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On a Quadrature Process of Birkhoff Type

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RIASSUNTO – *In questo lavoro viene data una formula di quadratura Gaussiana di tipo Birkhoff, e viene istituito un confronto tra questa ed altre formule di integrazione numerica lacunari.*

ABSTRACT – *In this paper a new Gaussian quadrature formula of Birkhoff type is obtained, and a comparison with other lacunary quadrature rules is presented.*

KEY WORDS – *Gaussian quadrature formulas - Birkhoff interpolation - Turán quadrature rules.*

A.M.S. CLASSIFICATION: 65D30 - 33A65

1 – Introduction

In recent years, interest about quadrature has shifted to Birkhoff quadrature formulas, in which the quadrature sums contain the values of the integrand function and some of its non consecutive derivatives.

The problem of lacunary interpolation was first considered by G.D. BIRKHOFF [3], and extensively studied by P. TURÁN and his students in a series of papers [1, 2, 9]. A thorough treatment of Birkhoff interpolation is the content of [5]; some results on Birkhoff type quadrature rules are contained, for instance, in [6, 7, 8, 12, 13].

P. TURÁN, in an important paper on approximation theory [11], raised many interesting questions, some of which are on Birkhoff interpolation. Among the others, the following problem on Birkhoff quadrature theory is posed.

PROBLEM XXXIII. *Determine the matrices A, if any, for which*

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} D_{0kn} f(x_{kn}) + \sum_{k=1}^{n} D_{2kn} f''(x_{kn}) \tag{1.1}$$

is valid for all polynomials of degree $\leq 2n$.

Here, A denotes an infinite triangular matrix, whose n-th row consists of the nodes x_{in}, $i = 1, 1, \ldots, n$, of the quadrature formula with

$$-1 \leq x_{1n} < x_{2n} < \ldots < x_{nn} \leq 1 .$$

The interesting feature of the quadrature rule (1.1) is that it is based on the values of a function and its second derivative.

The object of this paper is to give an answer to Turán's question.

Section 2 contains an approach to the above problem, some preliminary notations and properties of orthogonal polynomials. In Section 3 a quadrature rule, solution of the mentioned problem is provided; Section 4 is devoted to some remarks on the rule obtained in Section 3 and other quadrature formulas of Birkhoff type.

2 – Preliminaries and notations

As concerns the problem of finding a quadrature formula of type (1.1), it is evident that it can hold if and only if the $2n$ weights D_{ikn}, $i = 0, 2$; $k = 1, 2, \ldots, n$, satisfy the following system of linear equations

$$\sum_{k=1}^{n} D_{0kn} y_i(x_{kn}) + \sum_{k=1}^{n} D_{2kn} y_i''(x_{kn}) = \mu_i , \quad i = 1, 2, \ldots, 2n+1 , \tag{2.1}$$

where

$$y_i = x^{i-1} , \quad \mu_i = \int_{-1}^{1} y_i dx , \qquad i = 1, 2, \ldots, 2n+1 . \tag{2.2}$$

Then, consider the transposed homogeneous system associated to (2.1)

$$\sum_{j=1}^{2n+1} c_j y_j^{(h)}(x_{in}) = 0\,, \qquad h = 0,2; \quad i = 1,2,\ldots,n; \tag{2.3}$$

it is well known that, if $2n+1-q$, $q \geq 1$, is the rank of the matrix $B \in \mathbb{R}^{2n\times(2n+1)}$

$$B = \left\| y_j^{(h)}(x_{in}) \right\| \tag{2.4}$$

then system (2.3) has q linearly independent nontrivial solutions C^j, $j = 1,2,\ldots,q$, and system (2.1) is consistent if and only if each vector C^j is orthogonal to the vector $M := [\mu_1, \mu_2, \ldots, \mu_{2n+1}]$.

These observations can be seen also from another point of view. In fact, the matrix B is associated also with the following boundary value differential problem

$$\frac{d^{2n+1}}{dx^{2n+1}} Y(x) = 0; \tag{2.5}$$

$$Y(x_{kn}) = Y''(x_{kn}) = 0\,, \qquad k = 1,2,\ldots,n. \tag{2.6}$$

Since the general solution of (2.5) is given by $Y(x) = \sum_{i=1}^{n} c_i y_i(x)$ (see (2.2)), the boundary conditions (2.6) yield again system (2.3). Thus, if the matrix B in (2.4) has rank $2n+1-q$, the problem (2.5), (2.6) has q linearly independent nontrivial solutions Y_j, $j = 1,2,\ldots,q$, and (2.1) can hold if and only if

$$\int_{-1}^{1} Y_j(x)dx = 0\,, \qquad j = 1,2,\ldots,q; \tag{2.7}$$

In this case, if $q = 1$, a unique quadrature rule exists, while if $q > 1$ infinitely many quadrature formulas (1.1) can be constructed.

Now, the conditions (2.7) can be interpreted as conditions on the choice of the knots of the quadrature rule, once their number has been prefixed.

We shall now summarize some properties of orthogonal polynomials, which will be useful in the sequel.

Let $\left\{P_n^{(\alpha)}(x)\right\}_{n=0}^{\infty}$ denote the system of ultraspherical polynomials, orthogonal in $[-1,1]$ with respect to the weight function $(1-x^2)^\alpha$, $\alpha > -1$, normalized by $P_n^{(\alpha)}(1) = \binom{n+\alpha}{n}$, and $\left\{\pi_n(x)\right\}_{n=2}^{\infty}$ denote the system of polynomials, defined by

$$\pi_n(x) = -n(n-1)\int_{-1}^{x} P_{n-1}^{(0)}(t)dt = (1-x^2)P_{n-1}^{(0)\prime}(x). \tag{2.8}$$

It is well known that the polynomials $P_n^{(0)\prime}(x)$ are orthogonal with respect to the weight function $(1-x^2)$, in $[-1,1]$; more precisely, there results

$$P_n^{(1)}(x) = \frac{2}{n+2}P_{n+1}^{(0)\prime}(x). \tag{2.9}$$

The explicit representation of $P_n^{(1)}$ is the following [10, pg. 85]

$$P_n^{(1)}(x) = \sum_{h=0}^{\nu} a_{2h}x^{n-2h} \tag{2.10}$$

where

$$\begin{cases} \nu = [n/2], \\ a_{2h} = (-1)^h \dfrac{2}{n+2}\dfrac{(2n-2h+1)!!}{h!(n-2h)!2^h}, \qquad h = 0,1,\ldots,\nu. \end{cases} \tag{2.11}$$

From (2.8), (2.9) one gets

$$\begin{cases} \pi_n(x) = \dfrac{n}{2}(1-x^2)P_{n-2}^{(1)}(x), \\ \pi_n''(x) = -\dfrac{n^2(n-1)}{2}P_{n-2}^{(1)}(x). \end{cases} \tag{2.12}$$

We shall denote by ζ_{kn}, $k = 1,2,\ldots,n$, the zeros of $P_n^{(1)}$, and shall consider two quadrature processes based on these zeros: the Gauss-Jacobi rule associated with the weight function $(1-x^2)$

$$\int_{-1}^{1}(1-x^2)f(x)dx = \sum_{k=1}^{n}\lambda_{kn}f(\zeta_{kn}), \qquad f \in \mathbb{P}_{2n-1}, \tag{2.13}$$

and the Lobatto quadrature rule [4, pg.110]

$$(2.14)\quad \int_{-1}^{1} f(x)dx = \frac{2}{(n+1)(n+2)}[f(-1)+f(1)] + \sum_{k=1}^{n} \frac{\lambda_{kn}}{1-\zeta_{kn}^2} f(\zeta_{kn}),$$

valid for $f \in \mathbb{P}_{2n+1}$.

We also mention the following result: for any $F \in C^2[-1,1]$ we have

$$(2.15)\quad \int_{-1}^{1}(1-x^2)F''(x)dx = 2\Big[F(-1)+F(1) - \int_{-1}^{1} F(x)dx\Big],$$

which is obtained integrating by parts.

3 – A solution of Turán's problem

Turning to the formulation of the Turán's problem presented in Section 2, assume in (2.6) $x_{kn} = \zeta_{kn}$, $k = 1,2,\dots,n$, $n \geq 2$; thus the boundary value differential problem becomes

$$(3.1)\quad \begin{cases} \dfrac{d^{2n+1}}{dx^{2n+1}} Y(x) = 0, \\ Y(\zeta_{kn}) = Y''(\zeta_{kn}) = 0, \qquad k = 1,2,\dots,n. \end{cases}$$

A nontrivial solution of (3.1) is given by

$$Y_1(x) = \pi_{n+2}(x) = \frac{n+2}{2}(1-x^2)P_n^{(1)}(x);$$

in fact, from (2.12) there immediatly follows

$$Y_1(\zeta_{kn}) = Y_1''(\zeta_{kn}) = 0, \qquad k = 1,2,\dots,n.$$

Furthermore, $Y_1(x)$ satisfies condition (2.7), which now becomes

$$\int_{-1}^{1}(1-x^2)P_n^{(1)}(x)dx = 0.$$

These remarks suggest to take as matrix A the p-matrix [11, pg. 26] whose n-th row consists of the zeros of $P_n^{(1)}$.

Assuming such zeros as nodes of the quadrature process, a solution of problem XXXIII of Turán is provided by Theorem 1 below, where a formula is given which is exact for polynomials not only of degree at most $2n$, but also of degree at most $2n+1$.

THEOREM 1. *The quadrature rule*

$$\int_{-1}^{1} f(x)dx = \frac{(n+1)(n+2)}{n(n+3)} \sum_{k=1}^{n} \frac{\lambda_{kn}}{1-\zeta_{kn}^2} f(\zeta_{kn}) + \frac{1}{n(n+3)} \sum_{k=1}^{n} \lambda_{kn} f''(\zeta_{kn}) \tag{3.2}$$

is exact for $f \in \mathbb{P}_{2n+1}$, $n \geq 1$.

PROOF. The rule (3.2) is exact for $f \equiv 1$; in fact, by (2.14) the second side of (3.2) yields

$$\frac{(n+1)(n+2)}{n(n+3)} \sum_{k=1}^{n} \frac{\lambda_{kn}}{1-\zeta_{kn}^2} = \frac{(n+1)(n+2)}{n(n+3)} \left[\int_{-1}^{1} 1dx - \frac{2}{(n+1)(n+2)}2\right] = 2.$$

By symmetry, (3.2) holds also for each odd polynomial. Now, let f be an even polynomial of degree at most $2n$, vanishing at ± 1,

$$f(x) = (1-x^2)Q_{2n-2}(x)$$

where $Q_{2n-2} \in \mathbb{P}_{2n-2}$.

For such a function, the second member of (3.2) yields

$$\frac{(n+1)(n+2)}{n(n+3)} \sum_{k=1}^{n} \lambda_{kn} Q_{2n-2}(\zeta_{kn}) + \frac{1}{n(n+3)} \sum_{k=1}^{n} \lambda_{kn} f''(\zeta_{kn})$$

and using (2.13), (2.15) one gets

$$\frac{(n+1)(n+2)}{n(n+3)} \int_{-1}^{1} (1-x^2)Q_{2n-2}(x)dx + \frac{1}{n(n+3)} \int_{-1}^{1} (1-x^2)f''(x)dx =$$
$$= \frac{(n+1)(n+2)}{n(n+3)} \int_{-1}^{1} f(x)dx - \frac{2}{n(n+3)} \int_{-1}^{1} f(x)dx = \int_{-1}^{1} f(x)dx.$$

Now, any arbitrary $f \in \mathbb{P}_{2n+1}$ can be decomposed into $f = f_1 + f_2 + f_3$ where f_1 is a constant, f_2 is odd, f_3 is an even polynomial $\in \mathbb{P}_{2n}$, vanishing at ± 1; since (3.2) holds for each of the functions in the decomposition, the claim follows.

Finally, we remark that (3.2) is not exact for $f \in P_{2n+2}$. In fact, consider the polynomial F defined by

$$F(x) = (1 - x^2)\left[P_n^{(1)}(x)\right]^2 .$$

Using (2.9) and the following relations [10, pg. 352. pg. 84]

$$\lambda_{kn} = 8(n+1)(n+2)^{-1}(1-\zeta_{kn}^2)^{-1}\left[P_n^{(1)}(\zeta_{kn})\right]^{-2} ;$$

$$(1 - x^2)P_{n+1}^{(0)\prime}(x) = (n+1)(n+2)(2n+3)^{-1}\left[P_n^{(0)}(x) - P_{n+2}^{(0)}(x)\right] ;$$

it is easy to prove that the l.h.side and the r.h.side of (3.2) give, respectively,

$$8(n+1)[(n+2)(2n+3)]^{-1} \quad ; \quad 16(n+1)[(n+2)(n+3)]^{-1} . \qquad \square$$

4 – Some remarks on the obtained rule

We first remark that rule (3.2) is of Gaussian type, then we observe that the zeros of $P_n^{(1)}$ figure as nodes in several quadrature rules, obtained recently in the literature concerning Birkhoff-type quadrature processes. Here, we refer mainly to the formulas obtained in [7, 12, 13]. The formula in [7], however, is relative to weighted integrals, with weight function $(1-x^2)$, and makes use of *all* the zeros of π_{n+2}, while in [13] the quadrature rule below, based on the zeros of $P_n^{(1)}$ only, appears as an intermediate step in the development of the process yielding the conclusions of the paper; we mean the formula

$$(4.1) \qquad \int_{-1}^{1} f(x)dx = \frac{3}{2}\sum_{k=1}^{n}\lambda_{kn}f(\zeta_{kn}) + \frac{1}{8}\sum_{k=1}^{n}(1-\zeta_{kn}^2)\lambda_{kn}f''(\zeta_{kn}),$$

valid for $f \in \mathbb{P}_{2n-1}$.

Formula (4.1) is obtained applying the same n-nodes Gauss-Jacobi quadrature rule, associated with the weight function $(1-x^2)$, to evaluate integrals of two functions, namely $(1-x^2)f''(x)$ and $f(x)$.

It is worth observing that, when $n = 2$, formulas (3.2) and (4.1) coincide, resulting in

$$\int_{-1}^{1} f(x)dx = f\Big(-\frac{1}{\sqrt{5}}\Big) + f\Big(\frac{1}{\sqrt{5}}\Big) + \frac{1}{15}\Big[f''\Big(-\frac{1}{\sqrt{5}}\Big) + f''\Big(\frac{1}{\sqrt{5}}\Big)\Big],$$

valid for $f \in \mathbb{P}_5$.

So the question arises whether (4.1) is exact for $f \in \mathbb{P}_{2n+1}$ even for $n \geq 3$. The answer is negative.

What we wish to stress here is that (4.1) is but one of many quadrature rules based on the zeros of $P_n^{(1)}$ and having degree of exactness $2n-1$, as the following Theorem 2 establishes.

THEOREM 2. *There are infinitely many quadrature rules of the form*

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{n} B_{0kn} f(\zeta_{kn}) + \sum_{k=1}^{n} B_{2kn} f''(\zeta_{kn}), \qquad f \in \mathbb{P}_{2n-1}.$$

valid for $f \in \mathbb{P}_{2n-1}$.

PROOF. Imposing that the degree of exactness is $2n-1$, we get the following system

$$\text{(4.2)} \qquad \sum_{k=1}^{n} B_{0kn} y_i(x_{kn}) + \sum_{k=1}^{n} B_{2kn} y_i''(x_{kn}) = \mu_i, \qquad i = 1, 2, \dots, 2n,$$

which has at least the solution given by the weights of (4.1). Then, consider the matrix $B_1 \in \mathbb{R}^{2n \times 2n}$, of the associated transposed homogeneous system

$$B_1 = \begin{pmatrix} 1 \; \zeta_{1n} \; \zeta_{1n}^2 \dots & \zeta_{1n}^k & \dots & \zeta_{1n}^{n+2} & \dots & \zeta_{1n}^{2n-1} \\ 1 \; \zeta_{2n} \; \zeta_{2n}^2 \dots & \zeta_{2n}^k & \dots & \zeta_{2n}^{n+2} & \dots & \zeta_{2n}^{2n-1} \\ 1 \; \zeta_{nn} \; \zeta_{nn}^2 \dots & \zeta_{nn}^k & \dots & \zeta_{nn}^{n+2} & \dots & \zeta_{nn}^{2n-1} \\ 0 \;\; 0 \;\; 2 \; \dots & k(k-1)\zeta_{1n}^{k-2} & \dots & (n+2)(n+1)\zeta_{1n}^{n} & \dots & (2n-1)(2n-2)\zeta_{1n}^{2n-3} \\ 0 \;\; 0 \;\; 2 \; \dots & k(k-1)\zeta_{2n}^{k-2} & \dots & (n+2)(n+1)\zeta_{2n}^{n} & \dots & (2n-1)(2n-2)\zeta_{2n}^{2n-3} \\ 0 \;\; 0 \;\; 2 \; \dots & k(k-1)\zeta_{nn}^{k-2} & \dots & (n+2)(n+1)\zeta_{nn}^{n} & \dots & (2n-1)(2n-2)\zeta_{nn}^{2n-3} \end{pmatrix}$$

we shall prove that this matrix is singular, therefore system (4.2) has infinitely many solutions.

For the sake of brevity, we shall omit some unessential indices in the formulas, and will write ζ instead of ζ_{in}, since the only property of these numbers we need in the proof is that each of them is a zero of $P_n^{(1)}$; thus, from (2.10) it follows

$$(4.3) \qquad \zeta^n = \sum_{h=1}^{\nu} \gamma_{2h}\zeta^{n-h}$$

where

$$(4.4) \quad \gamma_{2h} = -a_{2h}/a_0 = (-1)^{h+1}\frac{(2n-2h+1)!!n!}{2^h h!(n-2h)!(2n+1)!!}, \quad h = 0,1,\ldots,\nu.$$

We introduce also the constants

$$(4.5) \qquad k_{2h} = \gamma_{2h} - \gamma_{2h-2}, \qquad h = 1,2,\ldots,\nu+1,$$

with

$$(4.6) \qquad \gamma_{2\nu+2} := 0;$$

from (4.5) and (4.4) one gets

$$(4.7) \qquad k_2 = \gamma_2 + 1,$$

$$(4.8) \qquad k_{2h} = \frac{(n+2)(n+1)}{(n-2h+2)(n-2h+1)}\gamma_{2h}, \qquad h = 1,2,\ldots,\nu.$$

We can now prove that matrix B_1 is singular, since its $(n+3)$-th column is a linear combination, with coefficients k_{2n}, of the columns $(n+3-2h)$-th, $h = 1,2,\ldots,\nu+1$. In fact, consider the first n rows of B_1, then the mentioned linear combination gives, recalling (4.5) and (4.7),

$$\sum_{h=1}^{\nu+1} k_{2h}\zeta^{n-2h+2} = \sum_{h=1}^{\nu} \gamma_{2h}\zeta^{n-2h+2} + \zeta^n - \sum_{h=1}^{\nu} \gamma_{2h}\zeta^{n-2h}$$

which, by (4.3) yields

$$\sum_{h=1}^{\nu+1} k_{2h}\zeta^{n-2h+2} = \zeta^{n+2}.$$

Consider now the remaining n rows, whose first and second entries are null: using (4.8) and (4.3) the linear combination gives

$$\sum_{h=1}^{\nu+1} k_{2h}(n-2h+2)(n-2h+1)\zeta^{n-2h} = (n+2)(n+1)\zeta^{n}. \qquad \square$$

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