. Finally, after the going up property of section 7, the proof is given for Souslin sets in section 8. The final section of the paper states a result for functions which is analogous to Theorem 1.1.

We have written a paper [5] parallel to this one which treats the same sort of regularity questions in a measurable, countably additive setting. The results are somewhat similar but the proofs are more difficult because of measurability problems. To overcome these problems we find it necessary to use effective descriptive set theory. Conventional set theory is adequate for the purposes of this paper.

2 – Terminology and notation

Our terminology and notation is based on that of DUBINS AND SAVAGE [2] and, for the most part, will be the same as theirs. This section reviews the essential definitions of [2] and introduces a few additional items.

A *stopping time* is a mapping t from H to $\{1,2,\ldots\} \cup \{\infty\}$ such that, if $t(h) = n < \infty$ and h' agrees with h in the first n coordinates, then $t(h') = n$. A *stop rule* is a stopping time which is everywhere finite. (Stopping times were called "incomplete stop rules" in [2]).

Let X^* be the set of all finite sequences of elements of X including the empty sequence. Let $p,q \in X^*$ and $h \in H$. Then pq is the member of X^* whose terms are the terms of p followed by the terms of q and ph is the member of H whose terms are the terms of p followed by those of h. If $A \subseteq H$, $Ap = \{h: ph \in A\}$ and $pA = \{ph: h \in A\}$. If g is a function defined on H, gp is the function on H defined by $(gp)(h) = g(ph), h \in H$. If t is a stopping time, $h = (h_1,h_2,\ldots) \in H$, and $t(h) = n < \infty$, then $h_t(h) = h_n$, $p_t(h) = p_n(h) = (h_1,\ldots,h_n)$, Ap_t is the set valued function defined by $(Ap_t)(h) = Ap_t(h)$, and gp_t is the function-valued function defined by $(gp_t)(h) = gp_t(h)$.

Let $p = (x_1,\ldots,x_n) \in X^*$ and let t be a stopping time and σ be a strategy. Define $t[p](h) = t(ph) - n$, $h \in H$. If $t(x_1,\ldots,x_n,\ldots) > n$, then $t[p]$ is again a stopping time and corresponds to the additional waiting time given that the first n coordinates are p. Define the *conditional strategy* $\sigma[p]$ by setting $\sigma[p]_0 = \sigma_n(p)$ and $\sigma[p]_m(q) = \sigma_{n+m}(pq)$ for each $m = 1,2,\ldots$ and $q \in X^m$. If σ is a strategy available at x in the house Γ, then $\sigma[p]$ is available at x_n, the last coordinate of p. Define $\sigma[p_t]$ at h to be $\sigma[p_t(h)]$ whenever $t(h) < \infty$.

Two strategies σ and σ' *agree prior to a stopping time* t if $\sigma_0 = \sigma'_0$ and whenever $h \in H$ and $t(h) > n$, then $\sigma_n(p_n(h)) = \sigma'_n(p_n(h))$.

Let $K \subseteq H$, g be a function with domain H, and let t be a stop rule. Say that K (respectively, g) is *determined by time* t if, whenever $h,h' \in H$ and $t(h) = t(h')$, then either h,h' are both in K or both are in the complement of K(respectively, $g(h) = g(h')$). Those sets K which are determined by some stop rule t are precisely the clopen subsets of H [2, Corollary 2.7.1].

It is not difficult to see that the open subsets of H are those sets of the form $[t < \infty]$ for some stopping time t. Thus (1.3) can be rewritten as

$$(2.1) \quad \Gamma^*(E)(x) = \inf\{\Gamma[t < \infty](x) : t \text{ a stopping time, } E \subseteq [t < \infty]\}$$

for $E \subseteq H$.

3 – Functional equations for Γ and Γ*

Each of the operators Γ and Γ^* satisfies a functional equation which is a version of the optimality equation of dynamic programming. These functional equations will be used repeatedly in our proof of Theorem 1.1.

Here is the equation for Γ.

PROPOSITION 3.1. *Let B be a Borel subset of H and let t be a stopping time such that $B \subseteq [t < \infty]$. Then, for every x,*

$$(3.1) \qquad \Gamma(B)(x) = \sup\left\{\int_{t<\infty} \Gamma(Bp_t)(h_t)d\sigma : \sigma \in S(x)\right\}.$$

PROOF. Let $\sigma \in S(x)$. By Lemma 5.1 of PURVES and SUDDERTH [9],

$$\sigma(B)(x) = \int_{t<\infty} \sigma[p_t](Bp_t)d\sigma \leq \int_{t<\infty} \Gamma(Bp_t)(h_t)d\sigma.$$

Take the supremum over σ in $S(x)$ to get that the left side of (3.1) is less than or equal to the right side.

To prove the reverse inequality, let $\sigma \in S(x)$, $\varepsilon > 0$, and, for each $p = (x_1, \ldots, x_n) \in X^*$, choose $\sigma(p) \in S(x_n)$ such that $\sigma(p)(Bp) > \Gamma(Bp)(x_n) - \varepsilon$. Now define the strategy $\hat{\sigma} \in S(x)$ to be the strategy which agrees with σ prior to t and satisfies $\hat{\sigma}[p_t] = \sigma(p_t)$ on $[t < \infty]$. Use [9, Lemma 5.1] again to get

$$\Gamma(B)(x) \geq \hat{\sigma}(B) = \int_{t<\infty} \sigma(p_t)(Bp_t)d\sigma \geq \int_{t<\infty} \Gamma(Bp_t)(h_t)d\sigma - \varepsilon$$

Take the supremum over $\sigma \in S(x)$ to complete the proof. □

Here is an immediate corollary which corresponds to Lemma 2.2 of [9].

COROLLARY 3.2. *If B is a Borel subset of H, τ is a stop rule and $x \in X$, then*

$$\Gamma(B)(x) = \sup\left\{\int \Gamma(Bp_\tau)(h_\tau)d\sigma : \sigma \in S(x)\right\}. \tag{3.2}$$

The operator Γ^* satisfies the same functional equation as Γ and it holds for arbitrary sets rather than just Borel sets.

PROPOSITION 3.3. *Let $A \subseteq H$ and let t be a stopping time such that $A \subseteq [t < \infty]$. Then, for every x,*

$$\Gamma^*(A)(x) = \sup\left\{\int_{t<\infty} \Gamma^*(Ap_t)(h_t)d\sigma : \sigma \in S(x)\right\}. \tag{3.3}$$

PROOF. Let $\varepsilon > 0$. For each $p = (x_1, \ldots, x_n) \in X^*$, use (2.1) to choose a stopping time $\bar{\tau}(p)$ such that $Ap \subseteq [\bar{\tau}(p) < \infty]$ and $\Gamma[\bar{\tau}(p) < \infty](x_n) < \Gamma^*(Ap)(x_n) + \varepsilon$. Define a stopping time τ by setting

$$\tau(h) = \begin{cases} t(h) + \bar{\tau}(p_t(h))(h_{t(h)+1}, h_{t(h)+2}, \ldots) & \text{if } t(h) < \infty, \\ \infty & \text{if } t(h) = \infty. \end{cases}$$

Notice that, if $h \in A$, then $t(h) < \infty$ and $(h_{t(h)+1}, h_{t(h)+2}, \ldots) \in Ap_t(h)$ so that $\bar{\tau}(p_t(h))(h_{t(h)+1}, h_{t(h)+2} \ldots) < \infty$. Hence, $A \subseteq [\tau < \infty]$. Notice also that $[\tau < \infty]p_t(h) = [\bar{\tau}(p_t(h)) < \infty]$ if $t(h) < \infty$. So, for every $\sigma \in S(x)$,

$$\begin{aligned} \sigma[\tau < \infty] &= \int_{t<\infty} \sigma[p_t(h)][\bar{\tau}(p_t(h)) < \infty]d\sigma \leq \\ &\leq \int_{t<\infty} \Gamma^*(Ap_t)(h_t)d\sigma + \varepsilon \end{aligned}$$

By (2.1), $\Gamma^*(A)(x) \leq \Gamma[\tau < \infty](x) = \sup\{\sigma[\tau < \infty] : \sigma \in S(x)\}$. So the proof that the left side of (3.3) is less than or equal to the right side is complete.

For the reverse inequality, it suffices to show that, for each open set O with $O \supseteq A$,

$$\Gamma(O)(x) \geq \sup\left\{\int_{t<\infty} \Gamma^*(Ap_t)(h_t)d\sigma : \sigma \in S(x)\right\}. \tag{3.4}$$

By (3.2) and the definition of Γ^*, for any stop rule r,

$$\Gamma(O)(x) = \sup\left\{\int \Gamma(Op_r)(h_r)d\sigma : \sigma \in S(x)\right\} \geq$$

$$\geq \sup\left\{\int \Gamma^*(Ap_r)(h_r)d\sigma : \sigma \in S(x)\right\}.$$

Replace r by the stop rule $t \wedge r$ to see that

$$\Gamma(O)(x) \geq \sup\left\{\int_{t\leq r} \Gamma^*(Ap_t)(h_t)d\sigma : \sigma \in S(x)\right\}.$$

Now take the supremum over r and apply Lemma 5.1 of [9] to get (3.4). □

COROLLARY 3.4. *If $A \subseteq H$, r is a stop rule, and $x \in X$, then*

$$\Gamma^*(A)(x) = \sup\left\{\int \Gamma^*(Ap_r)(h_r)d\sigma : \sigma \in S(x)\right\}. \tag{3.5}$$

4 – The measure of countable interesections

Call a Souslin set A *squeezable at x* if $\Gamma^*(A)(x) = \Gamma(A)(x)$ or, equivalently, if (1.2) holds. Our basic technique for proving that A is squeezable at x is to construct a closed set C inside A and a $\sigma \in S(x)$ such that $\sigma(C)$ is almost as large as $\Gamma^*(A)(x)$. Our main tool for these constructions is the result of this section.

The following assumptions are needed to state the result:

(i) $\{r_n\}$ is a sequence of stop rules such that $r_1(h) < r_2(h) < \dots$ for every $h \in H$.

(ii) $\{K_n\}$ is a sequence of clopen sets such that, for every n, K_n is determined by time r_n.

(iii) $0 \leq Q_0 \leq 1$ and, for $n \geq 1$, $Q_n : H \to [0,1]$ is determined by time r_n.

For the last two assumptions, fix $\varepsilon > 0, x \in X$ and set $q_n(h) = p_{r_n}(h)$ for every $h \in H$ and $n = 1,2\dots$.

(iv) $\sigma^1 \in S(x)$ and

$$\int_{K_1} Q_1 d\sigma^1 > Q_0 - \varepsilon/2.$$

(v) For every $h \in H$ and $n = 1,2,\dots,\overline{\sigma}^{n+1}(q_n(h)) \in S(h_{r_n}(h))$ and

$$\int_{K_{n+1}q_n(h)} (Q_{n+1}q_n(h))\, d\overline{\sigma}^{n+1}(q_n(h)) > Q_n(h) - \varepsilon/2^{n+1}$$

whenever $h \in \bigcap_{i=1}^{n} H_i$.

Proposition 4.1. *Let σ be the strategy which agrees with σ^1 prior to time r_1 and, for each $n \geq 1$ and $h \in H$, has a conditional strategy $\sigma[q_n(h)]$ which agrees with $\overline{\sigma}^{n+1}(q_n(h))$ prior to time $r_{n+1}[q_n(h)]$. Then $\sigma \in S(x)$ and*

$$\sigma\left(\bigcap_{n=1}^{\infty} K_n\right) \geq Q_0 - \varepsilon. \tag{4.1}$$

Proof. That $\sigma \in S(x)$ is clear from the definition of σ. The proof of the inequality can be reduced to Lemma 5.4 of [9]. The idea of the reduction is to replace X by the set X^* of finite sequences of members of X so that $Q_n(h) = Q_n(p_{r_n}(h))$ can be written in the form $Q^*(h^*_{r_n})$ in the new space. (A similar reduction is carried out in detail in [3]). Alternatively, one can easily imitate the proof in [9]. □

5 – Squeezing G_i's

Let $G^1, G^2, \ldots$ be open subsets of H and let $G = \cap G^n$.

PROPOSITION 5.1. *For every $x \in X, \Gamma^*(G)(x) = \Gamma(G)(x)$.*

The proof will be an application of Proposition 4.1. (The proof is analogous to that of Theorem 3 in Purves and Sudderth [9] which is a special case of Proposition 5.1) To start the construction of section 4, we need a lemma.

LEMMA 5.2. *Given $x \in X$, $\varepsilon > 0$, a stopping time t, and $A \subseteq [t < \infty]$, there is a strategy $\sigma = \bar{\sigma}(x, \varepsilon, A, t) \in S(x)$ and a stop rule $r = \bar{r}(x, \varepsilon, A, t)$ such that $r \leq t$ and*

$$\int_K \Gamma^*(Ap_r)(h_r)d\sigma > \Gamma^*(A)(x) - \varepsilon \tag{5.1}$$

where $K = [t = r]$ is a clopen subset of $[t < \infty]$ and is determined by time r.

PROOF. By Proposition 3.3, there exists $\sigma \in S(x)$ such that

$$\int_{t<\infty} \Gamma^*(Ap_t)(h_t)d\sigma > \Gamma^*(A)(x) - \varepsilon/2.$$

Now use the fact that $\sigma[t < \infty] = \sup\{\sigma[t = r] : r \text{ a stop rule}\}$ from [9, Lemma 5.1]. □

As was mentioned in section 2, for each of the open sets G^n, there is a stopping time t_n such that $G^n = [t_n < \infty]$. Also, we may assume for the proof of Proposition 5.1 that $G^1 \supseteq G^2 \supseteq \ldots$ and, by adding constants if necessary, that $t_1(h) < t_2(h) < \ldots$ for every h.

Fix $x \in X$ and $\varepsilon > 0$. We will use the lemma to define inductively $\{r_n\}, \{K_n\}, \{Q_n\}, \sigma^1$ and $\{\sigma^n\}$ satisfying properties (i) through (v) of section 4.

Set $Q_0 = \Gamma^*(G)(x)$. Since $G \subseteq [t_1 < \infty]$, we can apply Lemma 5.2 to get $r_1 = \bar{r}(x, \varepsilon/2, G, t_1)$, $K_1 = [t_1 = r_1]$, and $\sigma^1 = \bar{\sigma}(x, \varepsilon/2, G, t_1)$. Set

$Q_1(h) = \Gamma^*(Gq_1(h))(h_{r_1}(h))$ and notice property (iv) is an instance of (5.1).

Suppose now that $r_1, \ldots, r_n$; $K_1, \ldots, K_n$; $\sigma^1, \bar{\sigma}^2, \ldots, \bar{\sigma}^n$; and $Q_0, \ldots, Q_n$ have been defined and that $K_i = [t_i = r_i]$ for $i = 1, \ldots, n$. To define r_{n+1}, first suppose $h \in \bigcap_{i=1}^{n} K_i$.

Then

$$q_n(h) = p_{r_n}(h) = p_{t_n}(h) \text{ and } Gq_n(h) = \{h' : q_n(h)h' \in G\} \subseteq$$
$$\subseteq \{h' : t_{n+1}[q_n(h)](h') < \infty\}.$$

Use Lemma 5.2 to get

$$\bar{r}_n(q_n(h)) = \bar{r}\,(h_{r_n}(h), \varepsilon/2^{n+1}, Gq_n(h), t_{n+1}[q_n(h)]) .$$

If $h \notin \bigcap_{i=1}^{n} K_i$, let $\bar{r}_n(q_n(h)) = 1$. Now define

$$r_{n+1}(h) = r_n(h) + \bar{r}_n(q_n(h))(h_{r_n}(h) + 1, h_{r_n}(h) + 2, \ldots)$$

for every h. Set $K_{n+1} = [t_{n+1} = r_{n+1}]$ and notice that, for $h \in K_n = [t_n = r_n]$,

$$K_{n+1}q_n(h) = [t_{n+1}[q_n(h)] = \bar{r}_n[q_n(h))].$$

Set $Q_{n+1}(h) = \Gamma^*(Gq_{n+1}(h))(h_{r_{n+1}}(h))$. Finally, define

$$\bar{\sigma}^{n+1}(q_n(h)) = \bar{\sigma}\,(h_{r_n}(h), \varepsilon/2^{n+1}, Gq_n(h), t_{n+1}[q_n(h)]) \text{ , if } h \in \bigcap_{i=1}^{n} K_i$$

and let $\bar{\sigma}^{n+1}(q_n(h))$ be an arbitrary element of $S(h_{r_n}(h))$ if $h \notin \bigcap_{i=1}^{n} K_i$.

This completes the inductive definition and, with the aid of Lemma 5.2, properties (i) through (v) are easy to verify. So Proposition 4.1 gives a strategy $\sigma \in S(x)$ such that

$$\sigma\left(\bigcap_{n=1}^{\infty} K_n\right) \geq \Gamma^*(G)(x) - \varepsilon.$$

Furthermore

$$\bigcap_{n=1}^{\infty} K_n = \bigcap_{n=1}^{\infty}[t_n = r_n] \subseteq \bigcap_{n=1}^{\infty}[t_n < \infty] = G$$

so that

$$\Gamma(G)(x) \geq \sigma(G) \geq \Gamma^*(G)(x) - \epsilon.$$

This completes the proof of Proposition 5.1

6 – Squeezing $G_{\delta\sigma}$'s

A $G_{\delta\sigma}$ set is a countable union of countable intersections of open sets. The object of this section is to generalize Proposition 5.1 to such sets.

PROPOSITION 6.1. *If E is a $G_{\delta\sigma}$ subset of H and $x \in X$, then* $\Gamma^*(E)(x) = \Gamma(E)(x)$.

This result was proved in a countably additive setting in MAITRA, PURVES, and SUDDERTH [4]. The proof here will be similar, but there are some additional difficulties to overcome in the finitely additive case.

The proof will require a few lemmas and definitions. All of the sets occuring in the lemmas and definitions are assumed to be Borel subsets of H. (The results also hold for sets E which, like Borel sets, are in the domain of every strategy and have sections Ep, $p \in X^*$ with the same property.)

DEFINITION. *Say that E is Γ–null (Γ^*–null) if* $\Gamma(Ep)(x) = 0$ ($\Gamma^*(Ep)(x) = 0$) *for all $x \in X$, $p \in X^*$.*

The set functions $\Gamma^*(\cdot)(x)$ need not be countably subadditive, but do have a related property.

LEMMA 6.2. *If $E^1, E^2, \ldots$ are Γ^*-null then so is their union.*

PROOF. It suffices to show $\Gamma^*(\cup E^n)(x) = 0$ for a fixed x. (This is so because E^1p, $E^2p, \dots$ are Γ^*-null when $E^1, E^2, \dots$ are.) Let $\varepsilon > 0$.

For each $p \in X^*$ and $n = 1, 2, \dots$, choose an open set $O(p,n) \supseteq E^n p$ and such that $\Gamma(O(p,n))(l(p)) < \varepsilon/2^{|p|}$ where $l(p)$ denotes the last element of the finite sequence p and $|p|$ is the number of elements of p. Define

$$pO(p,n) = \{ph \colon h \in O(p,n)\},$$

and set

$$O^n = \cup\{pO(p,n) \colon |p| = n\}, \quad O = \cup O^n.$$

Then $O \supseteq \cup E^n$ because $O^n \supseteq E^n$ for every n.

Now for $\sigma \in S(x)$ and $p \in X^*$ with $|p| = n$,

$$\sigma[p](O^n p) = \sigma[p](O(p,n)) \leq \Gamma(O(p,n))(l(p)) < \varepsilon/2^n.$$

By Lemma 5.2 of [8], $\sigma(O) \leq \varepsilon$. Hence,

$$\Gamma^*(\cup E^n)(x) \leq \Gamma(O)(x) \leq \varepsilon.$$

□

LEMMA 6.3. *A $G_{\delta\sigma}$ set E which is Γ-null is also Γ^*-null.*

PROOF. Write $E = \cup E^n$ where the E^n are G_δ's. Every E^n is Γ-null because E is. By Proposition 5.1, every $E^n p$ is squeezable at every x. Hence, every E^n is Γ^*-null. Now use Lemma 6.2. □

Call a subset E of H *ε-squeezable at x* if $\Gamma^*(E)(x) < \Gamma(E)(x) + \varepsilon$.

LEMMA 6.4. *If D is ε-squeezable at x and $\Gamma^*(N)(x) = 0$, then $D \cup N$ is 2ε-squeezable at x.*

PROOF. Choose open sets O_1, O_2 such that $O_1 \supseteq D, O_2 \supseteq N$ and

$$\Gamma(O_1)(x) < \Gamma(D)(x) + 3\varepsilon/2, \Gamma(O_2)(x) < \varepsilon/2.$$

Then

$$\Gamma(O_1 \cup O_2)(x) \leq \Gamma(O_1)(x) + \Gamma(O_2)(x) < \Gamma(D)(x) + 2\varepsilon \leq$$
$$\leq \Gamma(D \cup N)(x) + 2\varepsilon.$$

□

For $E \subseteq H$ and $0 < \varepsilon < 1$, define $A_\varepsilon = A_\varepsilon(E)$ by

$$A_\varepsilon = \{p \in X^* : \Gamma(Ep)(l(p)) > 1 - \varepsilon\},$$

where $l(p)$ denotes the last element of p. Also define a stopping time $\tau_\varepsilon = \tau_\varepsilon(E)$ by

$$\tau_\varepsilon(h) = \inf\{n : p_n(h) \in A_\varepsilon\}$$

for $h \in H$.

LEMMA 6.5. *$E \cap [\tau_\varepsilon = \infty]$ is Γ-null.*

PROOF. Let $p = (x_1, \ldots, x_n) \in X^*$. If $\tau_\varepsilon(x_1, \ldots, x_n, \ldots) \leq n$, then

$$(E \cap [\tau_\varepsilon = \infty])p \subseteq [\tau_\varepsilon = \infty]p = \emptyset$$

and obviously

$$\Gamma((E \cap [\tau_\varepsilon = \infty])p)(x) = 0$$

for every x. If $\tau_\varepsilon(x_1, \ldots, x_n, \ldots) > n$, then the conditional stopping time $\tau_\varepsilon[p]$ is just $\tau_\varepsilon(Ep)$ and

$$(E \cap [\tau_\varepsilon = \infty])p = Ep \cap [\tau_\varepsilon(Ep) = \infty].$$

The final set is again of the form $E \cap [\tau_\varepsilon = \infty]$. So it sufficies to show $\Gamma(E \cap [\tau_\varepsilon = \infty])(x) = 0$ for every x. Suppose not. Then there is an x and a $\sigma \in S(x)$ such that $E \cap [\tau_\varepsilon = \infty]$ has positive measure under σ. By the finitely additive Levy zero-one law [10] there is an $h \in H$ and a positive integer n such that

$$\sigma[p_n(h)]((E \cap [\tau_\varepsilon = \infty])p_n(h)) > 1 - \varepsilon. \tag{6.1}$$

The set $[\tau_\varepsilon = \infty]p_n(h)$ must be nonempty, because it is a superset of a set of positive measure. This implies that $p_i(h) \notin A_\varepsilon$, $i = 1, \ldots, n$. (If, for some $i \leq n$, $(h_1, \ldots, h_i) \in A_\varepsilon$, then $\tau_\varepsilon(p_n(h)h')$ would be finite for all $h' \in H$ and $[\tau_\varepsilon = \infty])\, p_n(h)$ would be empty.)

On the other hand, by (6.1) and the fact that $\sigma[p_n(h)] \in S(h_n)$,

$$1 - \varepsilon < \sigma[p_n(h)](Ep_n(h)) \leq \Gamma(Ep_n(h))(h_n).$$

So $p_n(h) \in A_\varepsilon$. This is a contradiction. □

Lemma 6.6. *$E \cap [\tau_\epsilon < \infty]$ is ϵ-squeezable at every x.*

Proof. To simplify notation, write τ for τ_ϵ. The open set $[\tau < \infty]$ contains $E \cap [\tau < \infty]$. So it suffices to show that

$$\Gamma[\tau < \infty](x) \leq \Gamma(E \cap [\tau < \infty])(x) + \epsilon.$$

Now, on the set $[\tau < \infty]$, $\Gamma(Ep_\tau)(h_\tau) > 1 - \epsilon$. So, by Proposition 3.1,

$$\Gamma(E \cap [\tau_\epsilon < \infty])(x) = \sup \left\{ \int_{\tau<\infty} \Gamma(Ep_\tau)(h_\tau) d\sigma : \sigma \in S(x) \right\} \geq$$

$$\geq \sup \{(1-\epsilon)\sigma[\tau < \infty] : \sigma \in S(x)\}$$

$$= (1-\epsilon)\Gamma[\tau < \infty](x).$$

□

Here, at last, is the proof of the main result of this section.

Proof of Proposition 6.1. Fix $x \in X$ and $\epsilon \in (0,1)$. Set $D = E \cap [\tau_\epsilon < \infty]$ and $N = E \cap [\tau_\epsilon = \infty]$. Then D is ϵ-squeezable by Lemma 6.6. Also, N is Γ-null by Lemma 6.5 and is a $G_{\delta\sigma}$ because $[\tau_\epsilon = \infty]$ is closed. So, by Lemma 6.3, N is Γ^*-null. Apply Lemma 6.4 to see that $E = D \cup N$ is 2ϵ-squeezable. Since ϵ is arbitrary, the proof is complete.
□

7 – The "going up" property for Γ^*

Assume throughout this section that

$$A^1 \subseteq A^2 \subseteq \dots$$

is a sequence of subsets of H. If X is countable and all measures are countably additive, then, by [4, Corollary 2.10],

$$\Gamma^*(\cup A^n)(x) = \sup_n \Gamma^*(\cup A^n)(x)$$

for every x. This equality can fail quite easily in the presence of finitely additive measures. However, the main result of this section is an appropriate analogue.

PROPOSITION 7.1. *For every $x \in X$,*

$$\Gamma^*(\cup A^n)(x) = \sup\{\Gamma^*(A^r)(x) \colon r \text{ a stop rule }\}.$$

Here $A^r = \{h \colon h \in A^{r(h)}\}$.

The proof uses several lemmas. The first two are about squeezing sets uniformly in x. Their proof uses the following notation: For $p \in X^*$ and $B \subseteq H$, set $pB = \{ph \colon h \in B\}$.

LEMMA 7.2. *Given $A \subseteq H$ and $\varepsilon > 0$, there is an open set $O \supseteq A$ such that, for all $x \in X$,*

$$\Gamma(O)(x) \le \Gamma^*(A)(x) + \varepsilon.$$

PROOF. For each $y \in X$, choose $O(y)$ open with $O(y) \supseteq Ay$ and

$$\Gamma(O(y))(y) \le \Gamma^*(Ay)(y) + \varepsilon.$$

Define

$$O = \bigcup_y yO(y)$$

Apply Corollaries 3.2 and 3.4 with $r = 1$ to see that

$$\begin{aligned}\Gamma(O)(x) &= \sup\left\{\int \Gamma(O(y))(y)\gamma(dy) \colon \gamma \in \Gamma(x)\right\} \le \\ &\le \sup\left\{\int \Gamma^*(Ay)(y)\gamma(dy) \colon \gamma \in \Gamma(x)\right\} + \varepsilon = \\ &= \Gamma^*(A)(x) + \varepsilon.\end{aligned}$$

□

LEMMA 7.3. *Given $A \subseteq H$, there exists a G_δ set $G \supseteq A$ such that $\Gamma^*(Ap)(x) = \Gamma(Gp)(x)$ for all $p \in X^*$ and $x \in X^*$.*

PROOF. Let $\varepsilon > 0$. For each p, use Lemma 7.2 to get an open set $O(p) \supseteq Ap$ such that, for all x,

$$\Gamma(O(p))(x) \leq \Gamma^*(Ap)(x) + \varepsilon.$$

Let O^n be the open set

$$O^n = \cup\{pO(p) : |p| = n\}$$

and define

$$G^\varepsilon = \cap O^n.$$

Then $G^\varepsilon \supseteq A$ and, for every x and every p of length n,

$$\Gamma(G^\varepsilon p)(x) \leq \Gamma(O^n p)(x) = \Gamma(O(p))(x) \leq \Gamma^*(Ap)(x) + \varepsilon.$$

Finally, take

$$G = \bigcap_m G^{1/m}.$$

□

The next two lemmas are concerned with the uniform squeezing of all the A^n.

LEMMA 7.4. *There exist G_δ sets $G^1, G^2, \ldots$ such that, for all n, p, and x,*

(i) $G^n \subseteq G^{n+1}$

(ii) $A^n \subseteq G^n$

(iii) $\Gamma(G^n p)(x) = \Gamma^*(A^n p)(x)$.

PROOF. Use Lemma 7.3 to get G_δ's $\tilde{G}^n$ satisfying (ii) and (iii). Then let

$$G^n = \bigcap_{k \geq n} \tilde{G}^k.$$

□

A sequence $G = (G^1, G^2, \ldots)$ of G_i's as in Lemma 7.4 is called a *uniform squeeze* for $A = (A^1, A^2, \ldots)$.

LEMMA 7.5. *If G is a uniform squeeze of A, then*

$$\Gamma(G^r p)(x) = \Gamma^*(A^r p)(x)$$

for all stop rules r, $p \in X^*$, and $x \in X$.

PROOF. The proof is by induction on the structure of r [2, sections 2.7 and 2.9]. If r has structure zero, the desired equality is just property (iii) of the previous lemma. So assume r has structure $\alpha > 0$ and that the equality holds for stop rules of smaller structure.

By corollaries 3.2 and 3.4,

$$\Gamma(G^r p)(x) = \sup\left\{\int \Gamma(G^r py)(y)\gamma(dy) : \gamma \in \Gamma(x)\right\},$$

and

$$\Gamma^*(A^r p)(x) = \sup\left\{\int \Gamma^*(A^r py)(y)\gamma(dy) : \gamma \in \Gamma(x)\right\}.$$

So it sufficies to show that

$$\Gamma(G^r py)(y) = \Gamma^*(A^r py)(y) \tag{7.1}$$

for every $p = (x_1, \ldots, x_n)$ and y.

Consider first the case where

$$r(py\ldots) = r(x_1, \ldots, x_n, y, \ldots) = k \leq n + 1.$$

Then $G^r py = G^k py$ and $A^r py = A^k py$ so that (7.1) is an instance of Lemma 7.4 (iii). Next suppose

$$r(py\ldots) > n + 1$$

and recall that $r[py]$ is the stop rule given by

$$r[py](h) = r(pyh) - n - 1.$$

Introduce new sequences of sets

$$\tilde{G} = (G^{n+2}py, G^{n+3}py, \ldots)$$
$$\tilde{A} = (A^{n+2}py, A^{n+3}py, \ldots)$$

Now $\tilde{G}$ is a uniform squeeze for $\tilde{A}$ and it can be checked that

$$\tilde{G}^{r[py]} = (G^r)py\,,$$
$$\tilde{A}^{r[py]} = (A^r)py\,.$$

Equality (7.1) now follows from the inductive hypothesis because $r[py]$ has structure smaller than α. □

Our final lemma gives a "going up" property for Γ.

LEMMA 7.6. *Let $B^1 \subseteq B^2 \subseteq \ldots$ be Borel subsets of Π and let $x \in X$. Then*

$$\Gamma(\cup B^n)(x) = \sup\{\Gamma(B^r)(x)\colon r \text{ a stop rule}\}\,.$$

PROOF. Since $\cup B^n \supseteq B^r$ for every r, one inequality is obvious. The other inequality follows easily from Theorems 5.1 and 5.2 of [8]. □

PROOF OF PROPOSITION 7.1. Clearly,

$$\Gamma^*(\cup A^n)(x) \geq \sup_r \Gamma^*(A^r)(x)$$

because $\cup A^n \supseteq A^r$ for every r. To prove the opposite inequality, let $G = (G^1, G^2, \ldots)$ be a sequence of G_δ's which is a uniform squeeze for A. Then

$$\Gamma^*(\cup A^n)(x) \leq \Gamma(\cup G^n)(x) = \sup_r \Gamma(G^r)(x) = \sup_r \Gamma^*(A^r)(x),$$

where the inequality is from Proposition 6.1 and the equalities are by Lemma 7.6 and 7.5, respectively. □

8 – The proof of Theorem 1.1

Suppose that, for each finite sequence β of positive integers, $B(\beta)$ is a clopen subset of Π. Define

$$A = \bigcup_{\alpha} \bigcap_{n=1}^{\infty} B(\alpha_1, \dots, \alpha_k) \tag{8.1}$$

where the union is over all infinite sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ of positive integers. Because every closed subset of Π is a countable intersection of clopen sets, every Souslin set is of this form (cf. exercise I.2.2 of NEVEU [6]). Assume, without loss of generality that

$$B(\alpha_1, \dots, \alpha_n) \supseteq B(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \tag{8.2}$$

for every α and n.

Fix $x_0 \in X$ and $\varepsilon > 0$. It is enough to construct $\sigma \in S(x_0)$ such that

$$\sigma(A) \geq \Gamma^{*}(A)(x_0) - \varepsilon. \tag{8.3}$$

The construction uses Proposition 4.1 and techniques of Sierpinski [11, pp. 48-50] which have also been used to prove Choquet's capacitability theorem.

For each finite sequence $\beta = (\beta_1, \dots, \beta_k)$ of positive integers, introduce

$$A(\beta) = \bigcup_{i_1=1}^{\beta_1} \cdots \bigcup_{i_k=1}^{\beta_k} \bigcup_{\alpha} \bigcap_{n=1}^{\infty} B(i_1, \dots, i_k, \alpha_i, \dots, \alpha_n). \tag{8.4}$$

For m a positive integer, let βm be the concatenation $(\beta_1, \dots, \beta_k, m)$. Then clearly

$$A(\beta) = \bigcup_{m=1}^{\infty} A(\beta m), \tag{8.5}$$

and

$$A(\beta m) \uparrow A(\beta).$$

If $s_1, \ldots, s_k$ are stop rules, let

$$A(s_1, \ldots, s_k) = \{h : h \in A(s_1(h), \ldots, s_k(h))\}.$$

In order to apply Proposition 4.1, we must define $\{r_n\}, \{K_n\}, \sigma^1$, $\{\overline{\sigma}^n\}$, and $\{Q_n\}$. As part of the inductive definition, we will also need another sequence of stop rules $\{s_n\}$.

To start the induction, set $Q_0 = \Gamma^*(A)(x_0)$ and use (8.5) with β taken to be the empty sequence to see that

$$A(m) \uparrow A.$$

So, by Proposition 7.1, there is a stop rule s_1 such that

$$\Gamma^*(A(s_1))(x_0) > Q_0 - \varepsilon/4. \tag{8.6}$$

Let

$$K_1 = \bigcup_{i=1}^{s_1} B(i).$$

By (8.2) and (8.4), $K_1 \supseteq A(s_1)$. Check also that K_1 is clopen and let r_1 be a stop rule such that $s_1 \leq r_1$ and K_1 is dermined by time r_1. Use Corollary 3.4 to get $\sigma^1 \in S(x_0)$ such that

$$\int \Gamma^*(A(s_1)p_{r_1})(h_{r_1})d\sigma^1 > \Gamma^*(A(s_1))(x_0) - \varepsilon/4. \tag{8.7}$$

Because $K_1 \supseteq A(s_1)$ and is determined by time r_1,

$$\int_{K_1} \Gamma^*(A(s_1)p_{r_1})(h_{r_1})d\sigma^1 = \int \Gamma^*(A(s_1)p_{r_1})(h_{r_1})d\sigma^1. \tag{8.8}$$

Define $Q_1(h) = \Gamma^*(A(s_1)p_{r_1}(h))(h_{r_1}(h))$ and notice that property (iv) of section 4 is a consequence of (8.6), (8.7), and (8.8).

Suppose now that $r_1, \ldots, r_n; \sigma^1; \overline{\sigma}^2, \ldots, \overline{\sigma}^n; K_1, \ldots, K_n; Q_1, \ldots, Q_n$; and $s_1, \ldots, s_n$ have been defined. Assume properties (i) through (v) of section 4 hold and also that, for $k = 1, \ldots, n$.

$$s_k \leq r_k,$$

$$(8.9) \qquad K_k = \bigcup_{i_1=1}^{s_1} \cdots \bigcup_{i_k=1}^{s_k} B(i_1,\ldots,i_k)$$

$$Q_k(h) = \Gamma^*(A(s_1,\ldots,s_k)q_k(h))(h_{r_k}(h))$$

for all $h \in H$, where $q_k = p_{r_k}$.

For the inductive step, notice that, for every h,

$$A(s_1(h),\ldots,s_n(h),m)q_n(h) \uparrow A(s_1(h),\ldots,s_n(h))q_n(h).$$

By proposition 7.1, there is a stop rule $\overline{r}_n(q_n(h))$ such that

$$(8.10) \qquad \begin{aligned} &\Gamma^*(A(s_1(h),\ldots,s_n(h),\overline{r}(q_n(h))q_n(h))(h_{r_n}(h)) > \\ &>\Gamma^*(A(s_1(h),\ldots,s_n(h))q_n(h))(h_{r_n}(h)) - \epsilon/2^{n+2}. \end{aligned}$$

Define

$$s_{n+1}(h) = r_n(h) + \overline{r}_n(q_n(h))(h_{r_n}(h)+1, h_{r_n}(h)+2,\ldots)$$

and notice that

$$(8.11) \qquad A(s_1,\ldots,s_{n+1})q_n(h) \supseteq A(s_1,\ldots,s_n,\overline{r}(q_n(h))q_n(h)).$$

Next use (8.9) with $k = n+1$ to define K_{n+1} and Q_{n+1}. It is easy to check that K_{n+1} is clopen and, by (8.2) and (8.9),

$$(8.12) \qquad A(s_1,\ldots,s_{n+1}) \subseteq K_{n+1} \subseteq K_n.$$

Choose a stop rule r_{n+1} strictly larger than the maximum of r_n with s_{n+1} such that K_{n+1} is determined by time r_{n+1}. Then use Corollary 3.4 to choose, for every $h, \overline{\sigma}^{n+1}(q_n(h)) \in S(h_{r_n}(h))$ so that

$$(8.13) \qquad \begin{aligned} &\int \Gamma^*(A(s_1,\ldots,s_{n+1})q_n(h)p_{r_{n+1}}[q_n(h)])(h'_{r_{n+1}}[q_n(h)])d\overline{\sigma}^{n+1}(q_n(h))(h') > \\ &>\Gamma^*(A(s_1,\ldots,s_{n+1})q_n(h))(h_{r_n}(h)) - \epsilon/2^{n+2}. \end{aligned}$$

The integral above would be the same if taken over the set $K_{n+1}q_n(h)$ because this set contains $A(s_1,\ldots,s_{n+1})q_n(h)$ by (8.12) and is determined by time $r_{n+1}[q_n(h)]$.

Property (v) of section 4 now follows from (8.10), (8.11), (8.13) and the definition of Q_n in (8.9). Properties (i), (ii) and (iii) are clear from the construction and (iv) was checked above. So, by Proposition 4.1, there is a $\sigma \in S(x_0)$ such that

$$\sigma\left(\bigcap_{i=1}^{\infty} K_i\right) \geq Q_0 - \varepsilon.$$

The proof of (8.3), as well as that of Theorem 1.1, will be complete once we show that

$$A \supseteq \bigcap_{i=1}^{\infty} K_i.$$

To see this, fix $h \in \cap K_i$ and let S be the set of finite sequences of positive integers given by

$$S = \cup\{(\alpha_1,\ldots,\alpha_k) \colon \alpha_i \leq s_i(h), i = 1,\ldots,k, \text{ and } h \in B(\alpha_1,\ldots,\alpha_k)\}.$$

Then, by (8.2), S is closed under initial segments. Also, for every $k \geq 1$, S contains a sequence of length k because $h \in K_k$. Thus König's lemma applies to give an infinite sequence $\alpha_1, \alpha_2, \ldots$ such that $h \in B(\alpha_1,\ldots,\alpha_k)$ for all k. So $h \in A$. □

9 – Squeezing functions

Let g be a bounded, real-valued function defined on H which is *upper Souslin* in the sense that, for every real number a, $\{h\colon g(h) > a\}$ is a Souslin set. For example, g is upper Souslin if g is Borel measurable. Define

$$(\Gamma g)(x) = \sup\left\{\int g d\sigma \colon \sigma \in S(x)\right\}$$

for each $x \in X$. (The integral of g with respect to every σ is well-defined because g can be uniformly approximated by a linear combination of indicator functions of Souslin sets.) In view of Theorem 1.1 and classical results for functions, one expects that, given $\varepsilon > 0$, there will be a lower semi-continuous function $f \geq g$ such that $(\Gamma f)(x) \leq (\Gamma g)(x) + \varepsilon$. However, a simple counterexample in [5] shows that such an f need not exist. To obtain a squeezing result for functions, we introduce a new collection.

Say that the bounded function $f\colon H \to R$ is *upper* G_δ, if for every real number a, $\{h\colon g(h) > a\}$ is the intersection of a countable collection of open sets.

THEOREM 9.1. *If g is upper Souslin, then*

$$\Gamma g = \inf\{\Gamma f\colon f \geq g, f \text{ is upper } G_\delta\}.$$

We omit the proof because it is similar to and slightly simpler than that of an analogous result in the countably additive theory [5, Theorem 10.1].

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INDIRIZZO DEGLI AUTORI:

A. Maitra - W. Sudderth - School of Statistics - University of Minnesota - 270 Vincent Hall - 206 Church Street S.E. - Minneapolis, Minnesota 55455

R. Purves - Department of Statistics - University of California - Berkeley, California 94720