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Schauder's Fixed Point Theorem Using a Caccioppoli Theorem

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RIASSUNTO – *Si dà una nuova dimostrazione del teorema di punto fisso di J. Schauder, utilizzando un teorema di punto fisso di R. Caccioppoli per trasformazioni compatte in* $C[0,1]$.

ABSTRACT – *In this paper we give a new proof of the Schauder fixed point theorem using the Caccioppoli fixed point theorem.*

KEY WORDS – *Separable Banach space - Universal space - Fixed point.*

A.M.S. CLASSIFICATION: 47H10 - 54H25

– Introduction

In 1912 L.E.J. BROUWER [1] demonstrated his famous fixed point theorem, according to which every continuous function of the unitary n-dimensional ball in itself has a fixed point.

Successively, in 1922, G.D. BIRKHOFF and D. KELLOGG [2] gave a new proof of this fundamental theorem also extending its validity to the compact and convex subsets of the space $L^2[0,1]$ and $C^n[0,1]$. The extension of the theorem to the case of the space $C[0,1]$ was also dealt with by R. CACCIOPPOLI in 1930 [4]. He gave then and in successive

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works various applications of this theorem to the ordinary and partial differential equations. R. Caccioppoli himself however, recognized the priority [5] of G.D. BIRKHOFF and O.D. KELLOGG in applying fixed point topological methods to the study of functional equations. Also in 1930 J. SCHAUDER [3] extended the theorem of Brouwer and Birkhoff and Kellogg's relative generalizations to every Banach space, proving both the famous theorem which bears his name, and the following consequence:

"If H is a weakly closed, weakly compact, convex subset of a separable Banach space, then every weakly continuous function of H in hitself has a fixed point".

M. KREIN and V. SMULIAN [6] have generalized this result thus:

"If H is a closed convex subset of a Banach space, then every weakly continuous function $f: H \to H$ such that $f(H)$ is weakly compact and separable, has a fixed point".

In both this theorems, as in the Schauder theorem, the role of the separability is not completely clear. In this note, on the other hand, separability has a preminent role because not only does it lead the Schauder theorem to the separable Banach spaces but it even allows us, through the use of the fundamental theorem of Banach and Mazur which establishes the universality of the space $C[0,1]$ whith respect to the class of separable Banach spaces, to obtain the same Schauder theorem as a consequence of the Caccioppoli theorem or, if preferred, that of Birkoff and Kellogg.

1 – Preliminary results

If X is a subset in a normed space E, let us call span(X) the linear hull of X, that is the linear subspace of E generated by X, or the subset of the finite linear combinations of elements of X. Moreover, let X^* the completion of X because it is a metric subspace of E and $\overline{X}$ the closure of X because it is a subset of E. It is well known that a normed space is separable if and only if it admits a total sequence (which can be composed by linearly independent vectors), cf. J. DIEUDONNÉ [8] pag. 110. In particular we have:

THEOREM 1.1. *The linear hull of a separable subset of a normed space is a separable normed space.*

If we consider then the closed linear hull of a subset X, that is the closure in E of span(X), we note that in a Banach space results

$$\overline{\mathrm{span}(X)} = (\mathrm{span}(X))^{*}$$

so that we have the following

COROLLARY 1.2. *The closed linear hull of a separable subset of a Banach space is a separable Banach space.*

On the other hand, as a precompact metric space is a separable space we have also

COROLLARY 1.3. *The closed linear hull of a compact subset of a Banach space is a separable Banach space.*

EXAMPLE 1.3A. We note that the linear hull of a compact subset of a Banach space is not, generally speaking, a Banach space. For example in $C[0,1]$ the linear hull of the compact set

$$C = \{f_n | n \in N\}$$

where

$$f_0 = 0 \quad \text{and} \quad f_n(x) = x^n/n \quad \forall x \in [0,1]$$

is the subspace of the polinomials.

Moreover, given a function $f\colon X \to X$ and denoted by $\mathrm{Fix}(f)$ the set of fixed points of f, that is

$$\mathrm{Fix}(f) = \{x \in X | f(x) = x\}$$

the following is easily demonstrated

LEMMA 1.4. *Let X and Y sets, $f\colon X \to X$ a function, $j\colon X \to Y$ a bijection. Then the mapping $f'\colon Y \to Y$ defined by*

$$f' \colon = j \circ f \circ j^{-1}$$

has fixed points if and only if f has fixed points and it results

$$\mathrm{Fix}(f') = j(\mathrm{Fix}(f)).$$

2 – The theorem of Schauder

Let us call Σ, as in [4], the space of real-valued continuous functions on a compact interval $[a,b]$ with the usual norm. The previously mentioned Caccioppoli theorem (cf. [4] pag. 24-25) can be used thus expressed:

THEOREM 2.1. (R. CACCIOPPOLI [4]). *A continuous mapping $T: \Sigma \to \Sigma$ such that $T(\Sigma)$ is totally bounded, has a fixed point.*

Finally, we can demonstrate

THEOREM 2.1. (J. SCHAUDER [3]). *Let C be a nonempty compact convex subset of a real Banach space E and $T: C \to C$ continuous. Then T has a fixed point.*

PROOF. As C is compact, because of the corollary 1.3, the linear closed hull of C, $\overline{\text{span}(C)}$, is a separable Banach space, a subspace of E. Therefore, in virtue of the theorem of S. BANACH and S. MAZUR (cf. [10] pag. 169 or also [9] pag. 219), a linear isometry

$$\beta: \Sigma \to C[0,1]$$

exists such that $\beta(\overline{\text{span}(C)})$ is closed linear subspace of $C[0,1]$. Assuming $H = \beta(C)$, for the continuity of β and the supposed compactness of C, H is compact; moreover it is also convex because of the linearity of β. Now let us suppose that

$$\gamma: C \to H \quad \text{such that} \quad \gamma(x) = \beta(x) \quad \forall x \in C$$

and consider the continuous mapping

$$T_1 = \gamma \circ T \circ \gamma^{-1}$$

of H in itself. Then let T_1^* be J. DUGUNDJI's [7] extension of T_1 on $C[0,1]$. Therefore T_1^* is a continuous mapping of Σ in itself having its range included in the convex hull $\text{co}(T_1(H))$ of $T_1(H)$, which is precompact. In virtue of the Caccioppoli theorem, the equation $T_1^*(f) = f$, has at least one solution, and from

$$T_1^*(\Sigma) \subseteq \mathrm{co}(T_1(H)) \subseteq H$$

its results that the equation $T_1(f) = f$ can be solved. Thus it results that the equation $\mathrm{Fix}(T_1) \neq \emptyset$. Therefore, applying the lemma, the proof is complete. □

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