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Dominance Method for Plane Partitions: Skew Plane Partitions and Skew Tableaux

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RIASSUNTO - *Utilizzando la tecnica che si basa sulla relazione di dominanza fra partizioni, viene qui dedotto, in modo diretto ed elementare, il teorema che stabilisce il collegamento fra la funzione di Schur e la enumerazione delle partizioni piane.*

ABSTRACT - *Between algebra and combinatorics there exist many interesting connections. One important example is that the Schur fuction has remarkable relevance to the enumeration of plane partitions as well as Young tableaux. By means of "dominance technique", a direct and elementary derivation for the relations just mentioned is demonstrated which may be more accessible to the reader.*

KEY WORDS - *Schur function - Plane partition - Young tableau.*

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1 – Introduction

Let λ and μ be two partitions such that

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r), \quad \mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r); \quad \lambda_i \geq \mu_i \quad (i = 1, 2, \ldots, r). \tag{1.1a}$$

They are conjugated to λ' and μ', respectively, which satisfy the dual condition

$$\text{(1.1b)} \qquad \begin{aligned} &\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \ldots \geq \lambda'_c), \quad \mu' = (\mu'_1 \geq \mu'_2 \geq \cdots \geq \mu'_c); \\ &\lambda'_j \geq \mu'_j \quad (j = 1, 2, \ldots, c). \end{aligned}$$

In terms of Ferrers diagram or ordered ideal in the poset $\mathbf{N}^2$ (where $\mathbf{N}$ is the set of natural numbers), λ/μ can be expressed as a skew diagram:

$$\text{(1.2a)} \qquad \lambda/\mu := \Big((i,j) : \mu_i < j \leq \lambda_i \,,\, 1 \leq i \leq r\Big)$$

$$\text{(1.2b)} \qquad := \Big((i,j) : \mu'_j < i \leq \lambda'_j \,,\, 1 \leq j \leq c\Big)$$

A (column-strict) plane partition ($\underline{P}P$) of skew shape λ/μ is an array T of natural numbers, $T = (t_{ij} : (i,j) \in \lambda/\mu)$, where the entries of T satisfy the following conditions:

$$\text{(1.3a)} \qquad t_{ij} \geq t_{i,j+1} \quad \text{for} \quad \mu_i < j < \lambda_i \,, \quad 1 \leq i \leq r$$

and

$$\text{(1.3b)} \qquad t_{i,j} > t_{i+1,j} \quad \text{for} \quad \mu'_j < i < \lambda'_j \,, \quad 1 \leq j \leq c\,.$$

It is obvious that if the above inequalities are reversed, then the array T becomes an ordinary Young tableau (cf. [5]). We will discuss this topic in section 5.

Assume that (x_k) are commutative variables. For a given plane partition T with n_k parts equal to k, define its enumerator by the monomial

$$\text{(1.4)} \qquad x_T := \prod_{t \in T} x_t = \prod x_k^{n_k}\,.$$

Then the enumerative function (EF) for the set $P(\lambda/\mu)$ of $\underline{P}P$'s of skew shape λ/μ is defined by an enumerator-sum as follows:

$$\text{(1.5)} \qquad S_{\lambda/\mu}(x) = \sum_{T \in P(\lambda/\mu)} x_T\,,$$

which is a homogeneous function of degree equal to the number of parts involved in $\underline{P}P$'s. It is clear that the above definition can be carried over to the $\underline{P}P$'s with just one row or one column as well as to Young tableaux.

There is a striking known fact that $S_{\lambda/\mu}(x)$ is just the classical Schur function. The latter plays important part in the theory of group representations. The related problems have been solved and revisited several times. The successful methods, for example, are lattice permutation due to MACMAHON [8], algorithmic correspondence due to KNUTH [5], q-series computation due to ANDREWS [1], the denominator formulae of Lie theory due to MACDONALD [7] and the lattice path counting due to GESSEL (cf. [4]). But it has been dissatisfied by the author that all of these can be attributed to mathematical *verification*. Therefore a direct and elementary *derivation* for $S_{\lambda/\mu}(x)$ being the Schur function has been desirable. The dominance technique can fulfill this purpose which will be demonstrated in this paper.

2 – Symmetric functions and Schur functions

Let $e_k(s)$ and $h_k(s+1)$ be elementary and complete symmetric functions of degree k in variables $(x_i)_{i>s}$, with the boundary conditions $e_k := e_k(0)$, $h_k := h_k(1)$, $e_0(k) = h_0(k) = 1$ and $e_k(\infty) = h_k(\infty) = 0$, where k is a non-negative integer. Analogous to those for ordinary and Gaussian binomial coefficients, the following recurrences will play the vital role in our derivation.

LEMMA.

$$\text{(2.1)} \qquad \text{i.} \qquad \sum_{n<i\leq m} x_i e_k(i) = e_{k+1}(n) - e_{k+1}(m)$$

$$\text{(2.2)} \qquad \text{ii.} \qquad \sum_{n\leq j<m} x_j h_k(j) = h_{k+1}(n) - h_{k+1}(m).$$

PROOF. The generating functions for $\{e_k(i)\}$ and $\{h_k(j)\}$ with respect to the subscript k are respectively given by

$$E(i) = \sum_{k\geq 0} e_k(i) z^k = \prod_{r>i}(1 + zx_r)$$

and

$$H(j) = \sum_{k\geq 0} h_k(j) z^k = \prod_{r\geq j} (1 - z x_r)^{-1}.$$

Then (2.1) and (2.2) follow from

$$\sum_{n<i\leq m} x_i E(i) = \sum_{n<i\leq m} z^{-1}\Big(E(i-1) - E(i)\Big) = z^{-1}\Big(E(n) - E(m)\Big)$$

and

$$\sum_{n\leq j<m} x_j H(j) = \sum_{n\leq j<m} z^{-1}\Big(H(j) - H(j-1)\Big) = z^{-1}\Big(H(n) - H(m)\Big),$$

respectively.

Denote by $\det_{n\times n}(x_{ij})$ the determinant of the matrix $(x_{ij})_{1\leq i,j\leq n}$. Now we are ready to state the main theorem as follows.

THEOREM. *Assume the notation of section 1.*

i.

(2.3a) $$S_{\lambda/\mu}(x) = \det_{r\times r}(h_{j-i+\lambda_i-\mu_j}),$$

ii. *Dually:*

(2.3b) $$S_{\lambda/\mu}(x) = \det_{c\times c}(e_{j-i+\lambda'_i-\mu'_j}),$$

iii. *$S_{\lambda/\mu}(x)$ is symmetric and coincides with the classical skew Schur function.*

Motivated by the enumeration works of CARLITZ [2] on rectangular arrays and of MOHANTY [9] and NARAYANA [10] on lattice paths, we shall present dominance technique to derive the statements of the above theorem. Its peculiarity lies in utilizing sufficiently the information (i.e. dominance relation) between rows and columns for fixed type of plane partition. Contrary to the earlier methods, this approach may be more accessible to the reader.

3 – Derivation

For two column-partitions T and T', define the order relation $T \geq T'$ if and only if the array formed by the two partitions is in fact a plane partition. According to (1.2b) and (1.3), a plane partition T of shape λ/μ can be expressed as a *formal product* $T = T_1 \geq T_2 \geq \ldots \geq T_c$, in terms of its successive columns. This suggests the following separation:

$$S_{\lambda/\mu}(x) = \sum_{T_1 \geq T_2 \geq \cdots \geq T_c} x_{T_1} x_{T_2} \ldots x_{T_c}. \tag{3.1}$$

To make the derivation smooth, denote by λ^k the union of μ and the first k-columns of λ. Then from (3.1), $S_{\lambda/\mu}(x)$ can be recursively generated by column-summation as follows.

Initial value:

$$S_{\lambda^0/\mu} = 1. \tag{3.2a}$$

Recurrence:

$$S_{\lambda^k/\mu} = \sum_{T_k \geq T_{k+1}} x_{T_k} S_{\lambda^{k-1}/\mu} \qquad (k = 1, 2, \ldots, c-1) \tag{3.2b}$$

Final step:

$$S_{\lambda/\mu}(x) = S_{\lambda^c/\mu} = \sum_{T_c} x_{T_c} S_{\lambda^{c-1}/\mu}. \tag{3.2c}$$

Now substitute the initial value in (3.2a) by the determinant of the unitriangular matrix

$$\left(h_{j-i+\mu_i-\mu_j}(t_{i,\mu_i+1})\right)_{1 \leq i,j \leq r}.$$

Then from (3.2b), the summand of $S_{\lambda^1/\mu}$ can be rewritten, by absorbing the factor x_{T_1} in the determinant involved, as

$$\det_{r \times r} \begin{bmatrix} h_{j-i+\mu_i-\mu_j}(t_{i,\mu_i+1}) \\ x_{t_{i,j}} h_{j-i+\mu_i-\mu_j}(t_{i,\mu_i+1}) \end{bmatrix} \begin{matrix} (1 \leq i \leq \mu'_1) \\ (\mu'_1 < i \leq r), \end{matrix} \tag{3.3a}$$

where the brackets outside the determinant denote the ranges for the corresponding row-labels, and the summation-region $T_1 \geq T_2$ can be decomposed triangularly

$$\bigcup_{(\mu'_1 < i \leq r)} (t_{i,2} \leq t_{i,1} < t_{i-1,1}) \tag{3.3b}$$

where we assume that $t_{i,2} = 1$ for $\lambda'_2 < i \leq r$ and $t_{\mu'_1,1} = \infty$.

For determinant (3.3a), performing the operation by (2.2) with respect to $t_{i,1}$ over $t_{i,2} \leq t_{i,1} < t_{i-1,1}$, the i-th row becomes the difference of complete symmetric functions

$$h_{j-i+1-\mu_j}(t_{i,2}) - h_{j-i+1-\mu_j}(t_{i-1,1}), \qquad (j = 1, 2, \dots, r).$$

Multiplying the $(i-1)$-th row by the inverse of $x_{t_{i-1,1}}$ and adding to the i-th row, the latter reduces to

$$h_{j-i+1-\mu_j}(t_{i,2}), \quad (j = 1, 2, \dots, r) \quad \text{for} \quad \mu'_1 < i \leq \lambda'_1 = r.$$

Manipulating in this way for i from λ'_1 to $1 + \mu'_1$ with decrement 1, we finally arrive at

$$S_{\lambda^1/\mu} = \det_{r\times r} \begin{bmatrix} h_{j-i+\mu_i-\mu_j}(t_{i,\mu_i+1}) \\ h_{j-i+1-\mu_j}(t_{i,2}) \\ h_{j-i+1-\mu_j} \end{bmatrix} \begin{matrix} (1 \leq i \leq \mu'_1) \\ (\mu'_1 < i \leq \lambda'_2) \\ (\lambda'_2 < i \leq r). \end{matrix} \tag{3.3c}$$

The same manner yields

$$S_{\lambda^2/\mu} = \det_{r\times r} \begin{bmatrix} h_{j-i+\mu_i-\mu_j}(t_{i,\mu_i+1}) \\ h_{j-i+1+\mu_i-\mu_j}(t_{i,3}) \\ h_{j-i+2+\mu_i-\mu_j}(t_{i,3}) \\ h_{j-i+\lambda_i-\mu_j} \end{bmatrix} \begin{matrix} (1 \leq i \leq \mu'_2) \\ (\mu'_2 < i \leq \mu'_1) \\ (\mu'_1 < i \leq \lambda'_3) \\ (\lambda'_3 < i \leq r). \end{matrix} \tag{3.4}$$

Repeating this process c-times on recurrence (3.2) and noting that

$$\begin{aligned} \lambda_i^{k-1} &= \lambda_i^k \quad \text{for} \quad 1 \le i \le \mu'_k, \\ \lambda_i^k &= k \quad \text{for} \quad \mu'_k < i \le \lambda'_{k+1}, \\ \lambda_i^k &= \lambda_i \quad \text{for} \quad \lambda'_{k+1} \le i \le \lambda'_k, \\ t_{i,j} &= 1 \quad \text{for} \quad \lambda_i < j; \end{aligned}$$

one can establish the following intermediate expression

$$S_{\lambda^k/\mu}(x) = \det_{r\times r}\left(h_{j-i+\lambda_i^k-\mu_j}(t_{i,\lambda_i^k+1})\right) \quad \text{for} \quad 0 \le k \le c. \tag{3.5}$$

Thus we ultimately get (2.3a).

4 – Dual derivation

In a similar way to section 3, a plane partition T of shape λ/μ can be expressed as a *formal product* $T = T_1 > T_2 > \ldots > T_r$ of its successive rows according to (1.2a) and (1.3), after having defined the order relation $T > T'$ for two row-partitions T and T' if and only if the array formed by the two partitions is a column-strict $\underline{P}P$. Then (1.5) can be separated as

$$S_{\lambda/\mu}(x) = \sum_{T_1 > T_2 > \ldots > T_r} x_{T_1} x_{T_2} \ldots x_{T_r}. \tag{4.1}$$

Denote by $\underline{\lambda}^k$ the union of μ and the first k-rows of λ. Then from (4.1), $S_{\lambda/\mu}(x)$ can be recursively generated by row-summation as follows.

Initial value:

$$S_{\underline{\lambda}^0/\mu} = 1 \tag{4.2a}$$

Recurrence:

$$S_{\underline{\lambda}^k/\mu} = \sum_{T_k > T_{k+1}} x_{T_k} S_{\underline{\lambda}^{k-1}/\mu} \qquad (k = 1, 2, \ldots, r-1). \tag{4.2b}$$

Final step:

$$S_{\lambda/\mu}(x) = S_{\underline{\lambda}^r/\mu} = \sum_{T_r} x_{T_r} S_{\underline{\lambda}^{r-1}/\mu}. \tag{4.2c}$$

In the same approach as that for (3.2), replacing $S_{\underline{\lambda}^0/\mu}$ by the determinant

$$\det_{c\times c}\left(e_{j-i+\mu'_i-\mu'_j}(t_{\mu'_i+1,i})\right)=1$$

and then operating on the recurrence (4.2) by means of (2.1), we get the dual intermediate result as follows

$$S_{\underline{\lambda}^k/\mu}(x)=\det_{c\times c}\left(e_{j-i+\underline{\lambda}'^k_i-\mu'_j}(t_{\mu'^k_i+1,i})\right),\qquad (k=0,1,\ldots,r) \tag{4.3}$$

where $t_{i,j}=0$ for $i>\lambda'_j$. This provides a dual derivation for (2.3b).

From (2.3a) and (2.3b), we conclude that $S_{\lambda/\mu}(x)$ is a symmetric function and coincides with the skew Schur function.

5 – Young tableaux

Turning over skew diagram is in fact a bijection between Young tableaux of shape λ/μ and plane partitions with skew-shape parameters:

$$\underline{\lambda}_i:=c-\mu_{r-i+1}\,,\quad \underline{\mu}_i:=c-\lambda_{r-i+1}\quad\text{for}\quad 1\le i\le r;$$

$$\underline{\lambda}'_j:=r-\mu'_{c-j+1}\,,\quad \underline{\mu}'_j:=r-\lambda'_{c-j+1}\quad\text{for}\quad 1\le j\le c.$$

Hence the enumerative function $Y_{\lambda/\mu}(x)$ for Young tableaux of shape λ/μ is given by

$$Y_{\lambda/\mu}(x)=S_{\underline{\lambda}/\underline{\mu}}(x)=S_{\lambda/\mu}(x).$$

When $\mu=\Phi$ (empty partition) there is a useful determinantal quotient expression

$$S_\lambda(x)=\det_{n\times n}(x_j^{n-i+\lambda_i})/\det_{n\times n}(x_j^{n-i})$$

which has become the basis for further research on the enumeration theory of plane partitions and Young tableaux as well as on the Schur functions. The details may be refered to the monographs due to LITTLEWOOD [6] and MACDONALD [7] respectively.

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