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The Doubly Confluent Heun Equation: A Differential Equation Associated with the Naked Singularity in Cosmology

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RIASSUNTO – *L'equazione di Heun a doppia confluenza è un'equazione differenziale lineare del secondo ordine con due sole singolarità entrambe irregolari e di secondo tipo. L'equazione si risolve utilizzando la trasformata di Laplace. La funzione modulante è soluzione dell'equazione di Mathieu generalizzata e, utilizzando l'integrale di Pochhammer, si ottengono tutte le soluzioni significative sotto forma di serie convergenti anche se l''equazione di Heun a doppia confluenza non ha singolarità regolari. Sebbene il motivo originale nell'affrontare questo studio fosse legato alla teoria di un caso limite di un buco nero con raggio di Schwarzschild nullo (una singolarità nuda), l'aspetto piuttosto inconsueto dell'equazione differenziale in esame merita ulteriore approfondimento.*

ABSTRACT – *The doubly confluent Heun equation is a second order linear differential equation with only two singularities both of which are irregular and of the second type. This equation is solved by means of a Laplace transform. The modulating function is a solution of a generalised Mathieu equation, and by utilising a Pochhammer double-loop integral, all the relevant solutions are obtained in the form of convergent series in spite of the fact that the doubly confluent Heun equation has no regular singularities. While the original motivation in carrying out this study was the connection with the theory of a limiting case of a black hole with zero Schwarzschild radius (a naked singularity), the rather unusual aspect of the differential equation under consideration mentioned above merits further interest.*

KEY WORDS – *Doubly confluent Heun - Generalised Mathieu - Naked singularity.*

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1 – Introduction

A confluent form of the Heun equation occurs in various contexts including the theory of the Teukolsky black hole. See LEAVER [5]. If the doubly confluent Heun equation is formed by means of confluence, one interesting physical interpretation consists of the black hole with zero Schwarzschild radius, that is a naked singularity. Although thid notion is controversial from the point of view of certain aspects of current cosmological theory, the resulting equation has certain analytic properties in its own right which merit further study.

In the first place, we have the differential equation

$$x^2Y'' + (\alpha + \beta x)Y' + (\gamma + \delta x - \varepsilon^2 x^2)Y = 0\,. \tag{1.1}$$

This equation has only two singularities, both irregular of type two, one at the origin and the other at the point at infinity.

The Frobenius theory of linear differential equations is not directly applicable to dealing with irregular singularities except perhaps to deduce certain non-convergent solutions, and no further information is given. As indicated by LEAVER [5], it will be shown that convergent series solutions of (1.1) near the irregular singularities do exist.

A precedent for this situation is already implicit in INCE [4], page 503, where the eqution

$$z^2w'' + zw' - \left[a + \frac{k^2}{2} + \frac{k^2(z + z^{-1})}{4}\right]\frac{w}{4} = 0 \tag{1.2}$$

with two singularities, both irregular and of the first type, may be transformed into an algebraic form of Mathieu's equation by the substitution $z = \exp(2ix)$. Convergent series solutions relative to the irregular singularities of (1.2) are thus possible.

2 – An alternative form of (1.1). Symmetries

For convenience, replace Y by $\exp(\varepsilon x)y$ in (1.1) and obtain the equation

$$x^2y'' + (ax^2 + bx + c)y' + (dx + f)y = 0\,, \tag{2.1}$$

where $a = 2\varepsilon$, $b = \beta$, $c = \alpha$, $d = \delta + \beta\varepsilon$ and $f = \gamma + \alpha\varepsilon$.

The equation (2.1) will be taken as the standard form of the doubly confluent Heun equation.

Let $x = 1/\xi$ and $y = \xi^{d/a}\eta$, when (2.1) becomes

(2.2)

$$\xi^2\eta'' + \left[(2-c)\xi^2 + \left(2\frac{d}{a} - b\right)\xi - a\right]\eta' + \left[\frac{d}{a}\left(1 - c + \frac{d}{a}\right)\xi + f - \frac{bd}{a}\right]\eta = 0 .$$

This is of the same form as (2.1), so that if

$$Z(a,b,c,d,f;x) \tag{2.3}$$

is a solution of (2.1), then so also is

$$x^{-d/a}Z(2-c, 2d/(a-b), -a, d/a(a-c+d/a), f - bd/a; 1/x) . \tag{2.4}$$

Similarily, if in (2.1), we replace y by $\exp(-ax)y$, then a further solution of (2.1) is seen to be

$$\exp(-ax)Z(-a,b,c,d-ab,f-ac;x) . \tag{2.5}$$

On combining (2.5) and (2.4), we obtain the fourth solution

(2.6)

$$\exp(-ax)x^{\frac{d}{a}-b}Z(2-c, b-2\tfrac{d}{a}, a, (b-\tfrac{d}{a})(1-c+b-\tfrac{d}{a}), f-ac+b(\tfrac{d}{a}-1); \tfrac{1}{x}) .$$

From these symmetries, it follows that if a convergent series representation of (2.3) can be found, then fundamental systems of solutions of (2.1) can be deduced relative to both of the irregular singularities. All these solutions will be convergent series.

3 – The Laplace transform of (2.1)

As usual, write

$$y = \int_C e^{xt}u(t)\,dt , \tag{3.1}$$

where C is any closed contour on the Riemann surface of the integrand.

On substituting this expression into (2.1), it is found that the modulating function $u(t)$ is any solution of

$$(3.2)\qquad t(t+a)u'' + [2a-d+(a-b)t]u' - [b-2-f-ct]u = 0\,.$$

Replace t by $-as$, when (3.2) assumes its standard form as a generalised Mathieu equation:

$$(3.3)\qquad s(1-s)u'' + \left[2-\frac{d}{a}-(4-b)s\right]u' = [2+f-b-acs]u = 0$$

with Ince classification $[0,2,1_1]$. See INCE [4] page 497.

Explicit series solutions of (3.3) have been obtained by Exton [3] by the intermediate use of the inhomogeneous hypergeometric functions studied by Babister [1].

The standard solution of (3.3) near the origin is written as the uniformly and absolutely convergent series

$$(3.4)\qquad CH(A,B;C;K;s) = \sum_{r=0}^{\infty} K^r u_r(s)\,.$$

The parameters A, B, C and K are given by

$$A+B = 5-b\,,\quad AB = 2+f-b\,,\quad C = 2-\frac{d}{a}\quad \text{and}\quad K = -ac\,.$$

The function $u_r(s)$ denotes the multiple series

$$(3.5)\qquad \frac{s^{2r}}{4^r r!(\frac{c}{2}+\frac{1}{2},r)}\sum A_{m_0,m_1,\dots,m_r}(A,B;C)s^{m_0+m_1+\dots+m_r}$$

and

(3.6)

$$A_{m_0,m_1,\dots,m_r}(A,B;C) = \frac{(A,m_0)(B,m_0)(C+1,m_0)(2,m_0)}{(A+2,m_0)(B+2,m_0)(C,m_0)(1,m_0)} \cdot$$

$$\cdot \frac{(A+2,m_0+m_1)(B+2,m_0+m_1)(C+3,m_0+m_1)(4,m_0+m_1}{(A+4,m_0+m_1)(B+4,m_0+m_1)(C+2,m_0+m_1)(3,m_0+m_1)} \cdot$$

$$\cdot \;\dots$$

$$\cdot \frac{(A+2r-2,m_0+m_1+\dots+m_{r-1})(B+2r-2,m_0+m_1+\dots+m_{r-1})}{(A+2r,m_0+m_1+\dots+m_{r-1})(B+2r,m_0+m_1+\dots+m_{r-1}))} \cdot$$

$$\cdot \frac{(C+2r-1,m_0+m_1+\dots+m_{r-1})(2r,m_0+m_1+\dots+m_{r-1})}{(C+2r-2,m_0+m_1+\dots+m_{r-1})(2r-1,m_0+m_1+\dots+m_{r-1})} \cdot$$

$$\cdot \frac{(A+2r,m_0+m_1+\dots+m_r)(B+2r,m_0+m_1+\dots+m_r)}{(C+2r,m_0+m_1+\dots+m_r))(1+2r,m_0+m_1+\dots+m_r))},$$

where, as usual,

$$(a,m) = a(a+1)(a+2)\cdots(a+m-1) = \Gamma(a+m)/\Gamma(a)\,, \quad (a,0) = 1\,. \tag{3.7}$$

Here, and in what follows, the summation sign $\sum$ applies for each of the indicies of summation $m_0, m_1, \dots, m_r$ from 0 to ∞ unless otherwise indicated.

It has also been shown in Exton [3] that the function

$$s^{1-C}(1-s)^{C-A-B}CH(1-A,1-B;2-C;K;s) \tag{3.8}$$

is also a solution of (3.3), and this is the form most convenient for the present purpose.

4 – Convergent series solutions of (2.1)

In (3.1), suppose that the contour integration C is a Pochhammer double loop slung around the points 0 and $-a$ in the t-plane. Hence, apart from any constant multiupliers and noting that $t = -as$, we see that

$$(4.1)\quad y = \int_{[0;1]} e^{-axs} s^{1-C}(1-s)^{C-A-B} CH(1-A, 1-B; 2-C; K; s)\, ds .$$

Since the series (3.4) converges both absolutely and uniformly on the contour, we have

$$(4.2)\quad y = \sum_{r=0}^{\infty} \frac{K^r}{4^r r! (\frac{3}{2} - \frac{C}{2}, r)} \sum A_{m_0, m_1, \dots, m_r}(1-A, 1-B; 2-C)\Upsilon ,$$

where

$$(4.3)\quad \Upsilon = \int_{[0;1]} e^{-axs} s^{1-C+m_0+m_1+\cdots+m_r+2r}(1-s)^{C-A-B} ds .$$

This integral may readily be evaluated by means of term-by-term integration and we have

$$\Upsilon = \sum_{p=0}^{\infty} \frac{(-ax)^p}{p!} \int_{[0;1]} s^{1-C+m_0+m_1+\cdots+m_r+p+2r}(1-s)^{C-A-B} ds \propto$$

$$(4.4)\quad \propto (-1)^{m_0+m_1+\cdots+m_r} \frac{(2-C, m_0+m_1+\cdots+m_r+2r)}{3-A-B, m_0+m_1+\cdots+m_r+2r)} {}_1F_1\left(\begin{matrix} 2-C+m_0+m_1+\cdots+m_r+2r; \\ 3-A-B+m_0+m_1+\cdots+m_r+2r; \end{matrix} ax\right) .$$

See Exton [2] page 17, for example.

Hence, the solution of (2.1) takes the form of the convergent series

$$(4.5)\quad y = \sum_{r=0}^{\infty} \frac{K^r}{4^r r! (\frac{3}{2} - \frac{C}{2}, r)} \cdot$$

$$\cdot \sum A_{m_0, m_1, \dots, m_r}(1-A, 1-B; 2-C) \frac{2-C, m_0+m_1+\cdots+m_r+2r)}{3-A-B, m_0+m_1+\cdots+m_r+2r)} \cdot$$

$$\cdot (-1)^{m_0+m_1+\cdots+m_r} {}_1F_1\left(\begin{matrix} 2-C+m_0+m_1+\cdots+m_r+2r; \\ 3-A-B+m_0+m_1+\cdots+m_r+2r; \end{matrix} ax\right) .$$

The confluent hypergeometric function ${}_1F_1$ has been exhaustively studied by many authors, including Slater [6].

If (2.4), (2.5) and (2.6) are now employed, fundamental systems of solutions valid near both of the irregular singularities of (2.1) can be written down.

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