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Interpolation by Piecewise Weighted Mean Functions

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RIASSUNTO – *Nel lavoro viene introdotta una classe di funzioni "medie pesate a tratti" per l'interpolazione di una funzione reale $f(x)$ nota in un insieme finito di nodi $x_1 < x_2 < \ldots < x_n$. In ogni punto $x \in [x_i, x_{i+1}], i = 1, \ldots, (n-1)$, la media interpolante è una media pesata dei valori $f(x_i)$ ed $f(x_{i+1})$. È stata studiata la regolarità delle medie pesate a tratti nell'intervallo $[x_1, x_n]$ e si sono dimostrate le proprietà di "variation diminishing" e di conservazione della monotonia della sequenza di valori $f(x_1), \ldots, f(x_n)$.*

ABSTRACT – *In the present paper we introduce a class of "piecewise weighted mean functions" for the interpolation of a real function $f(x)$ given on a finite set of nodes $x_1 < x_2 < \ldots < x_n$. At any point $x \in [x_i, x_{i+1}], i = 1, \ldots, (n-1)$ the interpolating mean is a weighted mean of the values $f(x_i)$ and $f(x_{i+1})$. We analyse the smoothness of the piecewise mean functions in the interval $[x_1, x_n]$, and we show that they satisfy the variation diminishing property and they preserve the monotonicity of the sequence $f(x_1), \ldots, f(x_n)$.*

KEY WORDS – *Piecewise interpolation - Variation diminishing - Monotonicity preserving.*

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– Introduction

A wide class of weighted mean functions, possessing the interpolation property, is defined in [1]. For a given set of real values $f_i = f(x_i), i = 1, \ldots, n$, and distinct nodes $x_i, i = 1, \ldots, n$, arbitrarily distributed in

$I \subset R$, the interpolating mean is given by

$$u(f; x, n) = \sum_{i=1}^{n} f_i p_i(x; n) \tag{1.1}$$

where the weight functions $p_i(x; n)$, $i = 1, \ldots, n$, satisfy the conditions:

$$\begin{cases} p_i(x, n) \geq 0 \\ \sum_{i=1}^{n} p_i(x; n) = 1 \\ p_i(x_j; n) = \delta_{ij} \quad , \quad i, j = 1, \ldots, n \,. \end{cases} \tag{1.2}$$

We consider in particular the weight functions $p_i(x; n)$, which can be represented by the general formula:

$$p_i(x; n) = \frac{\left| \prod_{\substack{k=1 \\ k \neq i}}^{n} [\varphi(x) - \varphi(x_k)] \right|^{\alpha}}{\sum_{j=1}^{n} \left| \prod_{\substack{k=1 \\ k \neq j}}^{n} [\varphi(x) - \varphi(x_k)] \right|^{\alpha}}, \quad i = 1, \ldots, n \tag{1.3}$$

where $\alpha \in R^+$ and $\varphi(x)$ can be particularized as follows [1]:

$$\varphi(x) = x \quad , x \in I \subset R \tag{1.4}$$

in this case the (1.1) becomes the well known Shepard interpolation formula [2];

$$\varphi(x) = \cos x \quad , x \in [0, \pi) \tag{1.5}$$

$$\varphi(x) = e^x \quad , x \in I \subset R \tag{1.6}$$

more in general, we say that $\varphi(x)$ can be any function strictly monotone and at least C^1 in I.

In the present paper we introduce the "piecewise mean functions", to do this we suppose that the nodes are in increasing order, namely

$x_1 < x_2 < \ldots < x_n$, and we apply a formula of type (1.1) to the pairs of nodes x_i, x_{i+1}, $i = 1, \ldots, (n-1)$; the resulting interpolation scheme is

$$(1.7) \qquad u_2(f; x, n) = \sum_{j=i}^{i+1} f_j p_j(x; 2) \quad , i = 1, \ldots, (n-1)$$

where

$$(1.8) \qquad p_i(x; 2) = \frac{|\varphi(x) - \varphi(x_{i+1})|^{\alpha_i}}{|\varphi(x) - \varphi(x_i)|^{\alpha_i} + |\varphi(x) - \varphi(x_{i+1})|^{\alpha_i}}$$

$$(1.9) \qquad p_{i+1}(x; 2) = \frac{|\varphi(x) - \varphi(x_i)|^{\alpha_i}}{|\varphi(x) - \varphi(x_i)|^{\alpha_i} + |\varphi(x) - \varphi(x_{i+1})|^{\alpha_i}}$$

with $\alpha_i \in R^+$; we note that the conditions (1.2) are satisfied by the pair of weight functions defined by (1.8) and (1.9) and so $u_2(f; x, n)$ interpolates the values $f_1, \ldots, f_n$ and, in each subinterval $[x_i, x_{i+1}]$, is a weighted mean of the values f_i, f_{i+1}. We can immediately derive that the following properties hold:

$$(1.10) \qquad \min(f_i, f_{i+1}) \le u_2(f; x, n) \le \max(f_i, f_{i+1}) \quad , x \in [x_i, x_{i+1}];$$

and if $f_i = f_{i+1} = c$, then

$$(1.11) \qquad u_2(f; x, n) = c[p_i(x; 2) + p_{i+1}(x; 2)] = c \quad , x \in [x_i, x_{i+1}].$$

Moreover we can state the following lemma:

LEMMA 1.1.. *The weight functions defined by (1.8) and (1.9) with* $\alpha_i > 1$, $i = 1, \ldots, (n-1)$, *satisfy the conditions*

$$p'_k(x_j; 2) = 0 \quad ; k, j = i, (i+1).$$

PROOF. Let

$$B_i(x) = |\varphi(x) - \varphi(x_i)|^r$$

with $r = \alpha_i$, and

$$B_i'(x) = r|\varphi(x) - \varphi(x_i)|^{r-1}\,\mathrm{sign}[\varphi(x) - \varphi(x_i)]\varphi'(x);$$

calculating $B_i(x)$ and its derivative $B_i'(x)$ at the node x_i, we get

$$B_i(x_i) = 0 \tag{1.12}$$

$$B_i'(x_i) = 0. \tag{1.13}$$

The weight function $p_i(x;2)$ and its derivative $p_i'(x;2)$ can be expressed respectively by

$$p_i(x;2) = \frac{B_{i+1}(x)}{B_i(x) + B_{i+1}(x)}$$

$$p_i'(x;2) = \frac{B_{i+1}'(x)[B_i(x) + B_{i+1}(x)] - B_{i+1}(x)[B_i'(x) + B_{i+1}'(x)]}{[B_i(x) + B_{i+1}(x)]^2};$$

now, by (1.12) and (1.13) we get

$$p_i'(x_i;2) = \frac{B_{i+1}'(x_i)B_{i+1}(x_i) - B_{i+1}(x_i)B_{i+1}'(x_i)}{B_{i+1}^2(x_i)} = 0 \tag{1.14}$$

$$p_i'(x_{i+1};2) = 0. \tag{1.15}$$

Similarly we can obtain $p_{i+1}'(x_i;2) = p_{i+1}'(x_{i+1};2) = 0$ and the lemma is proved.

From the lemma 1.1 it follows that, if $\alpha_i > 1$, $i = 1,\dots,(n-1)$, then $u_2(f;x,n)$ has null derivative at the nodes $x_1,\dots,x_n$ and is at least C^1 in the whole interval (x_1,x_n); if $\alpha_i \leq 1$ for any i, then the relations (1.12), (1.13), (1.14) and (1.15) don't hold and, consequently, $u_2(f;x,n)$ can have cusps at the nodes x_i, x_{i+1}.

In order to avoid the null derivatives at the nodes $x_1,\dots,x_n$ we can consider the following piecewise mean:

$$u_2(L;x,n) = \sum_{j=i}^{i+1} L_j p_j(x;2) \quad ,i = 1,\dots,(n-1); \tag{1.16}$$

where $L_i = f(x_i) + (x - x_i)f'(x_i), i = 1, \dots, n$, as it has been already suggested for the Shepard formula [2]; the piecewise interpolating mean defined by (1.16) with $\alpha_i > 1$, $i = 1, \dots, (n-1)$, interpolates at the nodes $x_1, \dots, x_n$ both the values $f(x_1), \dots, f(x_n)$ and the derivatives $f'(x_1), \dots, f'(x_n)$.

In the next section we study more in detail the piecewise interpolating mean $u_2(\cdot; x, n)$ and we shall prove that this interpolation scheme enjoys the variation diminishing and the monotonicity preserving properties.

1 – Two properties

The (1.7) can be also expressed by

$$u_2(f; x, n) = \frac{f_i|\varphi(x) - \varphi(x_{i+1})|^{\alpha_i} + f_{i+1}|\varphi(x) - \varphi(x_i)|^{\alpha_i}}{|\varphi(x) - \varphi(x_i)|^{\alpha_i} + |\varphi(x) - \varphi(x_{i+1})|^{\alpha_i}}, \tag{2.1}$$

$$x \in [x_i, x_{i+1}]; \qquad i = 1, \dots, (n-1).$$

We say that $u_2(f; x, n)$ has a zero at a point $\zeta \in [x_i, x_{i+1}]$ provided that $u_2(f; \zeta, n) = 0$, we shall write [3]:

$$Z_{[x_i, x_{i+1}]}(u_2) = \text{ number of zeros of } u_2 \text{ in } [x_i, x_{i+1}].$$

Now we prove the following lemma:

LEMMA 2.1.. *The pair of weight functions defined by* (1.8) *and* (1.9) *satisfies this condition*

$$D\begin{pmatrix} t_1 & t_2 \\ p_i & p_{i+1} \end{pmatrix} > 0 \qquad \textit{for all} \quad x_i \le t_1 < t_2 \le x_{i+1}. \tag{2.2}$$

PROOF. Developping the determinant in (2.2) we get

$$\begin{vmatrix} p_i(t_1;2) & p_{i+1}(t_1;2) \\ p_i(t_2;2) & p_{i+1}(t_2;2) \end{vmatrix} = p_i(t_1;2)p_{i+1}(t_2;2) - p_{i+1}(t_1;2)p_i(t_2;2) =$$

$$= \frac{|\varphi(t_1) - \varphi(x_{i+1})|^{\alpha_i}|\varphi(t_2) - \varphi(x_i)|^{\alpha_i} - |\varphi(t_1) - \varphi(x_i)|^{\alpha_i}|\varphi(t_2) - \varphi(x_{i+1})|^{\alpha_i}}{[|\varphi(t_1) - \varphi(x_i)|^{\alpha_i} + |\varphi(t_1) - \varphi(x_{i+1})|^{\alpha_i}][|\varphi(t_2) - \varphi(x_i)|^{\alpha_i} + |\varphi(t_2) - \varphi(x_{i+1})|^{\alpha_i}]} > 0$$

since $x_i \leq t_1 < t_2 \leq x_{i+1}$, the last inequality follows from the strict monotonicity of $\varphi(x)$, and the lemma is proved.

By the lemma 2.1 we can derive that the pair of weight functions $p_i(x;2), p_{i+1}(x;2)$ forms a Tchebicheff system on $[x_i, x_{i+1}]$ and, consequently, it holds [3, pag. 29]:

$$(2.3) \qquad Z_{[x_i,x_{i+1}]}[\sum_{j=i}^{i+1} f_j p_j(x;2)] \leq 1\,, \quad \text{for all real } f_i, f_{i+1} \text{ not all } 0.$$

Following S.Karlin [4] we say that $u_2(f;x,n)$ is variation diminishing in $[x_1, x_n]$ if

$$S^-_{[x_1,x_n]}[u_2(f;x,n)] \leq S^-(f_1, f_2, \dots, f_n);$$

$$S^-_{[x_1,x_n]}[u_2(f;x,n)] = \sup S^-[u_2(f;t_1,n), \dots, u_2(f;t_m,n)]$$

where the supremum is extended over all sets $t_1 < t_2 < \dots < t_m (t_j \in [x_1, x_n])$, m is arbitrary but finite and $S^-(y_1, y_2, \dots, y_n)$ is the number of sign changes of the indicated sequence, zero terms being discarded.

We can now state the following theorem:

THEOREM 2.1. *The piecewise interpolating mean function $u_2(f;x,n)$ is variation diminishing in $[x_1, x_n]$ and, in particular, it holds*

$$(2.4) \qquad S^-_{[x_1,x_n]}[u_2(f;x,n)] = S^-(f_1, \dots, f_n)\,,$$

PROOF. Let the sequence $(f_1, \dots, f_n)$ has no zero terms, then (2.4) holds in the whole interval $[x_1, x_n]$ if it holds locally in every subinterval $[x_i, x_{i+1}]$, $i = 1, \dots, (n-1)$, namely if

$$(2.5) \qquad S^-_{[x_i,x_{i+1}]}[u_2(f;x,n)] = S_-(f_i, f_{i+1}).$$

If f_i and f_{i+1} have the same sign, then (2.5) follows immediately from (1.10). If f_i and f_{i+1} have different sign, then $S^-(f_i, f_{i+1}) = 1$; in order to get $S^-_{[x_i,x_{i+1}]}[u_2(f;x,n)]$ we note that the interpolation conditions at x_i and x_{i+1} guarantee that $u_2(f;x,n)$ has at least one sign change in

$[x_i, x_{i+1}]$, moreover this sign change is unique by virtue of the relation (2.3); so we can conclude that $S^-_{[x_i,x_{i+1}]}[u_2(f;x,n)] = 1$.

Let now the sequence $(f_1, \ldots, f_n)$ has r zero terms with $1 \le r \le (n-2)$, we suppose for simplicity that $f_i \ne 0, f_{i+1} = f_{i+2} = \ldots = f_{i+r} = 0, f_{i+r+1} \ne 0$, then the (2.4) is verified if it holds

$$S^-_{[x_i,x_{i+r+1}]}[u_2(f;x,n)] = S^-(f_i, f_{i+1}, \ldots, f_{i+r+1}) \tag{2.6}$$

by the (1.10) we derive that $u_2(f;x,n)$ in $[x_i, x_{i+1}]$ has the same sign of f_i, and in $[x_{i+r}, x_{i+r+1}]$ has the same sign of f_{i+r+1}; in the intermediate intervals $[x_{i+1}, x_{i+2}], \ldots, [x_{i+r-1}, x_{i+r}]$ we get from the (1.11) that $u_2(f;x,n) = 0$, we can so conclude that the (2.6) is verified; finally if $r = (n-1), n$, then $S^-(f_1, \ldots, f_n) = 0$ and, by (1.10) and (1.11) we can derive that $S^-_{[x_1,x_n]}[u_2(f;x,n)] = 0$ too, and the theorem is proved.

Now we shall prove that the interpolation scheme $u_2(f;x,n)$ preserves the monotonocity of the sequence $(f_1, f_2, \ldots, f_n)$

PROPOSITION 2.1. *If $f_1, f_2, \ldots, f_n$ are monotonic, then $u_2(f;x,n)$ it is monotonic in $[x_1, x_n]$.*

PROOF. [4, pag. 22]. Let d be any real number, and consider the relation

$$u_2(f;x,n) - d = \sum_{j=i}^{i+1} (f_j - d) p_j(x;2) \quad ; i = 1, \ldots, (n-1),$$

for any d, according to the hypotesis, the sequence $(f_1 - d, f_2 - d, \ldots, f_n - d)$ change sign at most once, the variation diminishing of $u_2(\cdot;x,n)$ (relation (2.4)) implies that $u_2(f;x,n) - d$ enjoys the same property, which in turn implies the monotonicity of $u_2(f;x,n)$.

Finally, to illustrate the behaviour of the proposed method, we present an example used in [5], where the interpolation values $f_1, \ldots, f_8$ are the vertices of the polygonal line reported in Figure 1. The interpolation curve $u_2(f;x,8)$ is obtained by setting $\varphi(x) = x$ in (2.1). In this case, if $\alpha_1 = \alpha_2 = \ldots = \alpha_7 = 1$, then $u_2(f;x,8)$ coincides exactly with the polygonal line, this can immediately be derived by the relation (2.1). Figure 1 shows the polygonal line and $u_2(f;x,8)$ with $\alpha_1 = \alpha_2 = \ldots =$

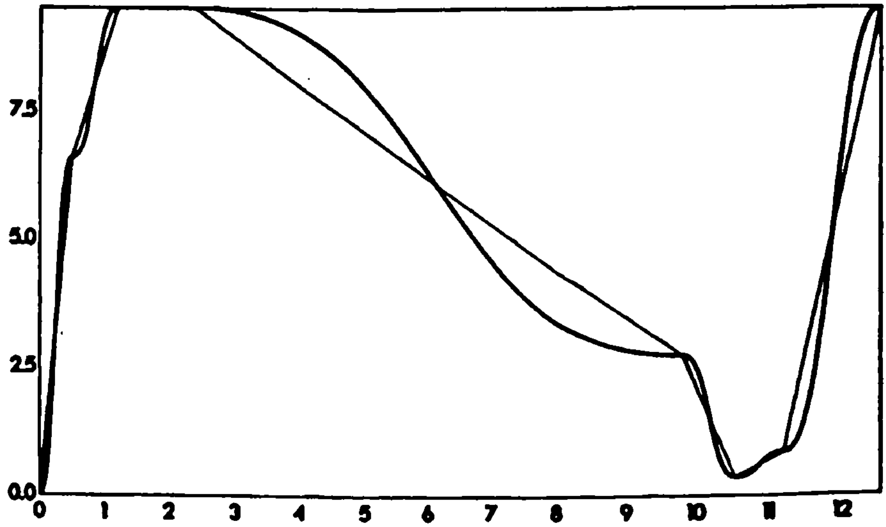

Fig. 1

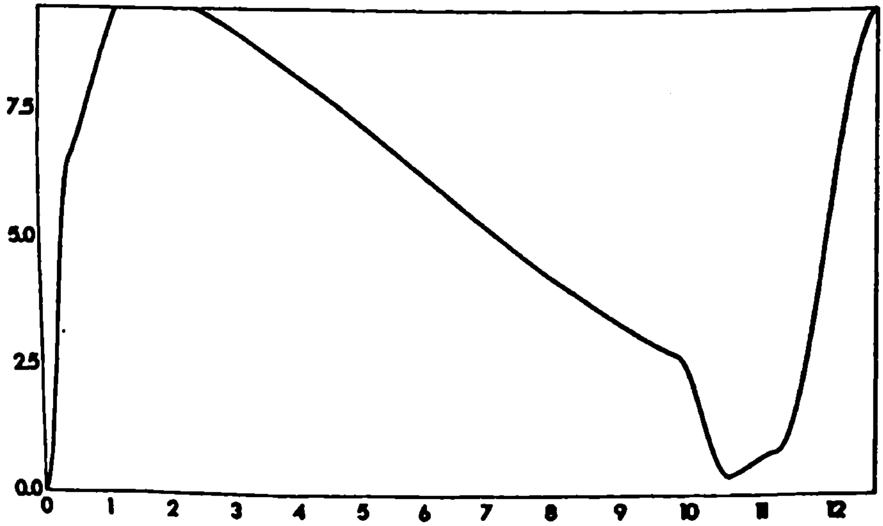

Fig. 2

$\alpha_7 = 2$; Figure 2 shows $u_2(f;x,8)$ with variable exponents such that $1 < \alpha_i < 2, i = 1,\ldots,7$.

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