Rendiconti di Matematica, Serie VII
Volume 11 , Roma (1991), 843-847

Some Remarks on Transitive Collineation Groups of Finite Projective Planes

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RIASSUNTO – *In* [3] *noi abbiamo generalizzato alcuni risultati di* J. ANDRÉ *(si veda* [1]*) e di* H. KARZEL *(si veda* [2]*), ed abbiamo provato una caratterizzazione riguardante i gruppi transitivi di collineazione di piani proiettivi finiti. In questo lavoro noi semplifichiamo quel risultato e grazie alla semplificazione conseguita miglioriamo un'altra caratterizzazione dovuta a* G. ZAPPA *(si veda* [5]*).*

ABSTRACT – *In* [3] *we generalized some results of* J. ANDRÉ *(see* [1]*) and* H. KARZEL *(see* [2]*), and proved a characterization of transitive collineation groups of finite projective planes. In this paper we simplify this result and consequently improve another characterization given in* [5].

KEY WORDS – *Finite automorphism groups of geometrics structures - Affine and projective planes.*

A.M.S. CLASSIFICATION: 20B25 - 51E15

Let B be a subset and Γ a subgroup of a group G. In [3] we said that B is a right (left) Γ-block of G if:

(1) $$\Gamma \subseteq B;$$

(2) $$\forall\, x, y \in B : Bx^{-1} = By^{-1} \quad \text{or} \quad Bx^{-1} \cap By^{-1} = \Gamma \Big(x^{-1}B = y^{-1}B \quad \text{or} \quad x^{-1}B \cap y^{-1}B = \Gamma \Big).$$

If B is a right Γ-block but not a subgroup of G, then Γ is the left stabilizer of B in G (see [3], remark 12).

REMARK 1. If $B \subseteq G$ and Γ and $\overline{\Delta}$ are the left and the right stabilizers of B in G respectively, then B is a right Γ-block of G if and only if B is a left $\overline{\Delta}$-block of G (cfr. propos. 13 of [3]).

In theorem 10 of [3] we proved a theorem of which we give here (see theorem 1), for the reader's facility, a simpler version and a simpler proof.

THEOREM 1. *Let B be a right Γ-block of a group G, $\overline{\Delta}$ the right stabilizer of B in G, let $P := \{\Gamma x | x \in G\}$, $\mathcal{B} := \{\Gamma x \in P | \Gamma x \subset B\}$ and $\mathcal{L} := \{\mathcal{B} \cdot g | g \in G\}$ where $\mathcal{B} \cdot g := \{\Gamma x g | \Gamma x \in \mathcal{B}\}$.*

Then $(P, \mathcal{L})$ is a projective plane of order n, $n \in \mathbb{N}$, if and only if

j1) $$(G\colon \Gamma) = n^2 + n + 1\,,$$

j2) $$(G\colon \overline{\Delta}) = n^2 + n + 1\,,$$

ji) $$|\mathcal{B}| = n + 1\,.$$

PROOF. It is obvious that if $(P, \mathcal{L})$ is a projective plane of order n then j1), j2) and ji) are true. Conversely let j1), j2) and ji) hold, moreover let $L := \{g \in G | \forall\ \Gamma x \in P\colon \Gamma x g = \Gamma x\}$. Then $L \trianglelefteq G$, $L \trianglelefteq \overline{\Delta} \cap \Gamma$ and since $(G\colon \Gamma) = n^2 + n + 1 \in \mathbb{N}$, $(G\colon L)$ is finite. Therefore if $G' := G/L$, $\Gamma' := \Gamma/L$, $\overline{\Delta'} := \overline{\Delta}/L$, then $B' := B/L = \{Lb | b \in B\}$ is a right Γ'-block of the finite group G' with $(G' : \Gamma') = n^2 + n + 1$, $|\{\Gamma' x | x \in B'\}| = n + 1$, $\overline{\Delta'}$ is the right stabilizer of B', $(G'\colon \overline{\Delta'}) = (G\colon \overline{\Delta}) = (G\colon \Gamma) = (G'\colon \Gamma')$ and $|\overline{\Delta'}| = |\Gamma'|$; thus B' includes exactly $n + 1$ left cosets of $\overline{\Delta'}$. As a consequence, B includes exactly $n + 1$ left cosets of $\overline{\Delta}$, and hence the set $\mathcal{A}$ of all the right Γ-blocks of the type Bb^{-1}, with $b \in B$, has $n + 1$ elements. From this our assertion follows immediately. In fact (since $|P| = n^2 + n + 1$, and $|\ell| = n + 1$ for every $\ell \in \mathcal{L}$, and $B_1 \cap B_2 = \Gamma$ for every pair of different elements $B_1, B_2 \in \mathcal{A}$) $\bigcup \mathcal{A} = G$, $(P, \mathcal{L})$ is a linear space and $|\ell_1 \cap \ell_2| = 1$ for every pair of different elements $\ell_1, \ell_2 \in \mathcal{L}$. □

THEOREM 2. *If B is a right Γ-block of a group G, then* j1) *and* ji) *implies* j2).

PROOF. Let $L, G', \Gamma', \overline{\Delta'}$ and B' as in the proof of theorem 1. Thus $(G': \overline{\Delta'}) = (G: \overline{\Delta})$; moreover, since $B'\overline{\Delta'} = B'$, $|\overline{\Delta'}|$ divides $|B'| = |\Gamma'| \cdot (n+1)$, hence $|\overline{\Delta'}|$ divides $|\Gamma'| = |G'| - |\Gamma'| \cdot (n+1) \cdot n$. Therefore, if $\nu + 1$ is the number of the left cosets of $\overline{\Delta'}$ included in B', then $n+1$ is a divisor of $\nu + 1$.

Now, since $\overline{\Delta'}$ is the right stabilizer of B' in G', the set $\{B'b^{-1}|b \in B'\}$ has exactly $\nu + 1$ elements; moreover, since B' is a right Γ'-block of G', if $B'x^{-1} \neq B'y^{-1}$ (where $x, y \in B'$), then $B'x^{-1} \cap B'y^{-1} = \Gamma'$; therefore $|\Gamma'| \cdot n \cdot (n+1) + |\Gamma'| = |G'| \geq |\cup \{B'b^{-1}|b \in B'\}| = |\Gamma'| \cdot n \cdot (\nu + 1) + |\Gamma'|$, and hence $n = \nu$. □

Now let G be a group, Γ a subgroup of G with $(G: \Gamma) = n^2 + n + 1$, $P := \{\Gamma x | x \in G\}$ the set of "points", and $\mathcal{B} := \{\Gamma\sigma_1, \ldots, \Gamma\sigma_{n+1}\}$ a subset of P with $|\mathcal{B}| = n+1$. We set $B := \cup\mathcal{B}$, $K_{\ell m} := \sigma_\ell^{-1}\Gamma\sigma_m$ for $\ell, m \in \{1, 2, \ldots, n+1\}$, $\Delta := \{\delta \in G | \exists\, i, j, \ell, m \in \{1, \ldots, n+1\}, (i,j) \neq (\ell, m) : \delta \in K_{ij} \cap K_{\ell m}\}$, $\overline{\Delta} := \{x \in G | Bx = B\}$, $\mathcal{L} := \{\mathcal{B} \cdot g | g \in G\}$ and call the elements of $\mathcal{L}$ lines.

G. ZAPPA proved in [5]:

The pair $(P, \mathcal{L})$ is a projective plane if and only if the following conditions are valid:

a) $G = \cup\left\{K_{\ell m} | \ell, m \in \{1, 2, \ldots, n+1\}\right\}$

b) Δ is a subgroup with $(G: \Delta) = n^2 + n + 1$

c) $\Delta \subset \overline{\Delta}$

By b) and c) we have $i := (G: \overline{\Delta}) \leq n^2 + n + 1$.

The result of Zappa can be improved:

THEOREM 3. *$(P, \mathcal{L})$ is already a projective plane if the condition* c) *is valid or the condition* a) *and* $i := (G: \overline{\Delta}) \leq n^2 + n + 1$.

PROOF. Since for each $g \in G$ the map $\hat{g}: P \ni \Gamma x \longrightarrow \Gamma x g \in P$ is an automorphism of the structure $(P\mathcal{L})$, we may assume $\sigma_1 = 1$, hence $1 \in \Gamma \subset B$.

Now let $x, y \in B$; then by $\Gamma B = B$, $\Gamma x, \Gamma y \subset B$, hence $\Gamma x, \Gamma y \in \mathcal{B}$, i.e. $\Gamma x = \Gamma\sigma_i$, $\Gamma y = \Gamma\sigma_j$ for some $i, j \in \mathbb{Z}_{n+1} := \{1, 2, \ldots, n+1\}$. Therefore $z := x^{-1}y \in K_{ij}$ and $\Gamma x z = \Gamma y \in \mathcal{B} \cap \mathcal{B}z$.

We have for $m \in \mathbb{Z}_{n+1}: \Gamma\sigma_m \in \mathcal{B}z \iff \exists\, \ell \in \mathbb{Z}_{n+1}: \Gamma\sigma_m = \Gamma\sigma_\ell z \iff \exists\, \ell \in \mathbb{Z}_{n+1}: z \in \sigma_\ell^{-1}\Gamma\sigma_m = K_{\ell m}$. Consequently $\mathcal{B} \cap \mathcal{B}z =$

$\{\Gamma\sigma_m \in B | \exists\, \ell \in \{1,\dots,n+1\} : z \in K_{\ell m}\} \ni \Gamma y = \Gamma\sigma_j$. Further: $Bx^{-1} \neq By^{-1} \iff Bz \neq B \iff z \in K_{ij} \setminus \overline{\Delta}$. If we assume c), i.e. $\Delta \subset \overline{\Delta}$, then $z \in K_{ij} \setminus \overline{\Delta}$ implies $z \notin \Delta$, hence (by definition of Δ) $z \notin K_{\ell m}$ for all $K_{\ell m} \neq K_{ij}$ and so $B \cap Bz = \Gamma\sigma_j$. Therefore: $Bx^{-1} \neq By^{-1} \implies Bx^{-1} \cap By^{-1} = \Gamma\sigma_j \cdot y^{-1} = \Gamma$, i.e. the condition (2) is valid and so B is a right Γ-block. Since $(G\colon \Gamma) = n^2+n+1$ and $|B| = n+1$ we have, by Th. 2, $(G\colon \overline{\Delta}) = n^2+n+1$; hence, by Th. 1, $(P,\mathcal{L})$ is a projective plane.

Now we assume a) and $i = (G\colon \overline{\Delta}) \leq n^2+n+1$. As in the proof of Th. 1 we proceed to the finite factor structure $G/L, \Gamma/L, \dots$, i.e. we may assume that G is finite. Then $|B| = |\Gamma|\cdot(n+1)$, $|G| = |\Gamma|\cdot(n^2+n+1) = |\Gamma|\cdot(n+1)n + |\Gamma| = |B|n + |\Gamma| = |\overline{\Delta}|\cdot i$ and by $B\overline{\Delta} = B$, $|\overline{\Delta}|\,\big|\,|B|$. Consequently $|\overline{\Delta}|$ is a divisor of $|\Gamma| = |G| - |B|n$, and since $i \leq n^2+n+1$, we have $|\overline{\Delta}| = |\Gamma|$ and $i = n^2+n+1$. If $\overline{\Gamma}$ denotes the left stabilizer of B, then $\Gamma \subset \overline{\Gamma}$, and $|\overline{\Gamma}|$ is a divisor of $|B|$ and of $|G|$, hence of $|\Gamma| = |G| - |B|\cdot n$. Consequently $\Gamma = \overline{\Gamma}$ and $|\Gamma| = |\overline{\Gamma}| = |\overline{\Delta}|$.

Now let $x, y \in B$ and $M = \cup\{x^{-1}B | x \in B\}$; then $\overline{\Delta} \subset x^{-1}B \cap y^{-1}B$ (since $1 \in x^{-1}B \cap y^{-1}B$ and $B\overline{\Delta} = B$) and "$x^{-1}B = y^{-1}B \iff xy^{-1} \in \Gamma$". Therefore $|\{x^{-1}B | x \in B\}| = \frac{|B|}{|\Gamma|} = n+1$, $|M| \leq (n+1)\big(|B| - |\overline{\Delta}|\big) + |\overline{\Delta}| = (n+1)\cdot\big((n+1)\cdot|\Gamma| - |\Gamma|\big) + |\Gamma| = |\Gamma|\cdot(n^2+n+1)$ and if there are $x, y \in B$ with $xy^{-1} \notin \Gamma$ and $x^{-1}B \cap y^{-1}B \neq \overline{\Delta}$, then $|M| < |\Gamma|\cdot(n^2+n+1)$, i.e. $M \neq G$.

But by a) $G = \cup\{\sigma_i^{-1}\Gamma\sigma_j | i,j \in \mathbb{Z}_{n+1}\} = \cup\{\sigma_i^{-1}B | i \in \mathbb{Z}_{n+1}\} \subset M$. Thus $x^{-1}B \cap y^{-1}B = \overline{\Delta}$ for all $x, y \in B$ with $xy^{-1} \notin \Gamma$, i.e. B is a left $\overline{\Delta}$-block of G and so by remark 1 and $\overline{\Gamma} = \Gamma$, B is a right Γ-block. This together with $(G\colon \Gamma) = (G\colon \overline{\Delta}) = n^2+n+1$ and $|B| = n+1$ implies by Th. 1 that $(P,\mathcal{L})$ is a projective plane.

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Lavoro pervenuto alla redazione il 7 novembre 1990
ed accettato per la pubblicazione il 29 gennaio 1991
su parere favorevole di H. Karzel e di P.V. Ceccherini

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