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Tame Intersection Cohomology

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RIASSUNTO – *In questo articolo dimostriamo che la comologia di intersezione di una pseudovarietà X può essere calcolata a partire da certi complessi di deRham di Cenkl e Porter per complessi simpliciali.*

ABSTRACT – *In this paper we prove that the intersection cohomology of a pseudomanifold X can be computed from certain deRham complex of Cenkl and Porter for a simplicial complex.*

KEY WORDS – *Tame cohomology - Intersection cohomology - stratified pseudomanifolds.*

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Let X be an n-dimensional pseudomanifold (i.e. a compact space with a closed subspace Σ, $\dim \Sigma \leq n-2$, such that $X - \Sigma$ is an oriented n-dimensional which is dense in X). Let $[X] \in H_n(X)$ be the fundamental class of X and let $\cap[X]: H^{n-i}(X) \to H_i(X)$ be the Poincaré map.

If X is a manifold (i.e. $\Sigma = \emptyset$), then

(1) $\cap[X]$ is an isomorphism,

(2) $H_i(X) \times H_{n-i} \to \mathbf{Z}$, is nonsingular when tensored by $\mathbf{Q}$.

If X is a pseudomanifold (i.e. $\Sigma \neq \emptyset$), then

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(1) and (2) are false in general,

$(2)_I$ holds for the intersection homology $IH_\bullet(X)$, [9].

PROBLEM: Find a deRham type complex $I\Omega^\bullet(X)$ with the cohomology $IH^\bullet(X)$ such that there is an isomorphism

$(1)_I \qquad IH^{n-i}(X) \to IH_i(X).$

If so, is the pairing in $(2)_I$ induced by the $\wedge$-product of the differential forms?

In this note we give a construction of $I\Omega^\bullet(X)$ based on the tame chain complex dual to the tame deRham complex of Cenkl and Porter of [5], [13]. The resulting complex is a complex of differential forms on a triangulation of a pseudomanifold. The cohomology of that complex also carries a torsion information of the intersection homology.

Over the reals there are two constructions of a deRham complex whose cohomology is isomorphic to the intersection homology [1],[2],[12].

None of these complex is closed under the product of forms. The new invariant which arise from this phenomenon will be studied elsewhere.

1 – Tame chain and cochain complexes

We use the following notation:

X stands for a simplicial set.

$A^{\bullet,q}_\bullet$ is a simplicial $\mathbf{Q}_q$-module such that for each n, $A^{\bullet,\bullet}_n$ is the tame deRham complex,

$C^\bullet_{\bullet,q}(X) = \mathrm{Hom}(A^{\bullet,q}_\bullet; \mathbf{Q}_q)$,

$C_{\bullet,q}(X) = \oplus_m S_m(X) \otimes C^m_{\bullet,q}/\sim$, where $\sim$ is an equivalence relation which makes $C_{\bullet,q}$ a differential graded $\mathbf{Q}_q$-module,

$A^{\bullet,q}(X) = \mathrm{Hom}(C_{\bullet,q}; \mathbf{Q}_q)$.

THEOREM 1.1 ([5], [13]). *There are the isomorphisms $H_\bullet(X, \mathbf{Q}_q) \simeq H_\bullet(C_{\bullet,q}(X))$ of modules, and the isomorphisms $H^\bullet(X, \mathbf{Q}_\bullet) \simeq H^\bullet(A^{\bullet,\bullet}(X))$ of algebras.*

2 – The tame intersection chain complexes

Let $X = X_n \supset \Sigma = X_{n-2} \supset X_{n-3} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$ be a stratification of an n-dimensional pseudomanifold X as in [7] or [9]. Let X_T be an n-dimensional pseudomanifold X together with a triangulation T, subordinate to the stratification. This means, in particular, that each X_k is a subcomplex of X and the set of vertices V of X_T is ordered. Therefore X_T is a simplicial complex.

With a simplicial complex there is associated a simplicial set constructed by adding to the simplices of the simplicial complex all the degenerate simplices. Such a simplicial set is called polyhedral ([6, p.111]). Thus with X_T there is associated the polyhedral simplicial set X^T. Here $X^T_n = \{(v_0, \ldots, v_n)/\ v_0 \leq \cdots \leq v_n, v_i \in V\}$, where $\leq$ stands for the ordering of vertices. The face and degeneracy operators are defined in the usual fashion. With each stratum X_k there is associated a subpolyhedral simplicial set $X^T_k = \{X^T_{k,(n)}; d_i, s_i\}$, and there are the inclusions of polyhedral simplicial sets corresponding to the stratifications of X; $X^T = X^T_n \supset \Sigma^T = X^T_{n-2} \supset X^T_{n-3} \supset \cdots \supset X^T_0 \supset X^T_{-1} = \emptyset$.

Let T' be the barycentric subdivision of T. Then with the polyhedral simplicial set X^T there is associated a simplicial set $X^{T'}$ corresponding to the barycentric subdivision of X_T. There is also a direct construction of $X^{T'}$ from X ([6, p.199]). Considering the set V of nondegenerate simplices of X^T and the partial ordering $\leq$ on V; $x \leq y$ when x is a face of y, set

$$(X^{T'})_n = \{(x_0, \ldots, x_n)\ /\ x_0 \leq x_1 \leq \cdots \leq x_n\}, \quad n = 0, 1 \ldots.$$

With the polyhedral simplicial set X^T we associate the topological space $\overline{X^T} = \bigcup (X^T_n \times \Delta_n)$, where X^T is given the discrete topology. Then the topological realization of X^T is the topological space $RX^T = \overline{X^T}\ /\sim$, where the equivalence relation $(x, a) \sim (y, b)$ is given by either one of the two conditions: (i) $d_i x = y$ and $a = \delta_i b$, (ii) $s_i x = y$ and $a = \sigma_i b$. Here δ_i, σ_i denote the face and degeneracy operators on the simplicial set Δ. Since X_T is a (geometric) simplicial complex, the geometric realization RX^T is homeomorphic to the topological space X_T ([6, p.118]). Denote this homeomorphism by $r: RX^T \longrightarrow X_T$. This homeomorphism can be chosen in such a way that the composition

$$\overline{X^T} \longrightarrow RX^T = \overline{X^T}\ /\sim\ \longrightarrow X_T$$

restricted to $c \times \Delta_m$, where c is an m-simplex with vertices $v_0, \ldots, v_m$, is a homeomorphism of $c \times \Delta_m$ onto the m-simplex $[v_0, \ldots, v_m]$ of X_T. Let T' be a refinement of T and let $c, c' \in X^{T'}$. If the simplex $rp(c' \times \Delta_m)$ of $X_{T'}$ is contained in the simplex $rp(c \times \Delta_m)$ of X_T, we say that c' is contained in c.

Let R be a commutative ring and let $C_*(X^T; R)$ and $C_*(X^{T'}; R)$ be the simplicial complexes of X^T and $X^{T'}$ respectively. Then there is a unique homomorphism of complexes

$$C_*(X^T; R) \longrightarrow C_*(X^{T'}; R)$$

defined as follows. To each $\xi \in C_i(X^T; R)$ we assign $\xi' \in C_i(X^{T'}; R)$ so that $\xi'(c') = 0$ for $c' \in X^{T'}$ not contained in an i-simplex $c \in X^T$ for which $\xi(c) \neq 0$, and $\xi'(c') = \xi(c)$ if c' is contained in an n-simplex $c \in X^T$ for which $\xi(c) \neq 0$. Let $C_*(X; R)$ be the direct limit of the complexes $C_*(X^T; R)$ over all the triangulations of the polyhedron X.

The support $|\xi|_T$ of $\xi \in C_i(X^T; R)$ is the subcomplex of X^T (i.e. the sub-polyhedral simplicial set). For a refinement T' of T the geometric realizations $R(|\xi|_T)$ and $R(|\xi|_{T'})$ are homeomorphic. We denote these realizations by $|\xi|$. $|\xi|$ is a subspace of X. A geometric chain ξ is said to have the support $|\xi|$.

A perversity $p = (p_2, p_3, \ldots, p_n)$ is a sequence of integers such that $p_2 = 0$ and $p_{k+1} = p_k$ or $p_{k+1} = p_k + 1$; [7].

Let

$$\text{(C.)} \qquad I_pC_{i,q}(X^T)$$

be the subgroup of $C_{i,q}(X^T)$ consisting of those i-chains ξ of weight q for which

$$(\mathrm{P}_1)_T \qquad \dim(|\xi|_T \cap X^T_{n-k}) \leq i - k + p_k$$

$$(\mathrm{P}_2)_T \qquad \dim(|\partial\xi|_T \cap X^T_{n-k}) \leq i - 1 - k + p_k.$$

From $(\mathrm{P}_1)_T$ it follows that $I_pC_{*,q}(X^T)$ is a subcomplex of $C_{*,q}(X^T)$. The homology group

$$I_pH_{i,q}(X^T)$$

of the complex $I_pC_{*,q}(X^T)$ is called the i-th **tame intersection homology group of perversity p.**

LEMMA 2.1. *The conditions $(P_1)_T$ and $(P_2)_T$ are independent of a particular triangulation.*

PROOF. Observe that the dimensions of the space involved are equal to the dimensions of their geometrical realizations. □

Let $Q_{p,i} = Q^T_{p,i}$ be the space $Q^{\bar{p}}_i$ in [7], and let

$$H_{\bullet,q}(Q^T_{p,i}) = H_\bullet(C_{\bullet,q}(Q^T_{p,i}))$$

be the tame homology of the complex $Q^T_{p,i}$.

LEMMA 2.2.

$$I_pH_{i,q}(X^T) = \mathrm{Im}\{H_{i,q}(Q^T_{p,i}) \longrightarrow H_{i,q}(Q^T_{p,i+1})\}.$$

PROOF. The proof is similar to that of [7]. □

COROLLARY 2.3. *There is an isomorphism of modules*

$$I_pH_{i,q}(X^T) \simeq I_pH_i(X^T; \mathbf{Q}_q).$$

PROOF. $I_pC_{\bullet,q}(X^T)$ is a subcomplex of $C_{\bullet,q}(X^T)$ and by [13], there is an isomorphism $H_\bullet(C_{\bullet,q}(X^T)) \simeq H_\bullet(X^T; \mathbf{Q}_q)$. Therefore

$$\mathrm{Im}\{H_i(C_{\bullet,q}(Q^T_{p,i})) \longrightarrow H_{i+1}(C_{\bullet,q}(Q^T_{p,i+1}))\} \simeq$$
$$\mathrm{Im}\{H_i(Q^T_{p,i}; \mathbf{Q}_q) \longrightarrow H_q(Q^T_{p,i+1}; \mathbf{Q}_q)\}$$

$= I_pH_{i,q}(X^T)$ by [7, p.160]. □

COROLLARY 2.4. *There are isomorphisms*

(i) $I_pH_i(X_T; \mathbf{Q}_q) \simeq I_pH_i(X^T; \mathbf{Q}_q)$,

(ii) $I_pH_i(X; \mathbf{Q}_q) \simeq I_pH_i(X_T; \mathbf{Q}_q)$.

PROOF. (i) This isomorphism is a composition of the isomorphisms $H_i(Q_{p,i}; \mathbf{Q}_q) \simeq H_i(RQ^T_{p,i}; \mathbf{Q}_q)$ and $H_i(Q^T_{p,i}; \mathbf{Q}_q) \simeq H_i(RQ^T_{p,i}; \mathbf{Q}_q)$.

Next, recall that $Q_{p,i}$ is a subspace of X_T and that

$$\begin{aligned} I_pH_i(X_T; \mathbf{Q}_q) &= \mathrm{Im}\{H_i(Q_{p,i}; \mathbf{Q}_q) \longrightarrow H_{i+1}(Q_{p,i+1}; \mathbf{Q}_q)\} \\ &\simeq \mathrm{Im}\{H_i(RQ^T_{p,i}; \mathbf{Q}_q) \longrightarrow H_{i+1}(RQ^T_{p,i+1}; \mathbf{Q}_q)\} \\ &\simeq \mathrm{Im}\{H_i(Q^T_{p,i}; \mathbf{Q}_q) \longrightarrow H_{i+1}(Q^T_{p,i+1}; \mathbf{Q}_q)\} \\ &= I_pH_i(X^T; \mathbf{Q}_q). \end{aligned}$$

(ii) The second isomorphism follows from the definition of intersection homology with coefficients in a ring ([7],[10]) □

Let R be a unitary ring with a unit. An R-orientation of an n-dimensional pseudomanifold X_T is an orientation of its regular part $X_T - \Sigma$. Such an orientation is given by an n-chain which associates with each oriented n-simplex the unit element of R. The corresponding geometric cycle is denoted by $[X]$. Let X^T be the polyhedral simplicial set associated with the polyhedron X_T ([6, p.199]). Let T' be the first barycentric subdivision of T and let $X^{T'}$ be the corresponding polyhedral simplicial set.

LEMMA 2.5. *A $\mathbf{Q}_q$-orientation $[X]$ of the pseudomanifold X_T induces a natural morphism of complexes*

$$A^{n-\bullet,q}(X^T) \xrightarrow{\cap[X]} C_{\bullet,q}(X^{T'}).$$

The map $\cap[X]$ induces the homomorphism of modules

$$H^{n-k,q}(X^T) \xrightarrow{\cap[X]} H_{k,q}(X^{T'}).$$

This isomorphism is not an isomorphism for a singular pseudomanifold X_T.

LEMMA 2.6. *There is a chain map*

$$\beta: A^{n-\bullet,q}(X^T) \longrightarrow C_{\bullet,q}(X^T).$$

such that the diagram

$$\begin{array}{ccc} A^{n-k,q}(X) & \xrightarrow{\cap[X]} & C_{k,q}(X) \\ & {\scriptstyle\beta}\searrow \quad \nearrow{\scriptstyle\iota} & \\ & I_pC_{k,q}(X) & \end{array}$$

commutes for every $q = 0,1,2\ldots$; $k = 0,1,2,\ldots$ and every perversity p. Here ι stands for the inclusion.

PROOF. The statement follows from the argument given in [7]. □

While the chain complex $I_pC_{\bullet,\bullet}(X)$ is a subcomplex of the ordinary chain complex $C_{\bullet,\bullet}(X)$, it was observed, already in the original paper of Goresky and MacPherson ([7]), that the intersection homology can be expressed in terms of ordinary homology of a flag of subcomplexes of X: $Q_{p,0} \subset Q_{p,1} \subset \cdots \subset Q_{p,n} = X$, for every perversity p. Let $\iota: Q_{p,i-1} \to Q_{p,i}$ and let $\iota_0: Q_{p,i} \to X$ be the inclusions and let $\{C_{\bullet,q}(Q_{p,i}), d\}$ be the chain complex associated with the subcomplex $Q_{p,i}$ of X. Set

$$(\hat{C}.) \qquad I_p\hat{C}_{i,q}(X) = \tilde{Z}_{i,q}(Q_{p,i+1}) \oplus W_{i,q}(Q_{p,i}),$$

where

$$\tilde{Z}_{i,q}(Q_{p,i+1}) = \{z \in C_{i,q}(Q_{p,i+1}) \;/\; z \in \iota_\# Z_{i,q}(Q_{p,i})\},$$

$$W_{i,q}(Q_{p,i}) = \{z \in C_{i,q}(Q_{p,i}) \;/\; dz \neq 0,\ \pi_\# dz = 0\},$$

and $\iota_\#$, $\pi_\#$ stand for the inclusion and projection in the exact sequence

$$0 \longrightarrow C_{\bullet,q}(Q_{p,\bullet}) \xrightarrow{\iota_\#} C_{\bullet,q}(Q_{p,\bullet+1}) \xrightarrow{\pi_\#} C_{\bullet,q}(Q_{p,\bullet+1}, Q_{p,\bullet}) \longrightarrow 0.$$

Note that the induced differential maps $\tilde{Z}_{i,q}(Q_{p,i+1})$ to zero and that

$$dW_{i+1,q}(Q_{p,i+1}) \subset \tilde{Z}_{i,q}(Q_{p,i+1}).$$

LEMMA 2.7. *The composition*

$$I_p\hat{C}_{i,q}(X) \longrightarrow C_{i,q}(Q_{p,i}) \longrightarrow C^p_{i,q}(X)$$

induces an inclusion of chain complexes

$$\psi\colon I_p\hat{C}_{\bullet,q}(X) \longrightarrow I_pC_{\bullet,q}(X)$$

which gives an isomorphism on homology.

PROOF. First of all we observe that the image of the composition lies in $I_pC_{i,q}(X)$ for every i, q and p. But, according to [7, p.148], $\dim(Q_{p,j} \cap X^{T}_{n-k}) \leq j - k + p_k$. Therefore for every $x \in C_{i,q}(Q_{p,i})$, $\iota_{0\#}x \in C^{q}_{i,q}(X)$, and $\dim(|\iota_{0\#}x| \cap X^{T'}_{n-k}) \leq i - k + p_k$ for every k because $|x| = |\iota_{0\#}x|$. Since $dx \in C_{i-1,q}(Q_{p,i-1})$, $\iota_{0\#}dx = d\iota_{0\#}x \in C_{i-1,q}(X)$ and $|d\iota_{0\#}x| = |dx|$ it follows that $\dim(|d\iota_{0\#}x| \cap X^{T'}_{n-k}) \leq i - k - 1 + p_k$ for every k. Hence the inclusion ι gives a homomorphism ψ of differential graded modules. In order to prove that ψ induces an isomorphism on homology it suffices to show that ψ induces an isomorphism $\psi_*\colon H(\{I_p\hat{C}_{\bullet,q}(X), d\}) \simeq \mathrm{Im}\{H_{i,q}(Q_{p,i}) \to H_{i,q}(Q_{p,i+1})\}$, for $i = 0, 1, \ldots, n$. Observe that

$$\begin{aligned}
\mathrm{Im}\{H_{i,q}(Q_{p,i}) &\longrightarrow H_{i,q}(Q_{p,i+1})\} = \\
&\iota_{\#}Z_{i,q}(Q_{p,i}) \,/\, dC_{i+1,q}(Q_{p,i+1}) \cap \iota_{\#}C_{i,q}(Q_{p,i}) = \\
&\tilde{Z}_{i,q}(Q_{p,i+1}) \,/\, dW_{i+1,q}(Q_{p,i+1}) = \\
&H(\{I_p\hat{C}_{\bullet,q}(X), d\}).
\end{aligned}$$

□

With the chain complexes

(C.) $\{I_pC_{\bullet,q}(X), d\}$

(Ĉ.) $\{I_p\hat{C}_{\bullet,q}(X), d\}$

there are associated the dual cochain complexes (A.) and (Â.);

(A.) $\{I_pA^{\bullet,q}(X), d\}$,

$I_pA^{i,q}(X) = A^{i,q}(X)/J_pA^{i,q}(X)$. Here $J_pA^{i,q}(X)$ is the $\mathbf{Q}_q$-submodule of those i-forms on X which vanish on the intersection i-chains

(Â.) $\{I_p\hat{A}^{\bullet,q}(X), d\}$,

$I_p\hat{A}^{i,q}(X) = \mathrm{Hom}(I_p\hat{C}_{i,q}(X), \mathbf{Q}_q)$, and the induced differential maps $\mathrm{Hom}(W_{i,q}(Q_{p,i}), \mathbf{Q}_q)$ to zero, and $\mathrm{Hom}(\tilde{Z}_{i,q}(Q_{p,i+1}), \mathbf{Q}_q)$ to $\mathrm{Hom}(W_{i+1,q}(Q_{p,i+1}), \mathbf{Q}_q)$.

LEMMA 2.8. (i) *The cohomology of the cochain complex $(\hat{A}.)$ and the intersection homology are related by the exact sequence*

$$0 \longrightarrow Ext(I_pH_{i-1,q}(X), \mathbf{Q}_q) \longrightarrow H^i(\{I_pA^{\bullet,q}(X), d\}) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}(I_pH_{i,q}(X), \mathbf{Q}_q) \longrightarrow 0.$$

(ii) *There are the isomorphisms of $\mathbf{Q}_q$-modules* $H^i(\{I_pA^{\bullet,q}(X), d\}) = H^i(\{I_p\hat{A}^{\bullet,q}(X), d\})$.

PROOF. Since the complexes (A.) and $(\hat{A}.)$ are the dual complexes of (C.) and $(\hat{C}.)$ respectively, the cohomologies ate related by the short exact sequence. The second part follows from the chain equivalences of the chain complexes $\{I_pC^{\bullet,q}(X), d\}$ and $\{I_p\hat{C}^{\bullet,q}(X), d\}$. □

3 – The tame intersection deRham complex

Although the cochain complexes (A.), $(\hat{A}.)$ are expressible in terms of differential forms and their quotients, they are basically just the duals of the chain complexes (C.) $(\hat{C}.)$. It seems more appropriate to construct a complex of differential forms on a triangulation of X whose cohomology is isomorphic to the intersection homology via the Poincaré map. Here we present a construction of such a complex (Ω) and then relate its cohomology to the cohomologies of the complexes (A.) and $(\hat{A}.)$.

Let X be an n-dimensional pseudomanifold together with its triangulation T and its first derived triangulation T'. Let T_{n-2} be the $(n-2)$-skeleton of T. Let p be a given perversity, p' its complementary perversity (i.e. $p + p' = t = (0, 1, 2, \ldots, n-2)$). Let i and q be integers; $0 \le i \le n$ and $0 \le q$. An (i, q)-form on X is an element of the $\mathbf{Q}_q$-module

$$I_p\Omega^{i,q}(X) = \Omega_1^{i,q}(X) \oplus \Omega_2^{i,q}(X),$$

where the modules $\Omega_j^{i,q}(X)$ $j = 1, 2$, are constructed in two stages.

First we construct the modules $\Omega_j^{i,q}(Q_{p,n})$ for $i = 1, 2, \ldots, n$ and $j = 1, 2$. Then the deformation retraction

$$\rho : X \longrightarrow (Q_1^{p'} \cap |T_{n-2}|) \longrightarrow Q_n^p$$

is used to "pull back" the constructed modules.

3.1 – Construction of $\Omega_j^{i,q}(Q_{p,n})$, $j = 1,2$.

$\Omega_1^{i,q}(Q_{p,n})$ for $i = 1,2,\ldots,n-1$, is the $\mathbf{Q}_q$-module of (i,q)-forms which are d-closed and whose restriction to $Q_{p,i}$ is non zero.

$\Omega_1^{n,q}(Q_{p,n}) = A_1^{n,q}(Q_{p,n})$.

$\Omega_2^{i,q}(Q_{p,n})$ for $i = 1,2,\ldots,n-1$, is the $\mathbf{Q}_q$-module of (i,q)-forms ω on each n-simplex, such that the restrictions to $Q_{p,i+1}$ are all the solutions ω of the equation $d\omega = \beta$ for any $\beta \in dA^{i,q}(Q_{p,i+1})$ [i.e. for any (global) coboundary of $A^{i+1,q}(Q_{p,i+1})$], such that the restriction of ω to $Q_{p,i}$ $\omega|_{Q_{p,i}} = \alpha \in dA^{i-1,q}(Q_{p,i})$. If $\beta = 0$ then $\omega = 0$. Note that a global solution ω does not exist in general. But locally such solutions exist. If $\alpha = 0$ for example, then the existence of local solutions follows from the (local) surjectivity of the differential $d\colon A^q(Q_{p,i+1}; Q_{p,i}) \to A^{i+1,q}(Q_{p,i+1})$. An explicit construction of ω is based on the results of [4]. Since for any nonzero $\omega \in \Omega_2^{i,q}(Q_{p,n})$ the restriction $d\omega$ to $Q_{p,i+1}$ is a nonzero closed form, it follows that d maps $\Omega_2^{i,q}(Q_{p,n})$ to $\Omega_1^{i+1,q}(Q_{p,n})$. Hence we have

LEMMA 3.2.

$$I_p\Omega^{\bullet,q}(Q_{p,n}) = \Omega_1^{\bullet,q}(Q_{p,n}) \oplus \Omega_2^{\bullet,q}(Q_{p,n})$$

is a differential module for each perversity p and every $q \geq 1$.

LEMMA 3.3. *The cohomology of the complex $I_p\Omega^{\bullet,q}(Q_{p,n})$, in dimension i, is isomorphic to*

$$\operatorname{Im}\{H^i(Q_{p,i+1}, \mathbf{Q}_q) \longrightarrow H^i(Q_{p,i}, \mathbf{Q}_q)\}.$$

PROOF. First of all we prove that $\operatorname{Im}\{\cdots\}$ is isomorphic to the cohomology $H^i(I_p\Omega^{\bullet,q}(Q_{p,i}))$. Let $\Omega_\bullet^{\bullet,q}(Q_{p,i})$ be the restriction of $\Omega_\bullet^{\bullet,q}(Q_{p,n})$ to $Q_{p,i}$. Then $H^i(I_p\Omega^{\bullet,q}(Q_{p,i})) \simeq \Omega_1^{i,q}(Q_{p,i}) \,/\, d\Omega_2^{i-1,q}(Q_{p,i})$, by definition of the tame deRham complex Ω. Then $H^i(I_p\Omega^{\bullet,q}(Q_{p,n})) \simeq H^i(I_p\Omega^{\bullet,q}(Q_{p,i}))$ because the cohomology modules $H^i(I_p\Omega^{\bullet,q}(Q_{p,n}, Q_{p,i}))$ and $H^{i+1}(I_p\Omega^{\bullet,q}(Q_{p,n}, Q_{p,i}))$ of the relative complexes are zero; as the relatives cocycles are zero. □

Let $\Omega_j^{\bullet,q}(X)$, $j = 1,2$, be the extensions of the $\mathbf{Q}_q$-modules $\Omega_j^{\bullet,q}(Q_{p,n})$ to X so that the cohomology of

$$I_p\Omega^{\bullet,q}(X) = \Omega_1^{\bullet,q}(X) \oplus \Omega_2^{\bullet,q}(X)$$

is isomorphic to the cohomology of $I_p\Omega^{\bullet,q}(Q_{p,n})$. Such a complex $I_p\Omega^{\bullet,q}(X)$ is called the **tame intersection deRham complex with perversity p and coefficients $\mathbf{Q}_q$**. Its cohomology $I_pH^\bullet(X,\mathbf{Q}_q)$ is called the **tame intersection cohomology of X with coefficients in $\mathbf{Q}_q$**. Let

$$\phi\colon I_pH^i(X,\mathbf{Q}_q) \longrightarrow \operatorname{Im}\{H^i(Q_{p,i+1},\mathbf{Q}_q) \longrightarrow H^i(Q_{p,i},\mathbf{Q}_q)\}$$

be the isomorphism of modules.

Recall that the key step in the proof of Generalized Poincaré Duality theorem ([7]) is the proof of the isomorphism

$$I_{p'}H_j(X,\mathbf{Q}_q) \simeq \operatorname{Im}\{H^i(Q_{p,i+1},\mathbf{Q}_q) \longrightarrow H^i(Q_{p,i},\mathbf{Q}_q)\},$$

where p and p' are complementary perversities, i.e. $p + p' = t = (0,1,2,\ldots,n-2)$ and $i + j = n$. That isomorphism is the composition of the following four isomorphisms

$$\begin{array}{c}
I_{p'}H_j(X,\mathbf{Q}_q)\\
\phi_3\Big\uparrow\\
\operatorname{Im}\{H_j(Q_{p'j}) \longrightarrow H_j(Q_{p'j+1})\}\\
\phi_2\Big\uparrow\\
\operatorname{Im}\{H_j(X-Q_{p,i+1}\cap|T_{n-2}|,X-Q_{p,i+1}) \longrightarrow H_j(X-Q_{p,i}\cap|T_{n-2}|,X-Q_{p,i})\}\\
\cap[X]\Big\uparrow\\
\operatorname{Im}\{H^i(Q_{p,i+1},Q_{p,i+1}\cap|T_{n-2}|) \longrightarrow H^i(Q_{p,i},Q_{p,i}\cap|T_{n-2}|)\}\\
\phi_1\Big\uparrow\\
\operatorname{Im}\{H^i(Q_{p,i+1}) \longrightarrow H^i(Q_{p,i})\}
\end{array}$$

where $i + j = n$, $p + p' = t$ and where all the homologies are with Q_q coefficients. The isomorphisms ϕ_1 and ϕ_2 come from the long exact sequences for the pairs; $\cap[X]$ is the classical Poincaré duality isomorphism in a relative form. The isomorphism ϕ_3 is constructed in [7]. Let

$$\mathring{\cap}[X] = \phi_3 \circ \phi_2 \circ \cap[X] \circ \phi_1 \circ \phi$$

be the composition of all these isomorphisms, together with our isomorphism ϕ, as constructed above. Then we get

THEOREM 3.4.

$$I_p H^i(X, Q_q) \xrightarrow{\mathring{\cap}[X]} I_{p'} H_{n-i}(X, Q_q)$$

is an isomorphism of Q_q-modules, for $i = 0, 1, \ldots, n$, $p + p' = t$ and any $q = 1, 2, \ldots$.

Note that this generalized Poincaré isomorphism $\mathring{\cap}[X]$ gives also the isomorphism of all the torsion in the intersection homology with the torsion in the tame intersection cohomology.

The complex $I_p \hat{A}^{\cdot,q}(X)$ can be "globalized" (following the construction of the complex $I_p \Omega^{\cdot,q}(X)$). The cohomology $I_p \hat{H}^{\cdot}(X, Q_q)$ of the resulting complex $I_p \Omega^{\cdot,q}(X)$ is isomorphic to the module $H^{\cdot}(I_p \hat{A}^{\cdot,q}(X))$. These modules are related to the intersection homology by the short exact sequence in Lemma 2.8. They are also related to the intersection cohomology modules $I_p H^{\cdot}(X, Q_q)$ in the following way.

PROPOSITION 3.5. *The cohomology modules $I_{\cdot} H^{\cdot}(X, Q_{\cdot})$ and $I_{\cdot} \hat{H}^{\cdot}(X, Q_{\cdot})$ are related by the exact sequence*

$$0 \longrightarrow Ext(I_p H^{n-i+1}(X, Q_q), Q_q) \longrightarrow I_{p'} \hat{H}^i(X, Q_q) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}(I_p H^{n-i}(X, Q_q), Q_q) \longrightarrow 0,$$

where $p + p' = t$.

Note that $I_p\Omega^{\cdot,\cdot}(X)$ is not closed under the wedge product. For example, let $\omega_k = \zeta_k + \alpha_k$, $k = 1,2$, be two elements of $I_p\Omega^{\cdot,\cdot}(Q_{p,n})$, such that $\zeta_k \in I_p\Omega_1^{i_k,q_k}(Q_{p,n})$, $\alpha_k \in I_p\Omega_2^{i_k,q_k}(Q_{p,n})$. Then $\omega_1 \wedge \omega_2 = \zeta_1 \wedge \zeta_2 + \xi$. The restriction of ξ to $Q_{p,i+j}$ is not an element of $dA^{i_1+i_2-1}(Q_{p,i_1+i_2})$ in general. Hence $\omega_1 \wedge \omega_2$ does not belongs to $I_p\Omega^{i_1+i_2,\cdot}(Q_{p,n})$. This difficulty does not disappear even when the perversities p are different for ω_1 and ω_2. However the wedge product of any two such forms is a well defined form on every simplex of T, and the integral of such product is also well defined. Using the symbol $\mathring{\cap}[X]$ for that integration we can define a pairing

$$I_{p_1}H_i(X,\mathbb{Q}_{q_1}) \times I_{p_2}H_{n-i}(X,\mathbb{Q}_{q_2}) \xrightarrow{\bullet} \mathbb{Q}_{q_1+q_2}$$

by setting $(a\mathring{\cap}[X]) \circ (b\mathring{\cap}[X]) = (a \wedge b)\mathring{\cap}[X]$. This pairing, when tensored with the rationals, for $p_1 + p_2 = t$, is the pairing of [7].

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