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Rotations and harmonicity in contact geometry

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RIASSUNTO – *Si considerano le φ-rotazioni intorno alle linee di flusso del campo ξ dei vettori caratteristici su una varietà sasakiana $(M, g, \varphi, \xi, \eta)$ e si prova che tutte queste rotazioni sono applicazioni armoniche se e solo se esse sono isometrie, cioè se e solo se la varietà è uno spazio localmente φ-simmetrico.*

ABSTRACT – *One considers the φ-rotations around the flow lines of the characteristic vector field ξ on a Sasakian manifold $(M, g, \varphi, \xi, \eta)$ and one proves that all these rotations are harmonic maps if and only if they are isometries, that is, if and only if the manifold is a locally φ-symmetric space.*

KEY WORDS – *Riemannian foliation, Sasakian manifold, rotations, φ-rotations, harmonic maps.*

A.M.S. CLASSIFICATION: 53B20 - 53C15 - 53C30 - 58E20

1 – Introduction

In this paper we continue our research about the relationship between the properties of local diffeomorphisms in normal and tubular neighborhoods on a Riemannian manifold (M, g) and the properties of the curvature of (M, g).

The geometry of *reflections* with respect to points, curves and submanifolds has been studied extensively. We refer to [18] for more details and further references. In particular, one investigated under what conditions the reflections are harmonic maps. It turns out that in all the situations studied in the different papers harmonic reflections are isometric maps.

(Local) reflections are special cases of (local) *rotations*. We refer to [10], [13], [14], [15] where these local diffeomorphisms are introduced and some aspects are studied. Also in the situations studied there one finds that harmonicity for rotations implies isometry.

Here we consider another natural situation for rotations around a curve and focus again on harmonicity. More precisely, let (M, g) be an (odd-dimensional) Riemannian manifold and let ξ be a unit Killing vector field on it such that the Riemannian curvature tensor R satisfies

$$R_{XY}\xi = \eta(X)Y - \eta(Y)X$$

for all vector fields X, Y on M. η denotes the dual one-form of the field ξ. Such a manifold is also known as a Sasakian manifold (see, for example, [1]). The integral curves σ of the characteristic vector field ξ are geodesics. Moreover, if ∇ denotes the Levi Civita connection of (M, g), then $\varphi = -\nabla\xi$ defines a particular tensor field of type (1,1) such that $\ker\varphi = \mathrm{span}\{\xi\}$ at each point of M. It follows that $S = \varphi + \eta \otimes \xi$ is a (1,1)-tensor field, parallel along the flow lines, which fixes ξ. Further, let $\exp_\sigma$ denote the exponential map of the normal bundle of the flow line σ. Then

$$s_\sigma = \exp_\sigma \circ S \circ \exp_\sigma^{-1}$$

defines the so-called (local) φ-rotation around σ. The main purpose of the paper is to prove

THEOREM A. *The φ-rotations along all flow lines σ of the characteristic Killing vector field ξ are harmonic maps if and only if they are isometries.*

As a consequence of this we obtain

THEOREM B. *A Sasakian manifold is a locally φ-symmetric space if and only if all φ-rotations are harmonic.*

The locally φ-symmetric spaces are the natural analogs in Sasakian geometry of the locally Hermitian symmetric spaces in Kähler geometry. They are introduced in [17] (see Section 2 for more details).

2 – Preliminaries

We start by giving a collection of basic formulas and results. Let (M, g) be a smooth (connected) Riemannian manifold with Levi Civita connection ∇ and associated Riemann curvature tensor R defined by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all vector fields $X, Y \in \chi(M)$. ($\chi(M)$ denotes the Lie algebra of smooth vector fields on M.) Further, we put

$$R_{XYZW} = g(R_{XY}Z, W)$$

and similarly for other covariant tensors such as ∇R, etc.

Now, let (M, g) be equipped with a *unit* Killing vector field ξ, with dual one-form η, such that

$$R_{XY}\xi = \eta(X)Y - \eta(Y)X \tag{1}$$

for all $X, Y \in \chi(M)$. Then (M, g) is equipped with a *Sasakian structure* (g, ξ, η, φ) and $(M, g, \xi, \eta, \varphi)$ is a *Sasakian manifold*. Here φ is the (1,1)-tensor field defined by

$$\varphi = -\nabla\xi\,. \tag{2}$$

The equivalence of this definition with the usual one follows at once from [1, p. 75]. We refer to [1], [20] for more details about contact and, in particular, about Sasakian geometry. There it is also proved that

$$\begin{cases} \varphi^2 = -I + \eta \otimes \xi\,, \\ \varphi\xi = \eta\varphi = 0\,, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)\,, \\ (\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X \end{cases} \tag{3}$$

for all $X, Y \in \chi(M)$. Moreover, since ξ is a Killing vector field, we have

$$R_{X\xi}Y = g(X,Y)\xi - \eta(Y)X \tag{4}$$

and finally

$$\begin{aligned} R_{XYZ\varphi W} + R_{XY\varphi ZW} = g(X,\varphi Z)g(Y,W) - g(X,\varphi W)g(Y,Z) \\ - g(Y,\varphi Z)g(X,W) + g(Y,\varphi W)g(X,Z)\,. \end{aligned} \tag{5}$$

Again, since ξ is Killing, (1), (2), (3) and (5) imply

$$\nabla_\xi R = 0\,. \tag{6}$$

The unit Killing vector field ξ generates a Riemannian foliation with geodesic leaves. Hence, the Sasakian geometry of the manifold may be treated via local Riemannian submersion theory. This feature has been developed in [16]. So, each point $m \in M$ has a neighborhood U such that ξ is regular on U and there exists a fibration $\pi\colon U \to \widetilde{U} = U/\xi$. π is a Riemannian submersion and $\widetilde{U}$ has an induced Kähler structure (G, J).

This property may be used to define the so-called *locally φ-symmetric spaces*. They have been introduced by T. TAKAHASHI in [17] as the analogs of locally Hermitian symmetric spaces. We refer to [2], [11], [12], [17] for a list of examples. We state the following result of [17] which may serve as a definition.

THEOREM 1. *A Sasakian manifold $(M, g, \xi, \eta, \varphi)$ is a locally φ-symmetric space if and only if each base manifold $\widetilde{U}$ of a local fibration $U \to \widetilde{U} = U/\xi$ is a locally Hermitian symmetric space.*

To prove Theorem A we will need the following characteristic property, proved in [3].

THEOREM 2. *A Sasakian manifold $(M, g, \xi, \eta, \varphi)$ is locally φ-symmetric if and only if*

$$\nabla_U R_{U\varphi UU\varphi U} = 0$$

for all horizontal vectors U (that is, U is orthogonal to ξ).

We refer to [4] for a survey on further results and other equivalent properties.

3 – Rotations, Fermi coordinates and expressions for the metric tensor

In this section we give a brief survey about rotations around a curve and about the method to describe these local diffeomorphisms by using Fermi coordinates. In addition we will use this to give expressions for (g_{ij}) in Fermi coordinates on a Sasakian manifold.

Let (M, g) be an n-dimensional smooth Riemannian manifold and let $\sigma\colon [a, b] \to M : t \mapsto \sigma(t)$ be a smooth, embedded curve in M. Further, denote by $N\sigma = T^{\perp}\sigma$ the normal bundle of σ and let $\exp_\sigma$ be the exponential map of $N\sigma$, that is,

$$\exp_\sigma(t, v) = \exp_{\sigma(t)}(v)$$

for all $t \in [a, b]$ and all $v \in T^{\perp}_{\sigma(t)}\sigma$. The set

$$\mathcal{U}_\sigma(s) = \left\{ \exp_\sigma(t, v) | t \in [a, b], v \in T^{\perp}_{\sigma(t)}\sigma, \|v\| < s \right\} \subset M$$

is called a tubular neighborhood of radius s about σ. There exists an $s > 0$ such that $\exp_\sigma$ is a diffeomorphism onto $\mathcal{U}_\sigma(s)$. In what follows we will restrict our treatment to such a neighborhood.

Next, a field of endomorphisms $t \mapsto S(t)$ along σ is said to be a *rotation field* along σ if S satisfies

$$S(t)\dot{\sigma} = \dot{\sigma}\,, \quad g(S(t)X, S(t)Y) = g(X, Y)\,,$$

for all X, Y orthogonal to $\dot{\sigma}(t)$ and all $t \in [a, b]$. Then

$$s_\sigma = \exp_\sigma \circ S \circ \exp_\sigma^{-1} \tag{7}$$

is a local diffeomorphism which is called a *(local) S-rotation around σ* (see [14], [15]). If $S - I$ is nonsingular on normal vectors, then it is said to be a *free rotation*. For $S = -I$ we get *the reflection with respect to σ*.

To describe the rotation

$$s_\sigma\colon \exp_\sigma(t,v) \mapsto \exp_\sigma(t, S(t)v) \tag{8}$$

we shall use a system of *Fermi coordinates* defined as follows. Let $\sigma\colon [a,b] \to M$ be a unit speed geodesic (that is, $\|\dot\sigma\| = 1$) and let $\{e_i, i = 1,\dots,n\}$ be an orthonormal basis of $T_{\sigma(a)}M$ such that $e_1 = \dot\sigma(a)$. Further, let $E_1 = \dot\sigma$ and let $E_2,\dots,E_n$ be the normal vector fields along σ which are parallel with respect to the normal connection $\nabla^\perp$ of the normal bundle $N\sigma$ and with $E_i(a) = e_i$. (Note that, when σ is a geodesic, this is just the parallel translation with respect to the Levi Civita connection.) Then the Fermi coordinates $(x^1,\dots,x^n)$ with respect to $\sigma(a)$ and $\{e_1,\dots,e_n\}$ are defined by

$$x^1\Big(\exp_{\sigma(t)} \sum_j t^j E_j\Big) = t - a\,,$$

$$x^i\Big(\exp_{\sigma(t)} \sum_j t^j E_j\Big) = t^i\,, \qquad i = 2,\dots,n\,.$$

We refer to [8], [9], [19] for more information about these Fermi coordinates.

With respect to this coordinate system the analytic expression for (8) becomes

$$s_\sigma\colon (x^1, x^2, \dots, x^n) \mapsto \Big(x^1, \sum_j S_j^2 x^j, \dots, \sum_j S_j^n x^j\Big)\,, \tag{9}$$

where $S_i^j(t) = g(S(t)E_i(t), E_j(t))$.

In the next section we will need some expressions for the matrix (g_{ij}) with respect to these Fermi coordinates. To obtain the required expressions, we proceed as follows. (An alternative method is described in [8].) Let

$$p = \exp_{\sigma(t)}(ru)\,, \quad u \in T^\perp_{\sigma(t)}\sigma\,, \quad \|u\| = 1\,,$$

and let $\gamma\colon s \mapsto \exp_{\sigma(t)}(su)$ be the unit speed geodesic connecting $\sigma(t)$ and p, with $\gamma(0) = \sigma(t), \gamma'(0) = u$. Next, we specify the frame field

$\{E_1, \ldots, E_n\}$ such that $E_n(t) = \gamma'(0)$. Further, let $\{F_1, \ldots, F_n\}$ be the frame along γ obtained by parallel translation of $\{E_1(t), \ldots, E_n(t)\}$ along γ. Finally, let $Y_1, Y_2, \ldots, Y_{n-1}$ be the Jacobi vector fields along γ with initial conditions

$$\begin{cases} Y_1(0) = E_1(t)\,, & Y_1'(0) = \Big(\nabla_{\gamma'} \dfrac{\partial}{\partial x^1}\Big)(\sigma(t))\,, \\ Y_a(0) = 0\,, & Y_a'(0) = E_a(t)\,, \quad a = 2, \ldots, n-1\,. \end{cases}$$

Note that

$$Y_1'(0) = -\kappa_u E_1(t)$$

where

$$\kappa_u = g\big(\ddot{\sigma}(t), u\big)\,.$$

Then we have

$$\text{(10)} \qquad \begin{cases} Y_1(s) = \dfrac{\partial}{\partial x^1}(\gamma(s))\,, \\ Y_a(s) = s\dfrac{\partial}{\partial x^a}(\gamma(s))\,, \quad a = 2, \ldots, n-1\,. \end{cases}$$

Now, we identify the spaces $\{\gamma'(s)\}^\perp$ with $\{\gamma'(0)\}^\perp$ via the parallel translated basis $\{E_1(t), \ldots, E_{n-1}(t)\}$ and define the automorphism-valued function $B\colon s \mapsto B(s)$ on $\{\gamma'(0)\}^\perp$ by

$$\text{(11)} \qquad Y_i(s) = (BF_i)(s)\,, \quad i = 1, 2, \ldots, n-1\,.$$

Then B satisfies the Jacobi equation

$$\text{(12)} \qquad B'' + R \circ B = 0\,,$$

where $R = R_{\gamma'.}\gamma'$, and it has the initial values

$$B(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B'(0) = \begin{pmatrix} -\kappa_u & 0 \\ 0 & I \end{pmatrix}. \tag{13}$$

Hence, using (10) and (11) we have for the components of g:

$$\begin{cases} g_{11}(p) = g\Big(\dfrac{\partial}{\partial x^1}, \dfrac{\partial}{\partial x^1}\Big)(s) = g(Y_1, Y_1)(s) = g(BF_1, BF_1)(s) \\ \qquad = ({}^tBB)_{11}(s), \\ g_{1a}(p) = g\Big(\dfrac{\partial}{\partial x^1}, \dfrac{\partial}{\partial x^a}\Big)(s) = \dfrac{1}{s} g(Y_1, Y_a)(s) = \dfrac{1}{s} g(BF_1, BF_a)(s) \\ \qquad = \dfrac{1}{s}({}^tBB)_{1a}(s), \\ g_{ab}(p) = g\Big(\dfrac{\partial}{\partial x^a}, \dfrac{\partial}{\partial x^b}\Big)(s) = \dfrac{1}{s^2} g(Y_a, Y_b)(s) = \dfrac{1}{s^2} g(BF_a, BF_b)(s) \\ \qquad = \dfrac{1}{s^2}({}^tBB)_{ab}(s) \end{cases} \tag{14}$$

for $a, b = 2, \dots, n-1$. Note that

$$g_{in}(p) = 0, \qquad i = 1, \dots, n-1.$$

In what follows we will use (14), (12) and (13) to write down some terms in the power series expansions of g_{ij}. But, since we only need this for Sasakian spaces with $\dot\sigma = \xi$, we shall only write down the results for that case. Note that then $\kappa_u = 0$.

After some detailed computations using the properties for the curvature tensor R of a Sasakian manifold, we get

THEOREM 3. *Let $(M, g, \xi, \eta, \varphi)$ be a Sasakian manifold and suppose we have a system of Fermi coordinates, as mentioned above, in a tubular neighborhood about a flow line σ of ξ. Then we have, at $p = \exp_{\sigma(t)}(ru)$,*

$$
\begin{aligned}
g_{11} =& 1 - r^2 + \frac{r^4}{6}(3 - R_{u\varphi uu\varphi u})(\sigma(t)) - \frac{r^5}{10}(\nabla_u R_{u\varphi uu\varphi u})(\sigma(t)) \\
&+ \frac{r^6}{180}\Big(7\sum_\alpha R^2_{u\varphi uu\alpha} - 6\nabla^2_{uu}R_{u\varphi uu\varphi u} - 15\Big)(\sigma(t)) + O(r^7)\,, \\
g_{1a} =& \frac{r^3}{4}(g(\varphi u, E_a) - R_{u\varphi uua})(\sigma(t)) - \frac{2r^4}{15}(\nabla_u R_{u\varphi uua})(\sigma(t)) \\
&+ \frac{r^5}{120}\Big(5\sum_\alpha R_{u\varphi uu\alpha}R_{u\alpha ua} - 5g(\varphi u, E_a) - 7\nabla^2_{uu}R_{u\varphi uua}\Big)(\sigma(t)) \\
&+ \frac{r^6}{1260}\Big(27\sum_\alpha R_{u\varphi uu\alpha}\nabla_u R_{u\alpha ua} + 26\sum_\alpha \nabla_u R_{u\varphi uu\alpha}R_{u\alpha ua} \\
&- 12\nabla^3_{uuu}R_{u\varphi uua} + 3\nabla_u R_{u\varphi uua}\Big)(\sigma(t)) + O(r^7)\,, \\
g_{ab} =& \delta_{ab} - \frac{r^2}{3}(R_{uaub})(\sigma(t)) - \frac{r^3}{6}(\nabla_u R_{uaub})(\sigma(t)) \\
&+ \frac{r^4}{180}\Big(8\sum_\alpha R_{uau\alpha}R_{ubu\alpha} - 9\nabla^2_{uu}R_{uaub}\Big)(\sigma(t)) \\
&+ \frac{r^5}{90}\Big(2\sum_\alpha R_{uau\alpha}\nabla_u R_{ubu\alpha} \\
&+ 2\sum_\alpha \nabla_u R_{uau\alpha}R_{ubu\alpha} - \nabla^3_{uuu}R_{uaub}\Big)(\sigma(t)) \\
&+ \frac{r^6}{5040}\Big(55R_{u\varphi uua}R_{u\varphi uub} + 34\sum_\alpha R_{uau\alpha}\nabla^2_{uu}R_{ubu\alpha} \\
&+ 34\sum_\alpha \nabla^2_{uu}R_{uau\alpha}R_{ubu\alpha} - 55g(\varphi u, E_a)R_{u\varphi uub} \\
&- 55g(\varphi u, E_b)R_{u\varphi uua} + 55g(\varphi u, E_a)g(\varphi u, E_b) \\
&+ 55\sum_\alpha \nabla_u R_{uau\alpha}\nabla_u R_{ubu\alpha} - 10\nabla^4_{uuuu}R_{uaub} \\
&- 16\sum_{\alpha,\beta} R_{uau\alpha}R_{u\alpha u\beta}R_{u\beta ub}\Big)(\sigma(t)) + O(r^7)
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
g^{11} &= 1 + r^2 + \frac{r^4}{6}(3 + R_{u\varphi uu\varphi u})(\sigma(t)) + O(r^5)\,, \\
g^{1a} &= \frac{r^3}{4}(R_{u\varphi uua} - g(\varphi u, E_a))(\sigma(t)) \\
&\quad + \frac{2r^4}{15}(\nabla_u R_{u\varphi uua})(\sigma(t)) + O(r^5)\,, \\
g^{ab} &= \delta_{ab} + \frac{r^2}{3}(R_{uaub})(\sigma(t)) + \frac{r^3}{6}(\nabla_u R_{uaub})(\sigma(t)) \\
&\quad + \frac{r^4}{60}\Big(4\sum_{\alpha} R_{uau\alpha}R_{ubu\alpha} + 3\nabla^2_{uu}R_{uaub}\Big)(\sigma(t)) + O(r^5)
\end{aligned}
\tag{16}
$$

where $a, b, \alpha, \beta = 2, \ldots, n-1$, *and* $R_{uau\alpha} = R_{uE_a uE_\alpha}$, *etc.*

4 – φ-rotations and harmonicity

Now we concentrate on the so-called *φ-rotation* around a flow line σ of the characteristic vector field ξ on a Sasakian manifold. This rotation is defined by (7) where

$$
S = \varphi + \eta \otimes \xi\,. \tag{17}
$$

It has already been considered in [10] and it is the normal analog of a J-rotation on an almost Hermitian manifold (see [13]). Note that a φ-rotation is a free rotation and moreover, using (3), we see that S is *parallel along* σ. Further, since $S^2 = -I$, s_σ^2 is the reflection with respect to σ.

In the proof of Theorem A we shall use the following result.

THEOREM 4. *Let $(M, g, \xi, \eta, \varphi)$ be a Sasakian manifold. Then it is a locally φ-symmetric space if and only if all the φ-rotations are isometries.*

A proof of this theorem has been given in [10]. In the analytic case, another proof is given in [5] by using power series expansions and a criterion for isometric rotations given in [15].

Next, we state some facts about harmonic maps and refer to [6], [7] for more details. Let (M^n, g) and (N^m, h) be Riemannian manifolds with metrics g and h respectively, and let $\psi\colon (M, g) \to (N, h)$ be a smooth map. The covariant differential $\nabla(d\psi)$ is a symmetric tensor field of order two with values in $\psi^{-1}(TN)$, called the *second fundamental form* of ψ. The trace of $\nabla(d\psi)$ is called the *tension field* of ψ and is denoted by $\tau(\psi)$. ψ is called a *harmonic map* if $\tau(\psi) = 0$.

To study these harmonic maps analytically, let $U \subset M$ be a coordinate neighborhood with coordinate system $(x^1, \dots, x^n)$ and $V \subset N$ a coordinate neighborhood with coordinate system $(y^1, \dots, y^m)$ such that $\psi(U) \subset V$. Then the map ψ may be expressed by $y^a = \psi^a(x^1, \dots, x^n)$, where $a = 1, \dots, m$. Further, the differential $d\psi(x)$ can be represented by the Jacobian matrix $\left(\frac{\partial\psi^a}{\partial x^i}\right)$ and finally, we see that

$$\left(\nabla(d\psi)\right)^a_{ij} = \frac{\partial^2\psi^a}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial\psi^a}{\partial x^k} + {}^N\Gamma^a_{\alpha\beta}\frac{\partial\psi^\alpha}{\partial x^i}\frac{\partial\psi^\beta}{\partial x^j} \tag{18}$$

where ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols for (M, g) and (N, h) respectively. Consequently, ψ is harmonic if and only if

$$\tau(\psi)^a = g^{ij}\left(\nabla(d\psi)\right)^a_{ij} = 0 \tag{19}$$

for all $a = 1, \dots, m$.

Now we turn to the

Proof of Theorem A. We start the proof by considering the expressions (18) when $\psi = s_\sigma$. Using (9) we then get, with our choice of Fermi coordinates:

$$\begin{aligned}
\left(\nabla(ds_\sigma)\right)^1_{11} &= -\Gamma^1_{11}(p) + \Gamma^1_{11}(s_\sigma(p)),\\
\left(\nabla(ds_\sigma)\right)^\gamma_{11} &= -\Gamma^k_{11}(p)S^\gamma_k(\sigma(t)) + \Gamma^\gamma_{11}(s_\sigma(p)),\\
\left(\nabla(ds_\sigma)\right)^1_{1j} &= -\Gamma^1_{1j}(p) + \Gamma^1_{1\beta}(s_\sigma(p))S^\beta_j(\sigma(t)),\\
\left(\nabla(ds_\sigma)\right)^\gamma_{1j} &= -\Gamma^k_{1j}(p)S^\gamma_k(\sigma(t)) + \Gamma^\gamma_{1\beta}(s_\sigma(p))S^\beta_j(\sigma(t)),\\
\left(\nabla(ds_\sigma)\right)^1_{ij} &= -\Gamma^1_{ij}(p) + \Gamma^1_{\alpha\beta}(s_\sigma(p))S^\alpha_i(\sigma(t))S^\beta_j(\sigma(t)),\\
\left(\nabla(ds_\sigma)\right)^\gamma_{ij} &= -\Gamma^k_{ij}(p)S^\gamma_k(\sigma(t)) + \Gamma^\gamma_{\alpha\beta}(s_\sigma(p))S^\alpha_i(\sigma(t))S^\beta_j(\sigma(t)),
\end{aligned} \tag{20}$$

for $i, j, k, \alpha, \beta, \gamma = 2, \ldots, n$ and where the Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

Now, we use the expressions (15) and (16) to obtain power series expansions for the quantities in (20) and hence, also for the components $\tau(s_\sigma)^k$ of the tension field $\tau(s_\sigma)$. In order to do this we put

$$ru = \sum_k x^k E_k$$

in (15) and (16). Then we may write

$$\tau(s_\sigma)^k = \sum_{i=2}^{5} C_i^{(k)} r^i + O(r^6)$$

and so, $\tau(s_\sigma) = 0$ implies $C_i^{(k)} = 0$, $i = 2, \ldots, 5$ as necessary conditions. A long calculation shows that these conditions involve only $R_{u \cdot} u$ and $\nabla^l_{u \ldots u} R_{u \cdot} u$ along the flow line σ. We now write down these conditions after putting $E_k(t) = (\varphi u)(\sigma(t))$. Then we get explicitly the following conditions along σ:

$$\sum_a (\nabla_u R_{uaua} - \nabla_{\varphi u} R_{\varphi u \varphi a \varphi u \varphi a}) = 0\,, \tag{21}$$

$$\sum_a (\nabla^2_{uu} R_{uaua} - \nabla^2_{\varphi u \varphi u} R_{\varphi u \varphi a \varphi u \varphi a}) = 0\,, \tag{22}$$

$$\begin{aligned} &45(\nabla_u R_{u\varphi u u \varphi u} - \nabla_{\varphi u} R_{u \varphi u u \varphi u}) \\ &- 3 \sum_{a,b} (R_{uaub} \nabla_u R_{uaub} - R_{\varphi u \varphi a \varphi u \varphi b} \nabla_{\varphi u} R_{\varphi u \varphi a \varphi u \varphi b}) \\ &+ 7 \sum_a (\nabla^3_{uuu} R_{uaua} - \nabla^3_{\varphi u \varphi u \varphi u} R_{\varphi u \varphi a \varphi u \varphi a}) = 0\,, \end{aligned} \tag{23}$$

$$
\begin{aligned}
(24) \qquad & 252(\nabla^2_{uu} R_{u\varphi uu\varphi u} - \nabla^2_{\varphi u\varphi u} R_{u\varphi uu\varphi u}) \\
& - 10 \sum_{a,b} (R_{uaub} \nabla^2_{uu} R_{uaub} - R_{\varphi u\varphi a\varphi u\varphi b} \nabla^2_{\varphi u\varphi u} R_{\varphi u\varphi a\varphi u\varphi b}) \\
& + 20 \sum_{a} (\nabla^4_{uuuu} R_{uaua} - \nabla^4_{\varphi u\varphi u\varphi u\varphi u} R_{\varphi u\varphi a\varphi u\varphi a}) \\
& - 35 \sum_{a,b} \nabla_u R_{uaub} \nabla_{\varphi u} R_{\varphi u\varphi a\varphi u\varphi b} + 65 \sum_{a,b} (\nabla_u R_{uaub})^2 \\
& + 110 \sum_{a,b} (\nabla_{\varphi u} R_{\varphi u\varphi a\varphi u\varphi b})^2 = 0\,,
\end{aligned}
$$

where all the summations are taken with respect to an orthonormal basis $\{f_1, \ldots, f_{n-1}\}$ of $\{\xi\}^\perp$ at each point of σ.

By replacing u by φu in (24) and by comparing with (24) we then get at once

$$\sum_{a,b} \{(\nabla_u R_{uaub})^2 + (\nabla_{\varphi u} R_{\varphi u\varphi a\varphi u\varphi b})^2\} = 0$$

and hence,

$$\nabla_u R_{uaub} = 0\,.$$

So,

$$\nabla_u R_{u\varphi uu\varphi u} = 0$$

which implies that the manifold is locally φ-symmetric (Theorem 2). This means that each φ-rotation is an isometry (Theorem 4).

The converse is trivial.

We finish with a

PROOF OF THEOREM B. The result follows at once from Theorem A and Theorem 4.

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