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A relation between total multigraphs and total multidigraphs

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RIASSUNTO – *Si considerano multigrafi e multidigrafi finiti, connessi con o senza cappi. Si definiscono i concetti di multigrafo e multidigrafo totale, caratterizzando quest'ultimo con l'uso di digrafi di suddivisione, generalizzando in modo naturale un risultato di Chartrand e Stewart [2]. Si stabilisce inoltre una relazione tra le nozioni di totalità per il caso orientato e quello non orientato simile a quella data per i grafi commutati [3].*

ABSTRACT – *We consider finite, connected multigraphs and multidigraphs with loops permitted. We define the notions of total multigraph and total multidigraph and we characterise the latter by means of subdivision digraphs, extending in a natural way a result of Chartrand and Stewart [2]. Furthermore, we obtain a relation between the notions of total in the directed and undirected cases analogous to that established in [3] for the line-graph concept.*

KEY WORDS – *Total graph - Line-graph.*

A.M.S. CLASSIFICATION: 05C

1 – Introduction

In 1932 WHITNEY [9] obtained some relations between the line-isomorphisms and the vertex-isomorphisms in graphs without loops. In order to give a new proof of some of these results, in 1943, KRAUSZ [8] defined and characterized the notion of line-graph.

This concept was later extensively studied and generalized in different ways to graphs with loops or multiple lines.

A similar notion for digraphs without loops was studied in 1960 by HARARY and NORMAN [5] and independently for arbitrary multidigraphs, in 1964, by HEUCHENNE [7].

The only attempt to relate the line-graph transformation with the line-digraph transformation, known to us, was done in 1979 by CHIAPPA [3]. There, two generalizations of the line-graph concept were considered and it was proved that each of them can be obtained by means of certain operations which involve the line-digraph transformation.

On the other hand, in order to study coloring problems and following the same idea that lead to the concept of line-graph BEHZAD [1] introduced in 1965 the concept of total graph. In 1966 CHARTRAND and STEWART [2] studied total digraphs.

As far as we know, total digraphs have not been studied since and all the papers on total graphs assume that there are neither multiple lines nor loops.

We next extend, in a natural way, the concept of total multigraphs and multidigraphs and we find a relation between them similar to that established for the line-graph notion in [3]. Also we characterize the total of a multidigraph D by means of the square of the subdivision digraph of D, as CHARTRAND and STEWART in [2].

We shall consider finite, connected multigraphs and multidigraphs with loops permitted and we shall use, in general, the terminology of [4], [5], [6].

1.1. Let $D = \langle V(D), U(D) \rangle$ be a multidigraph. We say that:

a) the point v_i is k-precedent to the point v_j if there exist k arcs (v_i, v_j).

b) the arc $x = (v_i, v_j)$ is precedent to the arc $y = (v_j, v_k)$.

c) the point v_i is precedent to the arc x and x is precedent to v_j if $x = (v_i, v_j)$.

Remark that in b) and c) precedent means k-precedent for $k = 1$, and k-precedent is meaningless for $k > 1$.

1.2. Let D be a multidigraph. The total multidigraph $T(D)$ of D is a multidigraph whose point are in one-to-one correspondence with the points and arcs of D, and such that the point u is k-precedent to the point v in $T(D)$ if and only if in D the element corresponding to u is k-precedent to the element corresponding to v. (See figure 1).

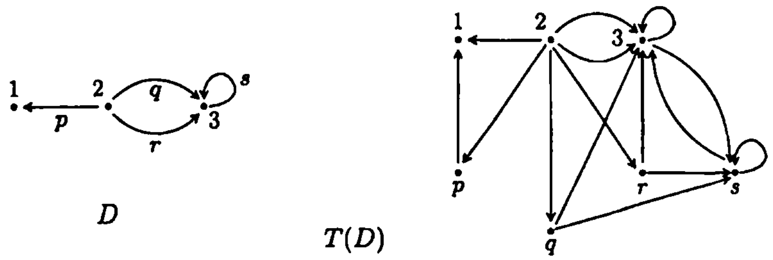


Fig. 1

A multidigraph H is a total multidigraph if there exists a multidigraph D such that $H = T(D)$.

The induced subdigraph of $T(D)$ whose points represent the arcs of D is the line-digraph or adjoint-digraph of D which we shall denote by $A(D)$.

1.3. It is easy to see that:

a) If D has p points and q arcs, then $T(D)$ has $p + q$ points and $3q + \sum_{i=1}^{p} \mathrm{id}(v_i) \cdot \mathrm{od}(v_i)$ arcs.
b) $T(D)$ is the arc-disjoint union of the submultidigraphs D, $A(D)$ and $M(D)$, where $M(D)$ is the digraph whose points are the same as those of $T(D)$ and whose arcs are $(v_i, x)(x, v_j)$ for each arc $x = (v_i, v_j)$.

1.4. Let $G = \langle V(G), U(G)\rangle$ be a multigraph. We say that:

a) the points v_i, v_j are k-adjacent if there exists k lines $[v_i, v_j]$.
b) two different lines are k-adjacent if they have k common points. In this case k can only be 1 or 2. Any loop is 1-adjacent with itself.
c) if $x = [v_i, v_j]$, $v_i \neq v_j$ then x is 1-adjacent to v_i and v_j. If $v_i = v_j$ then x and v_i are 1-adjacent.

We shall also say adjacent instead of 1-adjacent.

1.5. Let G be a multigraph. The total multigraph $\mathcal{T}(G)$ of G is a multigraph whose points are in one-to-one correspondence with the points and lines of G, and the point u is k-adjacent to the point v in $\mathcal{T}(G)$ if and only if in G the element corresponding to u is k-adjacent to the element corresponding to v (see figure 2).

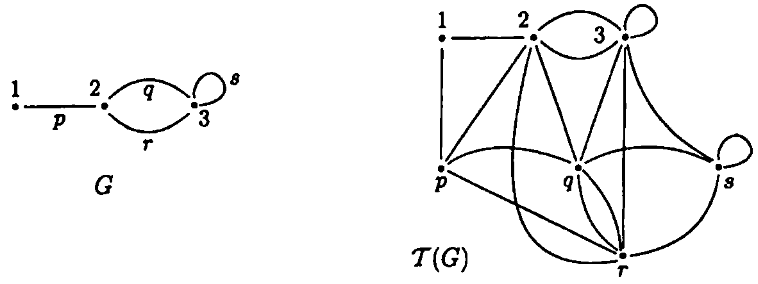


Fig. 2

A multigraph H is a total multigraph if there exists a multigraph G such that $H = \mathcal{T}(G)$.

The induced submultigraph of $\mathcal{T}(G)$ whose points represent the lines of G coincides with the v-line-graph of G defined in [3]. Now we simply say line-graph of G and we shall denote it $\mathcal{A}(G)$.

1.6. It is easy to see that if G has p points and q lines then $\mathcal{T}(G)$ has $p+q$ points and $3q+\sum_{i=1}^{p}\binom{a_i}{2}$ lines, where a_i denotes the number of incident lines at the point v_i.

2 – Characterization of total multidigraphs

Utilizing the so-called subdivision digraph, in the same way as [2], we shall give a necessary and sufficient condition for a multidigraph to be a total multidigraph.

2.1. The subdivision digraph $S(D)$ of a multidigraph D is that digraph obtained from D by replacing each of the k arcs (v_i, v_j) of D with a new point v_{ijt}, $1 \leq t \leq k$ and the two arcs (v_i, v_{ijt}), (v_{ijt}, v_j).

A digraph S is called a subdivision digraph if there exists a multidigraph D such that $S = S(D)$.

The point x of a multidigraph D is a carrier if $\mathrm{id}(x) = \mathrm{od}(x) = 1$. The points v_{ijt} of a subdivision digraph obtained as above are carriers.

2.2. THEOREM. *A digraph S is a subdivision digraph if and only if:*

a) *S is a cycle of even length n, $n \geq 2$; or*

b) *S is not a cycle and every semipath joining two noncarriers (distinct or not) has even length.*

PROOF. Necessity: Let S be a subdivision digraph. Since S is a cycle if and only if D is and since the subdivision process doubles the length of a cycle, it follows that the only cycles which are subdivision digraphs must be of even length.

If S is a subdivision digraph which is not a cycle there exist points that are not carriers; they are points of D. Then b) follows from the connectedness of D and 2.1.

Sufficiency: If S is a cycle of even length n, then there exists a cycle D of length $n/2$, such that $S = S(D)$.

If S is not a cycle, then S contains at least one noncarrier point v. Let V be the set of all points of S which are connected to v by a semipath of even length. This set is well defined since if a point u of V were connected to v by both an even and an odd semipath, this would imply that the noncarrier v or another one is connected to itself by a semipath of odd length, contradicting our hypothesis. Now if V is taken to be the point set of a digraph D such that a point v_i is k-precedent to a point v_j in D if and only if there are k paths of length two from v_i to v_j, it is easy to see that $S = S(D)$. □

2.3. The square D^2 of a digraph D is defined as that digraph whose points are those of D and such that a point u is precedent to a point v in D^2 if and only if u is connected to v by a path of length one or two in D.

2.4. THEOREM. *Let D be a multidigraph, then $[S(D)]^2 = T(D)$.*

PROOF. From the preceding definitions, if D has p points and q arcs, $S(D)$, $[S(D)]^2$ and $T(D)$ have $p+q$ points each. Since each of the k arcs (v_i, v_j) of D is replaced by a path v_i, (v_i, v_{ijt}), v_{ijt}, (v_{ijt}, v_j), v_j, $1 \le t \le k$, in $S(D)$, it follows that $[S(D)]^2$ has $2k$ arcs (v_i, v_{ijt}), (v_{ijt}, v_j), k arcs (v_i, v_j) and if $v_i = v_j$, a loop in each v_{ijt} $1 \le t \le k$. Furthermore if D has r arcs (v_j, v_s), $[S(D)]^2$ has kr arcs (v_{ijt}, v_{jsh}) $1 \le t \le k$, $1 \le h \le r$. Therefore the arcs of $[S(D)]^2$ are in one-to-one correspondence with the arcs of $M(D)$, D and $A(D)$ respectively. Hence $T(D) = [S(D)]^2$. □

2.5. COROLLARY. *A multidigraph T is a total multidigraph if and only if there exists a subdivision digraph S such that $S^2 = T$.*

3 – Relationship between total multigraphs and total multidigraphs

3.1. The symmetrized multidigraph G^s of a multigraph G is obtained as follows.

a) The points of G^s are those of G
b) Each line $u = [x, y]$ of G, $(x \neq y)$ is replaced by the arcs (x, y), (y, x). One of these shall be denoted u and the other u'.
c) Each loop u of G is preserved in G^s. In this case we consider $u = u'$.

This notion coincides with that of vertex-symmetrized multidigraph given in [3].

Let C be the mapping introduced in [3] which maps each digraph $A(G^s)$ to the multidigraph $C(A(G^s))$ obtained identifying each pair of points u, u' of $A(G^s)$ as one point u_0 and omitting the arcs (u, u'), (u', u) if $u \neq u'$.

3.2. Let C be the mapping which maps each multidigraph $T(G^s)$ to the multidigraph $C(T(G^s))$ which is the arc-disjoint union of G^s, $C(A(G^s))$ and $\mathrm{M}(G^s)$. Here $\mathrm{M}(G^s)$ is the digraph whose points are those of G^s and $C(A(G^s))$ and whose arcs are obtained in the following way:

(a) If $u \neq u'$, where $u = (v_i, v_j)$ in G^s, the arcs (v_i, u), (u', v_i), (u, v_j), (v_j, u') of $M(G^s)$ become (v_i, u_0), (u_0, v_i), (u_0, v_j), (v_j, u_0) of $\mathrm{M}(G^s)$.
(b) If $u = u'$ is a loop incident at the point x, the arcs (x, u), (u, x) of $M(G^s)$ become (x, u_0), (u_0, x) of $\mathrm{M}(G^s)$. (See figure 3).

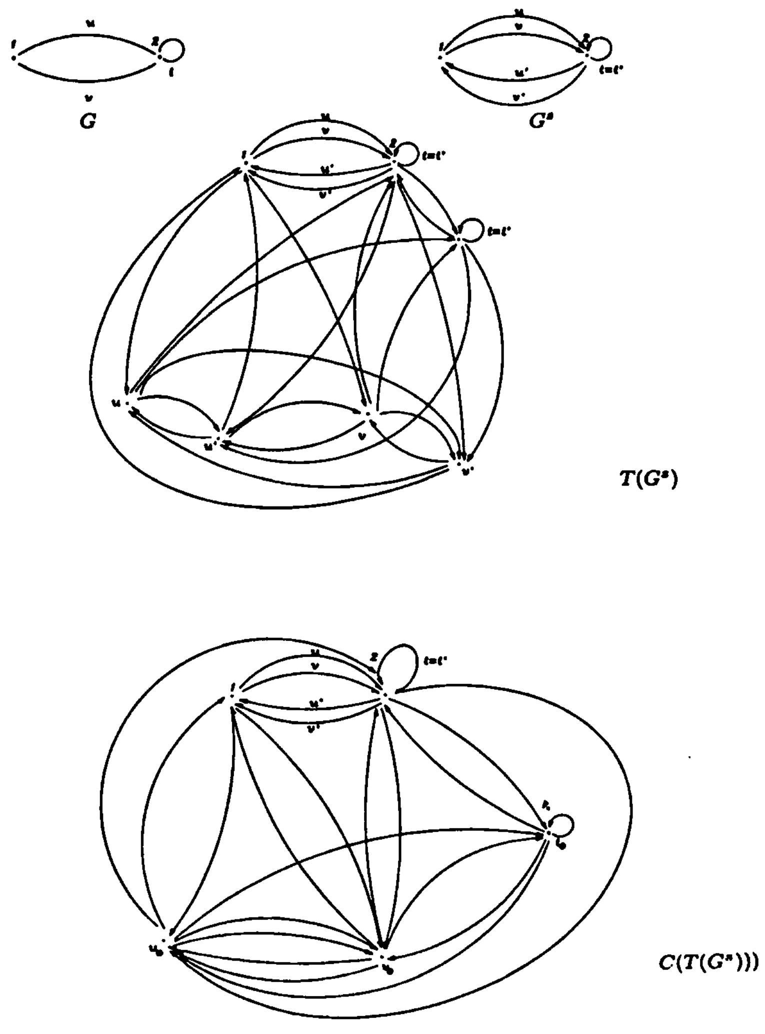
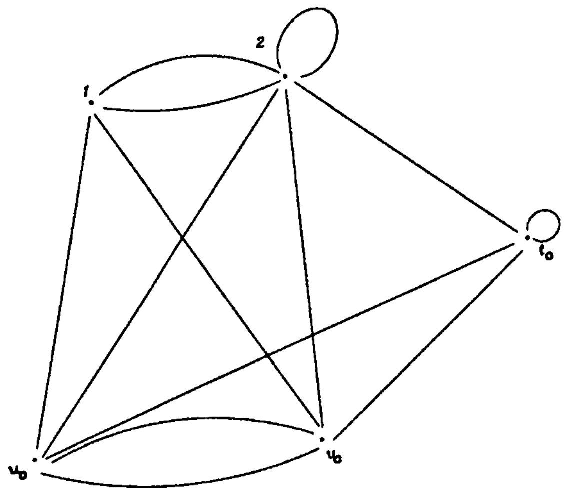


Fig. 3 $C(T(G^s))$ is a symmetrized multidigraph

3.3. Let $\mathcal{D}$ be the mapping which maps each symmetrized multidigraph to the multigraph obtained by leaving the loops unchanged and replacing each pair of opposed arcs by one line which incides at the same points that these arcs do.



$\mathcal{D}(C(T(G^s)))$

Fig. 4

3.4. REMARKS.

a) The mapping $\mathcal{D}$ is the inverse of the function $s : G \longrightarrow G^s$ which maps each multigraph G to the symmetrized multidigraph G^s.

b) According to the preceding definitions the points of $\mathcal{D}(C(T(G^s)))$ and $C(T(G^s))$ coincide. The points of $C(T(G^s))$ are the disjoint union of the points of G^s and $C(A(G^s))$.

c) There is a one-to-one correspondence between the points of $C(A(G^s))$ and the points of $A(G)$ which allows us to identify x_0 with x, since each point x_0 of $C(A(G^s))$ corresponds to the pair of points x, x' of $A(G^s)$ and these to the arc x of G.

d) Let x, y be k-adjacent points in $\mathcal{D}(C(T(G^s)))$ such that:

(i) if $x \in V(G^s)$, $y_0 \in V(C(A(G^s)))$ then $k = 1$

(ii) if $x_0, y_0 \in V(C(A(G^s)))$ then x_0, y_0 correspond to the pairs of the opposed arcs x, x' and y, y' of G^s respectively. If at least one of them is a loop, that is $x = x'$ or $y = y'$, we have $k = 1$ but if $x \neq x'$ and $y \neq y'$ we have $k = 1$ or $k = 2$.

3.5. THEOREM. *Let G be a multigraph. A multigraph H is the total multigraph of G if and only if $H = \mathcal{D}(C(T(G^s)))$.*

PROOF. From 3.1 and 3.4(b), (c) follows that the points of $\mathcal{T}(G)$ and $\mathcal{D}(C(T(G^s)))$ are the same.

Let $H = \mathcal{T}(G)$. We shall prove that $H = \mathcal{D}(C(T(G^s)))$.

Let x, y be the k-adjacent points in $\mathcal{T}(G)$. Since $V(\mathcal{T}(G))$ is the disjoint union of $V(G)$ and $V(\mathcal{A}(G))$ we consider the following cases:

a) Let $x, y \in V(G)$. If $x \neq y$ we have k arcs (x, y) and k arcs (y, x) which do not change in $T(G^s)$. Since C does not modify these arcs and $\mathcal{D}$ is the inverse of mapping $s\colon G \longrightarrow G^s$, then x and y are k-adjacent in $\mathcal{D}(C(T(G^s)))$.
   If $x = y$, there are in G k loops incident at x which do not change by the successive operations which transform G in $\mathcal{D}(C(T(G^s)))$.
b) Let $x, y \in V(\mathcal{A}(G))$. From proposition D of [3], we know that $\mathcal{A}(G) = \mathcal{D}(\mathcal{C}(A(G^s)))$, so that x, y are k-adjacent in $\mathcal{D}(\mathcal{C}(A(G^s)))$ and therefore in $\mathcal{D}(C(T(G^s)))$ which is the line-disjoint union of $\mathcal{D}(G^s)$, $\mathcal{D}(\mathcal{C}(A(G^s)))$ and $\mathcal{D}(\mathrm{M}(G^s))$.
c) Let $x \in V(G)$, $y \in V(\mathcal{A}(G))$. By 1.4 we have $k = 1$. Then, in $T(G^s)$ there exists an arc with initial point x and terminal point in $\{y, y'\}$ and another with initial point at $\{y, y'\}$ and terminal point x, if $y \neq y'$ the terminal point of the first arc is different from the initial point of the second one.
   If the arcs (x, y), (y', x) belong to $T(G^s)$, in $C(T(G^s))$ we have the arcs (x, y_0), (y_0, x) and in $\mathcal{D}(C(T(G^s)))$ x and y_0 are adjacents. Since y and y_0 represent the same point in $\mathcal{T}(G)$ and $\mathcal{D}(C(T(G^s)))$ we conclude that x and y are adjacent in $\mathcal{D}(C(T(G^s)))$.
d) Let $x \in V(\mathcal{A}(G))$, $y \in V(G)$. The proof in this case is analogous to the case c).

Now let $H = \mathcal{D}(C(T(G^s)))$. We shall prove that $H = \mathcal{T}(G)$.

Let x, y be points of $\mathcal{D}(C(T(G^s)))$. Taking into account 3.4 (b) and (c) we have the following cases: a) $x, y \in V(G^s)$; b) $x \in V(G^s)$, $y_0 \in V(\mathcal{C}(A(G^s)))$ and c) $x_0, y_0 \in V(\mathcal{C}(A(G^s)))$.

If x, y are k-adjacent in $\mathcal{D}(C(T(G^s)))$ then:

In case a) there are k arcs (x, y) and k arcs $(y, x.)$ in $C(T(G^s))$. If $x = y$ there are k loops incident at x. They are preserved in $T(G^s)$ and also in G^s, hence x and y are k-adjacent in G and also in $\mathcal{T}(G)$.

In case b), by 3.4. d) $k = 1$ and by 3.2. y_0 is obtained identifying the points y and y' of $T(G^s)$. If $y \neq y'$, y and y' are opposed arcs in G^s, otherwise y is a loop. In both cases x and y are adjacent in G and hence in $\mathcal{T}(G)$.

In case c) by 3.4 d) $k = 1$ or $k = 2$. If $k = 1$ and $x = x'$ or $y = y'$ in $T(G^s)$, then in G^s at least one of them is a loop. In G, x and y have only one common point and in $\mathcal{T}(G)$ they are adjacent. If $k = 1$, $x \neq x'$ and $y \neq y'$ in $T(G^s)$, then in G^s the arcs x and y have only one common point.

The lines x and y are 1-adjacent in G and so are in $\mathcal{T}(G)$. If $k = 2$, in G^s the two extremal points of x and y are the same, hence in $\mathcal{T}(G)$, x and y are 2-adjacents. □

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