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The System of Principal Connections

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RIASSUNTO – *Si studiano gli aspetti essenziali della teoria dei sistemi di connessioni e di sovraconnessioni nel caso dei fibrati principali, utilizzando l'algebra di Frölicher-Nijenhuis di forme a valori tangenti [3,4,6,7,11,12,14] già esaminata, in tale contesto, in [1]. Viene discussa la relazione tra gli approcci orizzontale e verticale e l'approccio tradizionale basato sulle forme a valori nell'algebra di Lie.*

ABSTRACT – *We study the basic aspects of the theory of systems of connections and overconnections for the case of principal bundles, using the Frölicher-Nijenhuis algebra of tangent-valued forms [3, 4, 6, 7, 11, 12, 14], which was examined in this context in [1]. We discuss the relation between the horizontal and vertical approaches and the traditional approach based on Lie algebra-valued forms.*

KEY WORDS – *Frölicher-Nijenhuis bracket - Principal bundles - Connections.*

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– Introduction

There exists a rich literature on principal bundles and invariant connections, stimulated by their importance in geometry and physics. Many important results have been achieved in the traditional approach based on the Maurer-Cartan formulas, in which the group action plays an essential role at any stage and at any level [8, 10]. However, in the recent years, a more general approach to the theory of connections, based on the Froelicher-Nijenhuis bracket of tangent-valued forms, has been developed [11, 12]. In a previous paper [1] we examined the theory of tangent-valued forms for the particular case of principal bundles, stressing how the general approach yields an important clarification of the picture; here we shall apply it to connections, overconnections and related topics. Furthermore, we make a detailed comparison with the traditional approach. All notations and preliminary results have been introduced in [1].

1 – G-invariant connections

1.1 – Jet prolongation of the group action and its quotient

Let $p_E: JE \longrightarrow E$ be the jet bundle of $p: E \longrightarrow B$. Using the jet functor J, the jet prolongation of the morphism $\tau: E \times G \longrightarrow E$ over B is defined as the fibred morphism over B

$$J\tau: J(E \times G) \longrightarrow JE.$$

This can be restricted naturally to a morphism $JE \times G \longrightarrow JE$. Actually

$$J(E \times G) \approx JE \underset{B}{\times} (T^*B \otimes TG)$$

which has the canonical subbundle

$$JE \times G \approx JE \underset{B}{\times} (0_{B\times G})$$

where $0_{B\times G} \subset T^*B \otimes TG$ is the image of the zero section $0: B \times G \longrightarrow T^*B \otimes TG$.

The restriction of $J\tau$ to this subbundle is the morphism over B

$$j\tau: JE \times G \longrightarrow JE$$

which can be characterized by

$$(j\tau)_g \equiv J(\tau_g): JE \longrightarrow JE \qquad \forall g \in G.$$

We now recall [12] that $p_E: JE \longrightarrow E$ can be viewed as the affine subbundle of $T^*B \underset{E}{\otimes} TE \longrightarrow E$ which projects over the identity $1_B \subset T^*B \underset{B}{\otimes} TB$. In other terms, JE is characterized by the "constraint equation" id $\otimes Tp = 1$, which in coordinate reads $\dot{x}_a \otimes \dot{x}^\lambda = \delta^\lambda_a$. The restriction of the coordinate $\dot{x}_\lambda \otimes \dot{y}^i$ to JE is denoted by y^i_λ. Then, the action $j\tau$ of G on JE can be viewed also as the action induced, via the inclusion $\lambda\colon JE \longrightarrow T^*B \underset{E}{\otimes} TE$, by the action of G on $T^*B \underset{E}{\otimes} TE$. Its coordinate expression is

$$\left(x^\lambda, y^i; y^i_\lambda\right) \circ j\tau = \left(x^\lambda, \tau^i; (\partial\tau^i/\partial y^j) y^j_\lambda\right).$$

1.2 – Quotient space

Consider the quotient space

$$C \equiv JE/G.$$

We denote by $(x^\lambda, a^i_\lambda)\colon (JE/G) \longrightarrow \mathbb{R}^{m+n}$ the induced fibred coordinate chart on C (a more particular chart is considered in [8]), defined by

$$\left(x^\lambda, a^i_\lambda\right)\left([k]_G\right) \equiv \left(x^\lambda(w), y^i_\lambda(k_0)\right);$$

where, $\forall\ k \in JE$, k_0 is the unique element of $[k]_G$ such that $p_E(k_0) = s_0 \circ p \circ p_E(k)$, i.e. $a^i_\lambda([k]_G) \equiv x_\lambda \otimes \dot{y}^i(k_0) \equiv (k_0)^i_\lambda \equiv k^j_\lambda((\sigma^{-1})^i_j(p_E(k))$.

PROPOSITION. *The affine bundle $JE \longrightarrow E$ projects over the affine bundle*

$$q_C\colon C \longrightarrow B$$

i.e. $p \circ p_E = q_C \circ p_C$, where $p_C\colon JE \longrightarrow C$ is the projection onto the quotient. □

Namely, the bundle $q_C\colon C \longrightarrow B$ is the affine subbundle of the bundle $(T^*B \underset{E}{\otimes} TE)/G \approx T^*B \underset{B}{\otimes} H \longrightarrow B$ which is projected onto $1_B \subset T^*B \underset{B}{\otimes} TB$. The associated vector bundle is $\overline{C} = T^*B \underset{B}{\otimes} A \longrightarrow B$.

The system $\mathrm{id}_{T^*B} \otimes \eta$ of G-invariant semi-basic 1-forms (see [1], 3.4, c) restricts naturally to the affine fibred morphism over B

$$\xi\colon C \underset{B}{\times} E \longrightarrow JE$$

i.e. $\xi([k]_G, a) \equiv j\tau(k, \bar{\tau}(p_0(k), a))$; thus, $\xi([k]_G, a)$ is the unique element of $[k]_G$ such that $p_0 \circ \xi([k]_G, a) = a$.

The coordinate expression of ξ is

$$\xi = d^\lambda \otimes \partial_\lambda + \sigma^i_j a^j_\lambda d^\lambda \otimes \partial_i .$$

We note that

$$\partial_\mu \xi^i_\lambda = 0 ; \quad \partial_k \xi^i_\lambda = (\partial_k \sigma^i_j) a^j_\lambda ; \quad \frac{\partial \xi^i_\lambda}{\partial a^k_\mu} = \sigma^i_k \delta^\mu_\lambda .$$

1.3 – Principal connections

A *principal connection* is a local section $\gamma \colon E \longrightarrow JE$ which is invariant under the action of the group, i.e. under the lift $j\tau \colon JE \times G \longrightarrow JE$. Moreover, we can also interpret any connection as a tangent valued form $E \longrightarrow T^*B \underset{E}{\otimes} TE$ projectable onto

$$1_B \colon B \longrightarrow T^*B \underset{B}{\otimes} TB .$$

Thus, a principal connection is a G-invariant 1-form and satisfies

$$\gamma^i_\lambda(a) = \gamma^j_\lambda(a') \frac{\partial \tau^i}{\partial y^j}\Big(a', \bar{\tau}(a', a)\Big) , \qquad \forall\, a, a' \in E \colon p(a) = p(a') .$$

In particular, putting $a' \equiv s_0 \circ p(a)$, we obtain:

$$\gamma^i_\lambda(a) = \sigma^i_j(a) a^j_\lambda\Big([\gamma]_G\Big) .$$

Then, the coordinate expression of a principal connection turns out to be

$$\gamma = d^\lambda \otimes \partial_\lambda + \sigma^i_j (\gamma_0)^j_\lambda d^\lambda \otimes \partial_i ,$$

where $(\gamma_0)^j_\lambda = a^j_\lambda([\gamma]_G)$.

We have a one-to-one correspondence among invariant connections and sections $B \longrightarrow C$. Namely, let $c \colon B \longrightarrow C$ be a local section. Then

$$\mathrm{c} \equiv \xi \circ (c \circ p, \mathrm{id}_E) \colon E \longrightarrow JE$$

is a principal connection. Conversely, let $\gamma: E \longrightarrow JE$ a principal connection. Then, there is unique section $c_\gamma: B \longrightarrow C$ such that

$$\gamma \equiv \xi \circ (c_\gamma \circ p, \mathrm{id}_E).$$

Hence:

DEFINITION. *The fibred morphism $\xi: C \underset{B}{\times} E \longrightarrow JE$ is called the system of principal connections.* □

1.4 – Covariant differential

Let $\gamma: E \longrightarrow JE$ be a principal connection, $\varphi: E \longrightarrow \Lambda^r T^* E \underset{E}{\otimes} TE$ any principal tangent valued form. According to the general definition [11, 12] which holds for any connection, the *covariant differential* of φ, with respect to γ, is the principal tangent-valued form

$$d_\gamma \varphi \equiv [\gamma, \varphi]: E \longrightarrow \Lambda^{r+1} T^* E \underset{E}{\otimes} TE.$$

In the particular case when φ is a projectable semi-basic form $\varphi: E \longrightarrow \Lambda^r T^* B \underset{E}{\otimes} TE$, then $d_\gamma \varphi$ is a vertical-valued semi-basic form

$$d_\gamma \varphi: E \longrightarrow \Lambda^{r+1} T^* B \underset{E}{\otimes} VE$$

whose coordinate expression, taking into account $\gamma^i_\lambda = \sigma^i_h (\gamma_0)^h_\lambda$ and the relation between the structure constants and the derivatives of σ^i_h, is

$$d_\gamma \varphi = \sigma^i_h \big(- \partial_{\lambda_1} \varphi^\mu_{\lambda_2 \ldots \lambda_{r+1}} (\gamma_0)^h_\mu - \partial_\mu (\gamma_0)^h_{\lambda_1} \varphi^\mu_{\lambda_2 \ldots \lambda_{r+1}} + \\ + \partial_{\lambda_1} (\varphi_0)^h_{\lambda_2 \ldots \lambda_{r+1}} + c^h_{kj} (\gamma_0)^j_{\lambda_1} (\varphi_0)^k_{\lambda_2 \ldots \lambda_{r+1}} \big) d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+1}} \otimes \partial_i .$$

Let $c: B \longrightarrow C$ and $f: B \longrightarrow \Lambda^r H^* \underset{B}{\otimes} H$ be sections. The *strong differential* of f with respect to c is

$$d_c f \equiv [c, f]: B \longrightarrow \Lambda^{r+1} H^* \underset{B}{\otimes} H.$$

Let $\mathbf{c}: E \longrightarrow JE$ and $\mathbf{f}: E \longrightarrow \Lambda^r T^* E \underset{E}{\otimes} TE$ be the connection and the tangent-valued form associated with c and f via η. Thus, $d_c f$ is the unique tangent-valued form associated with $d_{\mathbf{c}}\mathbf{f}$ via η. We write down explicitely its coordinate expression in the particular case when f is semi-basic:

$$
\begin{aligned}
f =& f^{\mu}_{\lambda_1\dots\lambda_r} d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r} \otimes \partial_\mu + f^{i}_{\lambda_1\dots\lambda_r} d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r} \otimes \partial_i \\
d_c f =& \big(- \partial_{\lambda_1} f^{\mu}_{\lambda_2\dots\lambda_{r+1}} c^i_\mu - \partial_\mu c^i_\lambda f^{\mu}_{\lambda_2\dots\lambda_{r+1}} + \partial_{\lambda_1} f^{i}_{\lambda_2\dots\lambda_{r+1}} + \\
& + c^i_{kj} c^j_{\lambda_1} f^{k}_{\lambda_2\dots\lambda_{r+1}} \big) d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+1}} \otimes \partial_i \,.
\end{aligned}
$$

1.5 – Connection and curvature

According to the general definition which holds for a general connection [3, 4, 11, 12, 14], the curvature of a principal connection γ is the principal projectable vertical-valued 2-form:

$$\rho \equiv \frac{1}{2}[\gamma, \gamma] \equiv \frac{1}{2} d_\gamma \gamma : E \longrightarrow \Lambda^2 T^* B \underset{E}{\otimes} VE \,.$$

Particularizing the coordinate expression of the previous subsection we obtain:

$$2\rho = \sigma^i_h \Big(\partial_\lambda (\gamma_0)^h_\mu + c^h_{kj} (\gamma_0)^j_\lambda (\gamma_0)^k_\mu \Big) d^\lambda \wedge d^\mu \otimes \partial_i \,.$$

Let $c: B \longrightarrow C$ be the unique section corresponding to γ via the system ξ. Then

$$\rho_c \equiv \frac{1}{2} d_c c : B \longrightarrow \Lambda^2 T^* B \underset{E}{\otimes} A$$

is the 2-form corresponding to ρ, given in coordinates by

$$2\rho_c = \Big(\partial_\lambda c^h_\mu + c^h_{kj} c^j_\lambda c^k_\mu \Big) e^\lambda \wedge e^\mu \otimes e_i \,.$$

1.6 – Universal connection and curvature

The universal connection was first introduced by GARCIA [8] and lately generalized by MANGIAROTTI and MODUGNO [11, 12](*). The basic idea is that the covariant differential of any given connection is the pull-back of a "universal" differential defined on the space of all connections of the system. In particular, the curvature is the pull-back of a universal curvature. Consider the bundle

$$\bar{p}\colon \tilde{C} \equiv C \underset{B}{\times} E \longrightarrow C$$

which is clearly a principal bundle fibrewise isomorphic to E. We indicate by $J_C\tilde{C}$ its jet space (with respect to $\bar{p}$). One sees easily that there is a natural monomorphism over $\tilde{C}$

$$i\colon C \underset{B}{\times} JE \hookrightarrow J_C\tilde{C}$$

which is characterized by the commutativity, $\forall\, s\colon B \longrightarrow E$, of the diagram

$$\begin{array}{ccc} C \underset{B}{\times} JE & \xrightarrow{\quad i \quad} & J_C\tilde{C} \\ & {\scriptstyle (js)^{\sim}}\nwarrow \qquad \nearrow {\scriptstyle j(\tilde{s})} & \\ & C & \end{array}$$

where "j" stands for jet prolongation of sections, $(js)^{\sim}$ and $\tilde{s}\colon C \longrightarrow \tilde{C}$ are pull-backs, respectively of $js\colon B \longrightarrow JE$ and of s, with respect to $p\colon E \longrightarrow B$. Its coordinate expression is

$$\left(x^\lambda, a^i_\lambda, y^i\,;\, y^i_\lambda, y^{\lambda i}_j\right) \circ i = \left(x^\lambda, a^i_\lambda, y^i\,;\, y^i_\lambda, 0\right).$$

It is now immediate to see that the map

$$\Gamma \equiv i \circ (\bar{p}, \xi)\colon \tilde{C} \longrightarrow J_C\tilde{C}$$

(*) The construction of "universal connection" given in [11] is different from ours, and holds in a much more particular case; however, the term "universal" has a similar meaning in the two constructions: in both cases, the individual connections arise from pull-backs of the universal connection.

is a section, hence a connection on $\tilde{p}\colon \tilde{C} \longrightarrow C$. Furthermore, Γ is a principal connection. Its coordinate expression is

$$\left(x^{\lambda}, a^{i}_{\lambda}, y^{i}\,;\, y^{i}_{\lambda}, y^{\lambda i}_{j}\right) \circ \Gamma = \left(x^{\lambda}, a^{i}_{\lambda}, y^{i}\,;\, \sigma^{i}_{h} a^{h}_{\lambda}, 0\right).$$

DEFINITION. $\Gamma\colon \tilde{C} \longrightarrow J_C \tilde{C}$ *is called the universal connection.* □

The word "universal" refers to the fact that any principal connection on $E \longrightarrow B$ can be obtained from Γ via a pull-back. Namely, for any section $c\colon B \longrightarrow C$ we have the commuting diagram:

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\quad \Gamma \quad} & J_C \tilde{C} \\ \Big\uparrow{\scriptstyle \bar{c}} & & \Big\uparrow{\scriptstyle i} \\ C \underset{B}{\times} E & \xrightarrow[\mathrm{id}_C \times c]{\qquad} & C \underset{B}{\times} JE \end{array}$$

It is well-known [12] that a connection can be viewed in several equivalent ways. It can be seen that the universality of Γ extends naturally to all these approaches. In §2 we shall examine explicitely the case of the so called (vertical-valued) connection form. For further details see [2].

The curvature of Γ is the principal projectable vertical-valued 2-form:

$$R \equiv \frac{1}{2} d_{\Gamma}\Gamma : \tilde{C} \longrightarrow \Lambda^{2} T^{*} C \underset{\tilde{C}}{\otimes} V\tilde{C}$$

whose coordinate expression can be easily written as a special case of the general formula for the curvature of a principal connection; we find

$$R = \sigma^{i}_{h}\left(c^{h}_{kj} a^{k}_{\lambda} a^{j}_{\mu} dx^{\lambda} \wedge dx^{\mu} + da^{h}_{\lambda} \wedge dx^{\lambda}\right) \otimes \partial_{i}\,.$$

We call R the *universal curvature* since the curvature of any given connection can be obtained from R via a pull-back. Namely, for any section $c\colon B \longrightarrow C$ we have the commuting diagram:

$$\begin{array}{ccc} \tilde{C} & \xrightarrow{\quad R \quad} & \Lambda^2 T^* C \underset{\tilde{C}}{\otimes} V_C \tilde{C} \\ \Big\uparrow{\scriptstyle \tilde{c}} & & \Big\uparrow{\scriptstyle i} \\ C \underset{B}{\times} E & \xrightarrow[\mathrm{id}_C \times \rho_c]{} & C \underset{B}{\times} \Lambda^2 T^* B \underset{E}{\otimes} VE \end{array}$$

(the monomorphism i which appears in the diagram is an immediate generalization of $i\colon C \underset{B}{\times} JE \hookrightarrow J_C \tilde{C}$; we leave this detail to the reader).

The reader will be already familiar with the special case when $E \equiv B \times \mathbb{R}$, $G \equiv \mathbb{R}$; then $C \cong T^* B$. Here, the universal connection and curvature can be identified, respectively, with the Liouville and symplectic form [2].

1.7 – Universal calculus

Consider a G-invariant morphism over E

$$\psi\colon J_k C \underset{B}{\times} E \longrightarrow \Lambda^r T^* E \underset{E}{\otimes} TE$$

where J_k stands for the k-order jet functor; this is a generalization of the concept of invariant tangent-valued form, that we call *graded* (with respect to C, index k). The interesting point is that for such forms one can define a *graded universal differential*. Namely [12]

$$d\psi\colon J_{k+1} C \underset{B}{\times} E \longrightarrow \Lambda^{r+1} T^* E \underset{E}{\otimes} TE$$

is characterized, for all sections $c\colon B \longrightarrow C$ and $s\colon B \longrightarrow E$, by the formula

$$d\psi \circ (j_{k+1} c, s) = d_c(\psi \circ j_k c)\,.$$

In particular, consider a projectable form of degree zero $\psi\colon C \underset{B}{\times} E \longrightarrow \Lambda^r T^* B \underset{E}{\otimes} TE$; then

$$d\psi\colon JC \underset{B}{\times} E \longrightarrow \Lambda^{r+1} T^* B \underset{E}{\otimes} VE$$

whose coordinate expression is

$$d\psi = \sigma^i_h \Big(- \partial_{\lambda_1} \psi^{\mu}_{\lambda_2 \ldots \lambda_{r+1}} a^h_\mu - a^h_{\mu,\lambda} \varphi^{\mu}_{\lambda_2 \ldots \lambda_{r+1}} +$$

$$+ \partial_{\lambda_1} (\psi_0)^h_{\lambda_2 \ldots \lambda_{r+1}} + c^h_{kj} a^j_{\lambda_1} (\psi_0)^k_{\lambda_2 \ldots \lambda_{r+1}} \Big) d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+1}} \otimes \partial_i .$$

Note that the system $\xi : C \underset{B}{\times} E \longrightarrow JE \hookrightarrow T^*B \underset{E}{\otimes} TE$ of principal connections is precisely such a 0-degree form. We have

$$d\xi : JC \underset{B}{\times} E \longrightarrow \Lambda^2 T^* B \underset{E}{\otimes} VE$$

and locally

$$d\xi = \sigma^i_h (a^h_{\lambda,\mu} + c^h_{kj} a^h_\lambda a^k_\mu) d^\lambda \wedge d^\mu \otimes \partial_i .$$

Clearly, there is a strict relation among ξ, $d\xi$, Γ and $R \equiv d_\Gamma \Gamma$, which can be conveniently expressed by commuting diagrams (for details see [12]).

Next we consider graded forms

$$f : J_k C \longrightarrow \Lambda^r H^* \underset{B}{\otimes} H ;$$

clearly, any such form corresponds to a unique graded invariant form, via the trivial extension of ξ. We thus have the *system of graded forms*. In a natural way we can define the *strong graded differential* of f

$$df : J_{k+1} C \longrightarrow \Lambda^{r+1} H^* \underset{B}{\otimes} H$$

by

$$df \circ j_{k+1} c \equiv d_c (f \circ j_k c) \qquad \forall\, c : B \longrightarrow C .$$

In particular, if $f : J_k C \longrightarrow \Lambda^r T^* B \underset{B}{\otimes} H$ we have

$$df : J_{k+1} C \longrightarrow \Lambda^{r+1} T^* B \underset{B}{\otimes} A ;$$

its coordinate expression can be written easily in the particular case $k = 0$:

$$df = (- \partial_{\lambda_1} f^{\mu}_{\lambda_2 \ldots \lambda_{r+1}} a^i_\mu - a^i_{\mu,\lambda} f^{\mu}_{\lambda_2 \ldots \lambda_{r+1}} + \partial_{\lambda_1} f^i_{\lambda_2 \ldots \lambda_{r+1}} +$$

$$+ c^i_{kj} a^j_{\lambda_1} f^k_{\lambda_2 \ldots \lambda_{r+1}}) d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+1}} \otimes \partial_i$$

We have already noted that the system ξ itself is a principal form of degree zero. The corresponding zero-form $f_\xi : C \longrightarrow C$ is the identity, and its differential

$$df_\xi : JC \longrightarrow \Lambda^2 T^* B \underset{B}{\otimes} A$$

gives us the curvature of any principal connection via the rule

$$df_\xi \circ jc = \rho_c .$$

Of course, f_ξ and df_ξ are strictly to the universal connection curvature (essentially they are equivalent objects). Actually, without entering all the details, one may note that the space of all forms $C \longrightarrow \Lambda H^* \underset{B}{\otimes} H$ gives rise, via ξ and the natural inclusion

$$i : C \underset{B}{\times} \Lambda^r T^* B \underset{B}{\otimes} TE \longrightarrow \Lambda^r T^* C \underset{\tilde{C}}{\otimes} T\tilde{C}$$

to a subsystem of the system of principal tangent-valued forms on $\tilde{C} \longrightarrow C$.

1.8 – Principal overconnections

The purpose of this subsection is to show the existence of a distinguished system of connections on the bundle $q_C : C \longrightarrow B$.

As a first step, we want to show that there is a distinguished system of vector fields on C

$$\zeta : JH \underset{B}{\times} C \longrightarrow TC .$$

(This system was first introduced by GARCIA [8]; the construction we present here is essentially the more general one which in [12] was shown valid for an arbitrary involutive system of connections). The construction of ζ is as follows. For any sections $h : B \longrightarrow H$ and $c : B \longrightarrow C$ consider the bracket $[h, c] : B \longrightarrow TC$, where h is viewed as an H-valued 0-form and c as an H-valued 1 form. The bracket of two sections depends on their first jet prolongation, thus we have a morphism

$$[\,,\,] : JH \underset{B}{\times} JC \longrightarrow TC$$

such that $[\,,\,]\circ(jh,jc)=[h,c]$ for any sections. From the coordinate expression one sees [12] that this is an affine morphism over C.

Next, we recall the affine bundle $\underline{\eta}\colon H\longrightarrow TB$, which induces another affine morphism over C, namely

$$\lambda\circ\Big((\underline{\eta}\circ(q_H)_0)\times \mathrm{id}_{JC}\Big)\colon JH\underset{B}{\times}JC\longrightarrow TC$$

where $(q_H)_0\colon JH\longrightarrow H$ is the natural projection. For simplicity, let us indicate this map again by λ. Then we can consider the difference of the two affine morphisms over C

$$\zeta\equiv\lambda-[\,,\,]\colon JH\underset{B}{\times}JC\longrightarrow TC\,.$$

The computation of the coordinate expression of ζ is straightforward:

$$\left(x^\lambda,a^i_\lambda\,;\,\dot x^\lambda,\dot a^i_\lambda\right)\circ\zeta=\left(x^\lambda,a^i_\lambda\,;\,\dot x^\lambda,z^i_\lambda-a^i_\mu\dot x^\mu_\lambda+c^i_{jk}a^j_\lambda z^k\right).$$

From this expression we see that this map is factorizable through the projection $JC\longrightarrow C$, or, in other terms, we actually have a fibred morphism over C

$$\zeta\colon JH\underset{B}{\times}C\longrightarrow TC$$

which moreover turns out to be linear.

Next, we consider the bundle

$$\bar p\colon\overline{C}\longrightarrow B$$

where $\overline{\chi}\colon\overline{C}\hookrightarrow T^*B\underset{B}{\otimes}JH$ is the subspace which projects over $\mathbf{1}_B$. Thus, $\bar p$ is an affine subbundle of $T^*B\underset{B}{\otimes}JH\longrightarrow B$. Now, we shall exhibit two further bundle structures with total space $\overline{C}$.

i) There is a unique fibred morphism over B

$$\bar q_C\colon\overline{C}\longrightarrow C$$

such that the following diagram commutes

$$\begin{array}{ccc} \overline{C} & \xrightarrow{\ \overline{\chi}\ } & T^*B \underset{B}{\otimes} JH \\ {\scriptstyle \bar{q}_C}\big\downarrow & & \big\downarrow \\ C & \xrightarrow[\ \chi\ ]{} & T^*B \underset{B}{\otimes} H \end{array}$$

Moreover, $\bar{q}_C$ is an affine bundle. Let $(x^\lambda, w^\lambda_{\mu\nu}, w^i_\lambda, w^i_{\lambda\mu})$ be the induced chart on $\overline{C}$. Then, the coordinate expression of q_C is

$$\left(x^\lambda, v^a_\lambda\right) \circ \bar{q}_C = \left(x^\lambda, w^a_\lambda\right).$$

ii) Let $\underline{\chi}: K \hookrightarrow T^*B \underset{B}{\otimes} JTB$ be the subspace which projects onto 1_B (i.e. the space of linear connections on $\pi_B: TB \longrightarrow B$). Then, there is a unique fibred morphism over B

$$q_K : \overline{C} \longrightarrow K$$

such that the following diagram commutes

$$\begin{array}{ccc} \overline{C} & \xrightarrow{\ \overline{\chi}\ } & T^*B \underset{B}{\otimes} JH \\ \big\downarrow & & \big\downarrow \\ K & \xrightarrow[\ \underline{\chi}\ ]{} & T^*B \underset{B}{\otimes} JTB \end{array}$$

Moreover, q_K is an affine bundle. Its coordinate expression is

$$\left(x^\lambda, u^\lambda_{\mu\nu}\right) \circ q_K = \left(x^\lambda, w^\lambda_{\mu\nu}\right).$$

We can now see that the system ζ of vector fields on C generates a system of connections on $q_C: C \longrightarrow B$. This system is given by

$$\bar{\xi} \equiv \mathrm{id}_{T^*B} \otimes \zeta : \overline{C} \underset{B}{\times} C \longrightarrow JC \hookrightarrow T^*B \underset{B}{\otimes} TC$$

and its coordinate expression is

$$\left(x^\lambda, a^i_\lambda, a^i_{\lambda,\mu}\right)\circ\bar{\xi} = \left(x^\lambda, a^i_\lambda, w^i_{\lambda\mu} - c^i_{jk} w^j_\lambda a^k_\mu - w^\nu_{\lambda\mu} a^i_\nu\right).$$

This is called the *canonical system of overconnections* (the word "overconnection" refers to the fibred structure $\bar{q}_C : \overline{C} \longrightarrow C$). Furthermore, $\bar{\xi}$ is a subsystem of the system of all affine connections $C \longrightarrow B$ (which, as we saw, is an affine bundle).

A very important feature of this system is that it is, in a sense, invertible. More precisely, consider the commuting diagram

$$\begin{array}{ccc}
\overline{C} \underset{B}{\times} C & \xrightarrow{\ \ \bar{\xi}\ \ } & JC \\
{\scriptstyle (q_K, q_C)\times \mathrm{id}_C}\big\downarrow & & \big\downarrow \\
K \underset{B}{\times} C \underset{B}{\times} C & \xrightarrow[\ \ \pi^3\ \ ]{} & C
\end{array}$$

Then, one sees that $\bar{\xi}$ is an isomorphism over π^3. Hence, we have the "inverse" fibred morphism

$$\bar{\pi} : K \underset{B}{\times} C \underset{B}{\times} JC \longrightarrow \overline{C}.$$

Let us indicate the induced fibred chart on $K \underset{B}{\times} C \underset{B}{\times} JC$ by

$$\left(x^\lambda, u^\lambda_{\mu\nu}; a^i_\lambda; a'^i_\lambda, a'^i_{\lambda,\mu}\right);$$

then the coordinate expression of $\bar{\pi}$ is

$$\left(x^\lambda, w^\lambda_{\mu\nu}, w^i_\lambda, w^i_{\lambda\mu}\right)\circ\bar{\pi} = \left(x^\lambda, u^\lambda_{\mu\nu}, a^i_\lambda, v'^i_{\lambda\mu} + u^\rho_{\lambda\mu} a'^i_\rho + c^i_{jk} a^j_\lambda a'^k_\mu\right).$$

REMARK. Take two sections $k : B \longrightarrow K$, $c : B \longrightarrow C$. These give rise to the section $\bar{c} \equiv (k, c, jc)\circ\bar{\pi} : B \longrightarrow \overline{C}$, hence to the connection $\bar{c} : C \longrightarrow JC$. It is interesting to note that $\bar{c}$ is characterized by the two conditions [12]

$$\nabla_{\bar{c}} c = 0; \quad q_K \circ \bar{c} = k.$$

Moreover, we have $\bar{q}_C \circ \bar{c} = c$. In other terms, $\bar{c}$ is a *prolongation* of (k, c). A calculation shows an interesting and unexpected result: the curvature of $\bar{c}$ dependes *algebraically* from the curvatures of k and c.

2 – Relation between the horizontal approach and the traditional vertical approach

In the traditional approach, principal connections and curvature are studied in the context of forms valued into the Lie algebra of the group (see [10], Ch. II), which are strictly related to invariant vertical-valued forms. We shall now exploit with some details the equivalence of the two approaches to connections.

2.1 – The canonical splitting of the vertical bundle

We remark that, since $\bar{\tau}: E \underset{B}{\times} E \longrightarrow G$, then

$$T_2\bar{\tau}: E \underset{B}{\times} TE \longrightarrow TG;$$

moreover, since $\bar{\tau}(a, a) = e \,\forall\, a \in E$ (e is the unit element of G), the restriction of $T_2\bar{\tau}$ to the "diagonal" submanifold

$$\{(a, v) \in E \underset{B}{\times} TE : a = \pi_E(v)\}$$

is valued into T_eG.

PROPOSITION. *The map*

$$T_2\bar{\tau} \circ (\pi_E, \mathrm{id}_{VE}) : VE \longrightarrow T_eG$$

is a linear isomorphism over E. In other terms, we have the global linear fibred splitting $\Phi: VE \longrightarrow E \times T_eG$ given by the commuting diagram

$$\begin{array}{ccc} VE & \xrightarrow{\quad\Phi\quad} & E \times T_eG \\ & {\scriptstyle (\pi_E, \mathrm{id}_{VE})} \searrow \quad \nearrow {\scriptstyle \mathrm{id}_E \times T_2\bar{\tau}} & \\ & E \times VE & \end{array}$$

□

The coordinate expression of Φ is

$$\Phi = \Phi^i_j dy^j \otimes \frac{\partial}{\partial u^i}(e)$$

where

$$\Phi^i_j(a) = \partial''_j \bar{\tau}^i(a,a)\,.$$

Identifying T_eG with the Lie algebra $\mathcal{G}$ of all right-invariant vector fields $G \longrightarrow TG$, the assignment of a vertical valued form $\varphi\colon E \longrightarrow \Lambda T^{*}E \underset{E}{\otimes} VE$ is equivalent to that of a $\mathcal{G}$-valued form $\tilde{\varphi} \equiv \Phi\varphi\colon E \longrightarrow \Lambda T^{*}E \otimes \mathcal{G}$.

2.2 – $\mathcal{G}$-valued forms

LEMMA. *The following diagram commutes* $\forall\, g \in G$

$$\begin{array}{ccc} E \underset{B}{\times} E & \xrightarrow{\ \bar{\tau}\ } & G \\ {\scriptstyle \tau_g \times \tau_g}\downarrow & & \downarrow{\scriptstyle \mathrm{ad}_g} \\ E \underset{B}{\times} E & \xrightarrow[\ \bar{\tau}\ ]{} & G \end{array}$$

where $\mathrm{ad}_g : G \longrightarrow G : h \longmapsto g^{-1}hg$. □

PROPOSITION. *The following diagram commutes* $\forall\, g \in G$.

$$\begin{array}{ccc} VE & \xrightarrow{\ \Phi\ } & \mathcal{G} \\ {\scriptstyle T\tau_g}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_g} \\ VE & \xrightarrow[\ \Phi\ ]{} & \mathcal{G} \end{array}$$

where $\mathrm{Ad}_g \equiv T(\mathrm{ad}_g) : TG \longrightarrow TG$. □

COROLLARY. *A vertical-valued principal form* $\varphi \colon E \longrightarrow \Lambda T^* E \underset{E}{\otimes} VE$ *is characterized by a* $\mathcal{G}$*-valued form* $\tilde{\varphi}$ *such that the following diagram commutes* $\forall g \in G$

$$\begin{array}{ccc} \Lambda TE & \xrightarrow{\ \tilde{\varphi}\ } & \mathcal{G} \\ {\scriptstyle \Lambda\tau_g}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_g} \\ \Lambda TE & \xrightarrow[\ \tilde{\varphi}\ ]{} & \mathcal{G} \end{array}$$

The corrispondence between φ *and* $\tilde{\varphi}$ *is via the relations*

$$\tilde{\varphi} = \Phi\varphi; \quad \varphi = \Phi^{-1}\tilde{\varphi}.$$

□

We write the principal coordinates expressions for the particular case of 1-forms:

$$\varphi = \sigma^i_h(\varphi_0)^h_\lambda dx^\lambda \otimes \partial y_i + \sigma^i_h(\sigma^{-1})^k_j(\varphi_0)^h_k dy^j \otimes \partial y_i\,;$$

$$\tilde{\varphi} = \tilde{\varphi}^i_\lambda dx^\lambda \otimes \partial u_i + \tilde{\varphi}^i_j dy^j \otimes \partial u_i$$

where

$$\tilde{\varphi}^i_\lambda = \Phi^i_k\sigma^k_h(\varphi_0)^h_\lambda\,; \quad \tilde{\varphi}^i_j = \Phi^i_m(\sigma^{-1})^k_j\sigma^m_h(\varphi_0)^h_k\,.$$

2.3 – Exterior differential and Lie Bracket of $\mathcal{G}$-valued forms

a) The exterior differential of $\mathcal{G}$-valued forms is defined for decomposable forms by the rule

$$d(\alpha \otimes u) = d\alpha \otimes u \colon E \longrightarrow \Lambda^{r+1}T^*E \otimes \mathcal{G} \qquad \alpha \colon E \longrightarrow \Lambda^r T^* E\,, \quad u \in \mathcal{G}$$

and extended naturally by linearity. In particular, let $\tilde{\varphi} \colon E \longrightarrow T^*E \otimes \mathcal{G}$ be any $\mathcal{G}$-valued 1-form. The coordinate expression of its exterior differential is

$$\begin{aligned}
d\tilde{\varphi} =& d\tilde{\varphi}^i_\lambda \wedge dx^\lambda \otimes \partial u_i + d\tilde{\varphi}^i_j \wedge dy^j \otimes \partial u_i = \\
=& \Phi^i_k \sigma^k_h \partial_\mu (\varphi_0)^h_\lambda dx^\mu \wedge dx^\lambda \otimes \partial u_i + \\
&+ \left(\partial_\ell \Phi^i_k \sigma^k_h + \Phi^i_k \partial_\ell \sigma^k_h\right)(\varphi_0)^h_\lambda dy^\ell \wedge dx^\lambda \otimes \partial u_i + \\
&+ \Phi^i_m (\sigma^{-1})^k_j \sigma^m_h \partial_\lambda (\varphi_0)^h_k dx^\lambda \wedge dy^j \otimes \partial u_i + \\
&+ \left(\partial_\ell \Phi^i_m (\sigma^{-1})^k_j \sigma^m_h + \Phi^i_m \partial_\ell (\sigma^{-1})^k_j \sigma^m_h + \right. \\
&\left. + \Phi^i_m (\sigma^{-1})^k_j \partial_\ell \sigma^m_h\right)(\varphi_0)^h_k dy^\ell \wedge dy^j \otimes \partial u_i \,.
\end{aligned}$$

b) The Lie bracket of two G-valued forms is defined by the rule

$$[\alpha \otimes u, \beta \otimes v] \equiv \alpha \wedge \beta \otimes [u, v] \colon E \longrightarrow \Lambda^{r+s} T^* E \otimes \mathcal{G}$$

$$\alpha \colon E \longrightarrow \Lambda^r T^* E\,, \ \beta \colon E \longrightarrow \Lambda^s T^* E\,, \quad u, v \in \mathcal{G}$$

and extend naturally by linearity. In particular, let $\tilde{\varphi}, \tilde{\psi} \colon E \longrightarrow T^* E \otimes \mathcal{G}$ be $\mathcal{G}$-valued 1-forms. The coordinate expression of their Lie bracket is

$$[\tilde{\varphi}, \tilde{\psi}] = \left(\tilde{\varphi}^i_\lambda dx^\lambda + \tilde{\varphi}^i_j dy^j\right) \wedge \left(\tilde{\psi}^h_\mu dx^\mu + \tilde{\psi}^h_k dy^k\right) c^l_{ih} \partial u_l \,.$$

2.4 – Connection forms

A principal connection $\gamma \colon E \longrightarrow JE$ is equivalent [4, 5, 8] to the G-invariant vertical valued form, called *connection form*

$$\omega_\gamma \colon E \longrightarrow T^* E \underset{E}{\otimes} VE$$

which is characterized by the identity

$$\gamma + \omega_\gamma = 1_{TE} \colon E \longrightarrow T^* E \underset{E}{\otimes} TE\,;$$

its coordinate expression is

$$\omega_\gamma = d^i \otimes \partial_i - \sigma^i_j (\gamma_0)^j_\lambda d^\lambda \otimes \partial_i .$$

We remark that any invariant vertical-valued 1-form which is a linear projection $\omega: TE \longrightarrow VE$ over E (that is whose restriction to VE is the identity) is a connection form, i.e. corresponds to a (unique) invariant connection.

PROPOSITION. *We have*

$$2\rho = [\gamma, \gamma] = [\omega_\gamma, \omega_\gamma] = -d_\gamma \omega_\gamma$$

(here the bracket are F-N brackets). □

PROOF. It follows from a direct calculation, taking into account the identity

$$0 = [1_{TE}, 1_{TE}] = [\gamma + \omega_\gamma, \gamma + \omega_\gamma] .$$

□

We then recover the Maurer-Cartan identity:

PROPOSITION. *Let $\omega: E \longrightarrow T^*E \underset{E}{\otimes} VE$ be an invariant connection form, $\rho \equiv \frac{1}{2}[\omega, \omega]: E \longrightarrow \Lambda^2 T^*E \underset{E}{\otimes} VE$ its curvature. Let $\tilde{\omega}: E \longrightarrow T^*E \otimes \mathcal{G}$ and $\tilde{\rho}: E \longrightarrow \Lambda^2 T^*E \otimes \mathcal{G}$ be the corresponding $\mathcal{G}$-valued forms. We then have:*

$$\tilde{\rho} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] .$$

□

PROOF. Let $u, v: E \longrightarrow TE$ be invariant fields. From the definition of Froelicher-Nijenhuis bracket we have

$$\begin{aligned}[\omega, \omega](u, v) = &\frac{1}{2}\Big([\omega(u), \omega(v)] - [\omega(v), \omega(u)] - \omega[u, \omega(v)] + \omega[v, \omega(u)] + \\ &- \omega[u, \omega(v)] + \omega[v, \omega(u)] + \frac{1}{2}\omega(\omega([u, v])) - \frac{1}{2}\omega(\omega([v, u])) + \\ &+ \frac{1}{2}\omega(\omega([u, v])) - \frac{1}{2}\omega(\omega([v, u]))\Big) = \\ = &[\omega(u), \omega(v)] - \omega[u, \omega(v)] + \omega[v, \omega(u)] + \omega([u, v]) ;\end{aligned}$$

as $\omega(v)$ is vertical then $[u,\omega(v)]$ is vertical, thus $\omega([u,\omega(v)]) = [u,\omega(v)]$. Finally, we have

$$[\omega,\omega](u,v) = [\omega(u),\omega(v)] - [u,\omega(v)] + [v,\omega(u)] + \omega([u,v])$$

which, taking into account the definition of d and $[\,,\,]$ for $\mathcal{G}$-valued forms gives us the result. □

2.5 – The system of invariant connection forms

The methodology of systems can be easily applied to the vertical approach to connections. We give here only a short exposition, leaving the details to the reader. Let

$$W \hookrightarrow H^* \underset{B}{\otimes} A$$

be the subbundle of all linear projections $H \longrightarrow A$, i.e. the subbundle characterized by the constraints

$$(e^i \otimes e_j)_{|W} = \delta^i_j \,.$$

Then a section $w\colon B \longrightarrow W$ characterizes a unique invariant connection form

$$\mathbf{w}\colon E \longrightarrow T^*E \underset{E}{\otimes} VE \,.$$

Namely, the restriction to W of the system $\Lambda\eta\otimes\bar{\eta}$ of invariant vertical-valued forms ([1],3.6,b) is a system, the *system of invariant connection forms*. The semi-basic A-valued form

$$\frac{1}{2}[w,w] \equiv \rho_w\colon B \longrightarrow \Lambda^2 T^* B \underset{B}{\otimes} A$$

corresponds, via the system of semi-basic vertical-valued forms ([1],3.6,d), to the curvature of w.

The *universal connection-form* is connection form

$$\omega_\Gamma\colon \tilde{C} \longrightarrow T^*\tilde{C} \underset{\tilde{C}}{\otimes} VE \subset T^*\tilde{C} \underset{\tilde{C}}{\otimes} V_C\tilde{C}$$

associated with the universal connection (1.6). It can be characterized by the relation

$$\langle \omega_\Gamma, (u,v) \rangle = (k, \langle \omega_k, v \rangle)$$

where $k \equiv \pi_C(u) \in C$, $(u,v) \in T\tilde{C} = TC \underset{TB}{\times} TE$. The universal connection-form is a particular invariant connection form on the principal bundle $\tilde{C} \longrightarrow C$, hence from the first proposition in 2.4 we see that the universal curvature is given by

$$\rho_\Gamma = \frac{1}{2}[\omega_\Gamma, \omega_\Gamma] \,.$$

Further details on various equivalent ways to view the universal connection and curvature can be found in [2].

A final remark concerns the application of the system methodology to the approach based on $\mathcal{G}$-valued forms. An invariant $\mathcal{G}$-valued form is *not* associated with a form $B \longrightarrow \Lambda H^* \underset{B}{\otimes} \mathcal{G}$. Rather, from the commuting diagram of the proposition in 2.2 we see that an invariant $\mathcal{G}$-valued form is associated with a form

$$B \longrightarrow \Lambda H^* \underset{B}{\otimes} ((E \times \mathcal{G})/G)$$

where $(E \times \mathcal{G})/G$ is the quotient of $E \times \mathcal{G}$ by the *adjoint* action. But this is *canonically* isomorphic to $A \equiv VE/G$. Thus the system (i.e. quotient) descriptions of the vertical and $\mathcal{G}$-valued approaches coincide.

However, we remark that the choice of a local gauge $s_0 : B \longrightarrow E$ induces also another (not canonical) local description. Consider the local fibred isomorphism over E

$$\Phi_0 : VE \longrightarrow E \times \mathcal{G} : v \mapsto (s_0 \circ p \circ \pi_E(v), \Phi(v_0))$$

where v_0 is ([1],2.2) the unique element of $[v]_G$ such that $\pi_E(v_0) = s_0 \circ p \circ \pi_E(v)$. Then we have, for all $g \in G$, the commuting diagram

$$\begin{array}{ccc} VE & \xrightarrow{\Phi_0} & \mathcal{G} \\ {\scriptstyle T\tau_g} \downarrow & & \downarrow {\scriptstyle Id_{\mathcal{G}}} \\ VE & \xrightarrow[\Phi_0]{} & \mathcal{G} \end{array}$$

Identifying a vertical-valued form with a $\mathcal{G}$-valued form via Φ_0, and performing the quotient with respect to the group action, we then see that an invariant vertical-valued form is characterized uniquely by a section $B \longrightarrow \Lambda H^* \otimes \mathcal{G}$. Actually, it is known [8] that the choice of a gauge determines locally a fibred isomorphism $A \cong B \times \mathcal{G}$ over B.

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