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Nijenhuis and bi-Hamiltonian manifolds with symmetries

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RIASSUNTO – *Si completa la discussione della riduzione di varietà di Nijenhuis e bihamiltoniane con deformazione e simmetrie, considerata in un precedente lavoro [1]. Si applica tale schema alla struttura di integrabilità di alcune gerarchie di equazioni di evoluzione in una e in due dimensioni spaziali.*

ABSTRACT – *The reduction scheme for Nijenhuis and bi-Hamiltonian manifolds with deformation and symmetries, previously considered in [1], is completed. Some applications to the integrability structures of NLEE's in one and two spatial dimensions are given.*

KEY WORDS – *Nijenhuis manifolds. Bi-Hamiltonian manifolds. Symmetries. Deformation. Integrable non linear evolution equations.*

A.M.S. CLASSIFICATION: 58F05 - 35Q58

1 – Introduction

In a previous paper [1] the reduction of bi-Hamiltonian manifolds with deformation and symmetries has been considered. The aim of this paper is to complete the discussion of the reduction scheme and to give some applications to integrable non linear evolution equations. The main

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emphasis is on the geometric objects, i.e. Poisson and Nijenhuis tensor fields, symmetries, deformations, rather than on the explicit construction of the equations.

As for the notations, definitions, theoretical results we refer to [1]. However, in view of the applications we explicitly recall a result discussed in [1], which is related with the so-called reduction technique to a transversal submanifold. If M is a Nijenhuis manifold, we say that $S \subset M$ is a transversal submanifold if the Riesz index of N is constant at any point of S: $\operatorname{ind}(N) = r$, and if the splitting property holds:

$$T(S,M) = TS \oplus \operatorname{Ker} N^r \qquad \varphi(u) = \varphi_s(u) + \varphi_k(u) \tag{1.1}$$

$T(S,M)$ being the space of vector fields evaluated at any point $u \in S$. On account of (1.1), one can define the mapping

$$\Pi\colon T(S,M) \longrightarrow TS\colon \quad \varphi \longmapsto \varphi_s \tag{1.2}$$

and show that N, its graded symmetries τ and its symmetries $\varphi_a (a \in \mathcal{G}$, the symmetry algebra of N) can be reduced to S:

LEMMA 1.1. *The transversal submanifold S is a Nijenhuis manifold. The reduced structure in S is defined by*

$$\overline{N} = \Pi N \Pi^{-1} \quad \overline{\tau} = \Pi\tau \quad \overline{\varphi}_a = \Pi\varphi_a \tag{1.3}$$

Moreover, let us assume that M be a PN manifold, that $L_\tau(P) = NP$, where $L_\tau(\cdot)$ is the Lie derivative w.r.t. τ, and that the vector fields φ_a be symmetries of both P and N: $L_{\varphi_a}(P) = L_{\varphi_a}(N) = 0$. By introducing the adjoint map Π^* of Π w.r.t. a pairing $< , >$: $T^*M \times TM \to \mathbb{R}$, one can show that:

LEMMA 1.2. *The transversal submanifold $S \subset M$ is a PN manifold, with $\overline{P}$ given by*

$$\overline{P} = \Pi P \Pi^*$$

and $\overline{N}, \overline{\tau}, \overline{\varphi}_a$ given by (1.3).

Here, the reductions of P and N are considered separately, differently from [1]: this is made in view of the applications. Indeed, even if from an abstract point of view it is possible to consider the structures N, P, τ, φ_a all together, the application of the abstract scheme to the consctruction of hierarchies of integrable evolution equations points out the difficulty of simultaneously defining a Nijenhuis and a Poisson structure together with a symmetry algebra and a master-symmetry. In many cases, one can rigorously construct a $\mathcal{G}N\tau$ structure, i.e. a manifold with a symmetry algebra, a Nijenhuis tensor and a master-symmetry; by its reduction and realization, the recursion scheme for a given hierarchy of evolution equations is obtained. In this regard, we recall that an iterative construction of vector fields in involution does not require a Poisson structure, but only a Nijenhuis tensor and a symmetry algebra (e.g., this is the case of the well-known Burgers' hierarchy). Another possibility is to obtain a $\mathcal{G}PN$ structure (a Poisson-Nijenhuis manifold with a symmetry algebra) without a master-symmetry.

The origin of these difficulties will appear clearly from the applications in the following sections. Roughly speaking, this is due to the fact that one needs to introduce a derivation operator which is both invertible and skew-symmetric and that, on the other hand, the master-symmetries of the abstract model are *constant* elements w.r.t. this derivation. Moreover, the definition of a Poisson tensor field requires the introduction of a vector space in duality with the tangent space; in many applications it cannot be identified with the tangent space itself, so that a derivation operator cannot be interpreted as a mapping from the cotangent to the tangent bundle; an example of this situation is given by the algebra of pseudo-differential and of differential operators, on account of the well-known Adler's result [2]. This paper is organized as follows. In Sect. 2, sufficient conditions are determined for the existence of a Nijenhuis structure. They are suggested by the abstract Zakharov-Shabat (ZS) and chiral models [3], and they are based on the existence of two compatible actions of a Lie group on the manifold. Under particular assumptions, they can be interpreted as the conditions for the existence of a bi-Hamiltonian structure. The remaining sections 3-5 contain three explicit applications. In Sect. 3, the chiral model is introduced in order to obtain the so-called second representation of KP and KdV hierarchies [4, 5]. This structure is studied as a $\mathcal{G}N\tau$ structure: the Lenard bicomplex of vector fields in involution is obtained, giving rise to the KP and KdV hierarchies by two particular realizations. In Sect. 4, the integrability structure of the WKI hierarchy [6] is analyzed by means of the reduction technique on a transversal submanifold (Lemmas 1.1, 1.2). By a suitable choice of the reduction submanifold, the $\mathcal{G}PN$ structure is constructed; moreover, by considering a suitable "enlarged" submanifold, a $\mathcal{G}N\mathcal{T}$ structure is obtained. In the last Sect. 5, a realization of the ZS model is considered, furnishing the recursion scheme for the $AKNS$ hierarchy in two spatial dimensions and in particular for the Davey-Stewartson (DS) hierarchy [7].

2 – Lie groups and Nijenhuis manifolds

Let M be a differentiable manifold, G a Lie group and $\Phi\colon G \times M \to M$, $\Psi\colon G \times M \to M$ two (left) actions of G on M. The corresponding infinitesimal actions are denoted by $X\colon \mathcal{G} \to \mathcal{X}(M)\colon X_u\xi \equiv \varphi_\xi(u)$, $Y\colon \mathcal{G} \to \mathcal{X}(M)\colon Y_u\xi \equiv \psi_\xi(u)$, where $u \in M$, $\xi \in \mathcal{G}$. So, one has:

$$[\varphi_\xi, \varphi_\eta] = \lambda\varphi_{[\xi,\eta]} \qquad (\lambda \in \mathbb{C}) \tag{2.1}$$

$$[\psi_\xi, \psi_\eta] = \mu\psi_{[\xi,\eta]} \qquad (\mu \in \mathbb{C}) \tag{2.2}$$

where $[\xi, \eta]$ and $[\varphi_\xi, \varphi_\eta] \equiv L_{\varphi_\xi}(\varphi_\eta)$ are respectively the commutators of the Lie algebra $\mathcal{G}$ of G and of the vector fields in $\mathcal{X}(M)$. If an action, e.g. Ψ, is Abelian, it is $\mu = 0$; if $\mu \neq 0$, one can normalize $\mu = 1$. We say that Φ and Ψ are *compatible actions* if the following relation holds:

$$[\varphi_\xi, \psi_\eta] + [\psi_\xi, \varphi_\eta] = \lambda\psi_{[\xi,\eta]} + \mu\varphi_{[\xi,\eta]} \tag{2.3}$$

(If an action is Abelian, eqs. (2.1)-(2.3) define a semi-direct sum of the Lie algebras defined in $\mathcal{X}(M)$ by φ and ψ).

Now if the vector fields φ_ξ are a basis in $\mathcal{X}(M)$, the tensor field N given by $N\varphi_\xi = \psi_\xi$ is well-defined and it endows the manifold M with a Nijenhuis structure:

PROPOSITION 2.1. *If a Lie group G acts on a differentiable manifold M by two compatible actions Φ and Ψ with generators φ_ξ and ψ_ξ, and if φ_ξ are a basis in $\mathcal{X}(M)$, then the manifold M is a Nijenhuis manifold with N defined by $N\varphi_\xi = \psi_\xi$.*

PROOF. The Nijenhuis condition $T(N) = 0$ follows directly from eqs. (2.1)-(2.3), since for any pair of vector fields $\varphi_\xi, \varphi_\eta$ it is:

$$
\begin{aligned}
T(N)(\varphi_\xi, \varphi_\eta) &= [N\varphi_\xi, N\varphi_\eta] - N[\varphi_\xi, N\varphi_\eta] - \\
&- N[N\varphi_\xi, \varphi_\eta] + N^2[\varphi_\xi, \varphi_\eta] = \\
&= [\psi_\xi, \psi_\eta] - N([\varphi_\xi, \psi_\eta] + [\psi_\xi, \varphi_\eta]) + \lambda N^2 \varphi_{[\xi,\eta]} = \\
&= \mu\psi_{[\xi,\eta]} - N(\lambda\psi_{[\xi,\eta]} + \mu\varphi_{[\xi,\eta]}) + \lambda N\psi_{[\xi,\eta]} = \\
&= \mu(\psi_{[\xi,\eta]} - N\varphi_{[\xi,\eta]}) \\
&= 0
\end{aligned}
\tag{2.4}
$$

□

As a particular case, we say that Ψ is obtained by a *deformation* of Φ if there is a vector field $\tau \in \mathcal{X}(M)$ such that

$$
\psi_\xi = [\tau, \varphi_\xi] \qquad (\forall \xi \in \mathcal{G})
\tag{2.5}
$$

Since on account of eqs. (2.1), (2.5) it is:

$$
\begin{aligned}
[\varphi_\xi, \psi_\eta] + [\psi_\xi, \varphi_\eta] &= [\varphi_\xi, [\tau, \varphi_\eta]] + [[\tau, \varphi_\xi], \varphi_\eta] = \\
&= [\tau, [\varphi_\xi, \varphi_\eta]] = \\
&= \lambda\psi_{[\xi,\eta]}
\end{aligned}
\tag{2.6}
$$

the compatibility condition (2.3) can be verified only with $\mu = 0$. So, any action Ψ which is a deformation of Φ is compatible with Φ iff it is an Abelian action.

If $M = \mathcal{G}$, a solution of eqs. (2.1)-(2.3) is obtained by considering *affine* actions of $\mathcal{G}$ onto itself [8], i.e. by assuming that φ_ξ and ψ_ξ take the form

$$
\varphi_\xi(u) = D\xi + a_\xi u \qquad \psi_\xi(u) = D'\xi + a'_\xi u
\tag{2.7}
$$

Indeed, eqs. (2.1), (2.2) are verified iff a_ξ and a'_ξ are linear representations of $\mathcal{G}$ and D, D' are one-cocyles w.r.t. them, i.e. if

$$a_\xi a_\eta - a_\eta a_\xi = -\lambda a_{[\xi,\eta]} \qquad a'_\xi a'_\eta - a'_\eta a'_\xi = -\mu a'_{[\xi,\eta]} \tag{2.8}$$

and if

$$a_\xi D\eta - a_\eta D\xi = -\lambda D[\xi,\eta] \qquad a'_\xi D'\eta - a'_\eta D'\xi = -\mu D'[\xi,\eta] \tag{2.9}$$

Furthermore, the compatibility condition (2.3) is verified if a_ξ and a'_ξ are compatible representations (i.e. if their linear combination is itself a representation) and if D and D' are cocycles also w.r.t. a'_ξ and a_ξ respectively. A solution of the entire set of conditions is obtained by taking $a_\xi u = a'_\xi u = -ad_\xi u$ and for any pair of derivations D and D' in $\mathcal{G}$. As a particular case, if the action Φ is linear ($D = 0$) and if the action Ψ is constant ($a'_\xi u = 0$), then a Nijenhuis structure is defined in $\mathcal{G}$ for any representation a_ξ and for any cocycle D' w.r.t. a_ξ. A non trivial example of this situation will be considered in Example III of this section.

REMARK I. Let us assume that $M = \mathcal{G}$, that $\mathcal{G}$ can be identified with its dual space $\mathcal{G}^*$ by a suitable pairing $<,> : \mathcal{G} \times \mathcal{G}^* \to \mathbb{R}$ and that Φ and Ψ be affine actions. Then eq. (2.7) can be seen as defining two linear mappings $P\colon \mathcal{G} \times \mathcal{G}^* \to \mathcal{G}$ and $Q\colon \mathcal{G} \times \mathcal{G}^* \to \mathcal{G}$ given by $P_u\xi = \varphi_\xi(u)$ and $Q_u\xi = \psi_\xi(u)$, so that one can associate in $M = \mathcal{G}$ two tensor fields P and Q of type $(2,0)$ with the actions Φ and Ψ. Let us consider the tensor P; eqs. (2.1), (2.7) entail that the Schouten condition $[P,P]_s\ (\xi,\eta) = 0$, which has to be verified by any Poisson tensor [1], is equivalent to

$$\begin{aligned} &P'_u(\xi, P_u\eta) - P'_u(\eta, P_u\xi) + P_u P'^{*}_u(\xi,\eta) = \\ &=(a_\xi D\eta - a_\eta D\xi - Da^*_\xi\eta) + (a_\xi a_\eta - a_\eta a_\xi - a_{a^*_\xi\eta})u = 0 \end{aligned} \tag{2.10}$$

where a^*_ξ is defined by $< u, a^*_\xi\eta > = < a_\xi u, \eta >$.

If $\lambda = 1$, eqs. (2.8), (2.9) for a_ξ and D are verified by taking $a_\xi u = ad_\xi u$ and by D equal to any coadjoint cocycle. Moreover, if D is skew-symmetric w.r.t. $<,>$, P itself is skew-symmetric and eq. (2.10) entails that P is a Poisson tensor. The same conclusion can be drawn for the tensor Q; moreover, the condition that the action Φ and Ψ be compatible can be interpreted also as the condition defining a bi-Hamiltonian structure in M.

If $\lambda = 0$, there is the particular solution $a_\xi u = 0$, so that P is a constant tensor field in $M = \mathcal{G}$, and it is clearly a Poisson tensor under the further assumption that D be skew-symmetric.

Summarizing, compatible actions entail a Nijenhuis structure, but the existence of two compatible Poisson tensors requires more restrictive conditions; in particular, for $M = \mathcal{G}$ and for affine actions Φ and Ψ, the introduction of a suitable pairing $<,>$ and the skew-symmetry of D w.r.t. $<,>$ are required. □

REMARK II. Under the assumptions of Remark I, the existence of a deformation τ for the action Φ is generally not equivalent to the existence of a deformation for the tensor P, at least if τ is assumed to be the same in both cases and if it is not given by a constant vector field. Indeed, if Q is related with Ψ one obtains:

$$Q_u\xi = \psi_\xi(u) = [\tau, \varphi_\xi](u) = L_\tau(P_u)\xi + P_u\tau_u'^{*}\xi \tag{2.11}$$

so that the tensor field which is related with Ψ is not the deformation of the tensor field related with Φ. □

At last, we apply the previous abstract approach to three well-known examples of Nijenhuis structures, which can be constructed by means of compatible actions on a manifold. They are strictly related with three abstract integrability structures which have been discussed in [1], i.e. the Zakharov-Shabat (ZS), the chiral model and the structure of the non-periodic Toda chain respectively.

EXAMPLE I. Let $A, K \subset A, \mathcal{G} \subset A$ be respectively an associative algebra with unit and with a derivation D, the kernel of D and a subalgebra where D is invertible. Let $M = \mathcal{G} + \{c\}$ be an affine hyperplane modelled on $\mathcal{G}, c \in K$ being such that $[c, \mathcal{G}] \subset \mathcal{G}$. The infinitesimal generators of the actions Φ and Ψ are the vector fields

$$\varphi_\xi(u) = D\xi + [u, \xi] \qquad \psi_\xi(u) = [a, \xi] \tag{2.12}$$

where $u \in M, \xi \in \mathcal{G}, [u, \xi] \equiv u\xi - \xi u, a \in A$ is such that $[a, \mathcal{G}] \subset \mathcal{G}$. The conditions (2.1)-(2.3) are verified with $\lambda = 1$, $\mu = 0$, so that M is a Nijenhuis manifold on account of Prop. 2.1. Moreover, Ψ is a deformation of Φ, since it is $\psi_\xi = [\tau, \varphi_\xi]$ with $\tau(u) = a$ (actually, this statement is rigorous if $a \in \mathcal{G}$, it is only formal if $a \in A/\mathcal{G}$: this point will be considered in more detail in the applications of the following sections). As is known [9], this manifold is also a bi-Hamiltonian manifold if it is realized by the algebra of polynomials.

Indeed, let B be an associative algebra with unit and with a trace-form $Tr\colon B \to \mathbb{R}$ (in the applications, B is $\mathbb{R}$ or $gl(n, \mathbb{R})$), A the algebra of polynomials of any fixed order n whose coefficients are $C^\infty(B)$, $D = \partial/\partial x$, K and $\mathcal{G}$ be the polynomials with constant and with rapidly vanishing coefficients, respectively:

$$\xi \in \mathcal{G}\colon \xi = \sum_{i=0}^{n} \xi_{n-i} z^i \qquad (z \in \mathbb{C}, \xi_i \in \mathcal{S}(B)) \tag{2.13}$$

A structure of associative algebra is given by the following product:

$$(\xi \cdot \eta)(z) = \sum_{i=0}^{n} \left(\sum_{k=0}^{i} \xi_{n-k} \eta_{n+k-i} \right) z^i \tag{2.14}$$

and $\mathcal{G}$ can be identified with G^* by putting

$$\mu \in \mathcal{G}^*\colon \mu = \sum_{i=0}^{n} \mu_i z^i \qquad (z \in \mathbb{C}, \mu_i \in \mathcal{S}(B)) \tag{2.15}$$

and by introducing the pairing

$$< \mu, \xi >= \sum_{i=0}^{n} \int_{-\infty}^{+\infty} Tr(\mu_i \xi_i) dx \tag{2.16}$$

EXAMPLE II. Under the assumptions of the previous example, let us consider the compatible actions with the following infinitesimal generators:

$$\varphi_\xi(u) = D\xi \qquad \psi_\xi(u) = [u, \xi] \tag{2.17}$$

fulfilling eqs. (2.1)-(2.3) with $\lambda = 0$, $\mu = 1$. In contrast with the previous example, these actions have no deformation. Indeed, for a vector field τ such that $\psi_\xi = [\tau, \varphi_\xi]$ it should be

$$\tau'_u \chi = [D^{-1}\chi, u] \Rightarrow \tau''_u(\chi, \overline{\chi}) = [D^{-1}\chi, \overline{\chi}] \tag{2.18}$$

for any $\chi, \overline{\chi} \in \mathcal{X}(M)$, but this equation has no solution, since $\tau''_u(\chi, \overline{\chi})$ is not symmetric w.r.t. $\chi, \overline{\chi}$. Also in this case, the results of the previous example for the algebra of polynomials hold true (they are clearly trivial if B is an Abelian algebra); so, one can define a bi-Hamiltonian structure. In contrast with the non-existence of a deformation for the actions, one has a deformation of Poisson manifold, since it is straightforward to verify that the vector field τ: $3\tau(u) = [D^{-1}u, u]$ is such that

$$P_u\xi = D\xi \qquad Q_u\xi = [u, \xi] \qquad Q = L_\tau(P) \tag{2.19}$$

EXAMPLE III. Let A be the algebra of the sequences with coefficients in the associative algebra B:

$$\varphi \in A \qquad \varphi = \{\varphi_i\}_{i \in \mathbb{Z}}, \qquad \varphi_i \in B \tag{2.20}$$

Let $\mathcal{G}, K$ be the subalgebras of the sequences such that $\varphi_i \to 0 (i \to \pm\infty)$ and of the "constant" sequences respectively. The manifold M is the affine hyperplane $M = \mathcal{G} + \{c\}, c \in K$, and the infinitesimal generators of the actions Φ and Ψ are

$$\varphi_\xi(u) = \{u_i\xi_i - \xi_{i-1}u_i\} \qquad \psi_i(u) = \{a_i\xi_i - \xi_{i-1}a_i\} \tag{2.21}$$

with $u \in M$, $a \in K$; these actions are compatible, since eqs. (2.1)-(2.3) are verified with $\lambda = 1$, $\mu = 0$, and they admit the (formal) deformation $\tau(u) = a$. So, M is a Nijenhuis manifold. In this case, the two actions do not endow M with a bi-Hamiltonian structure, if $\mathcal{G}^*$ is identified with $\mathcal{G}$ by the pairing $< \mu, \xi >= \sum_i \mu_i\xi_i$. Indeed, since in this case it is

$$D = 0 \qquad a_\xi u = \{u_i\xi_i - \xi_{i-1}u_i\} \tag{2.22}$$

for the action Φ and

$$D'\xi = \{a_i\xi_i - \xi_{i-1}a_i\} \qquad a'_\xi u = 0 \tag{2.23}$$

for the action Ψ, these mappings are not skew-symmetric w.r.t. the previous pairing.

REMARK III. The previous construction of a Nijenhuis structure on a differentiable manifold M can be generalized by considering a vector space U which is endowed with two Lie algebraic structures $\mathcal{G}$ and $\mathcal{G}'$, whose commutators are denoted by $[\xi,\eta]_{\mathcal{G}} \equiv ad_\xi \eta$ and $[\xi,\eta]_{\mathcal{G}'} \equiv ad'_\xi \eta$ respectively. Then eqs. (2.1)-(2.2) and the compatibility condition (2.3) become respectively

$$(2.24) \qquad [\varphi_\xi, \varphi_\eta] = \lambda \varphi_{ad_\xi \eta} \qquad [\psi_\xi, \psi_\eta] = \mu \psi_{ad'_\xi \eta}$$

$$(2.25) \qquad [\varphi_\xi, \psi_\eta] + [\psi_\xi, \varphi_\eta] = \mu \varphi_{ad'_\xi \eta} + \lambda \psi_{ad_\xi \eta}$$

and it is straighforward to verify (as well as in Prop. 2.1) that the tensor N: $N\varphi_\xi = \psi_\xi$ endows M with a Nijenhuis structure. As a particular case, if $M = U$ and affine actions of the form (2.7) are considered, then eq. (2.24) has the particular solution $a_\xi u = -ad_\xi u$, $a'_\xi u = -ad'_\xi u$, with D, D' derivations w.r.t. ad, ad' respectively. The compatibility condition (2.25) entails that D, D' be derivations w.r.t. both ad and ad' and that ad, ad' be compatible, i.e. that their linear combinations be themselves Lie algebraic structures.

As an example, let us consider the vector space U_n of the polynomials with degree n in $z \in \mathbb{C}$ with the product $\xi \cdot \eta$ defined by (2.14); let us endow U_n with another structure of associative algebra, given by the product

$$(2.26) \qquad (\xi * \eta)(z) := \sum_{i=0}^{n-1} z^i \left(\sum_{j=i+1}^{n} \xi_{j-i-1} \eta_{n-j} \right)$$

Then U_n is endowed with two Lie algebraic structures $\mathcal{G}$ and $\mathcal{G}'$ by defining

$$(2.27) \qquad ad_\xi \eta = \xi \cdot \eta - \eta \cdot \xi \qquad ad'_\xi \eta = \xi * \eta - \eta * \xi$$

(Indeed, U_n can be endowed with $n+1$ non trivial Lie algebraic structures; they are the starting point for the construction of the multi-Hamiltonian structures characterizing the hierarchies of integrable equations considered by FLASHKA-NEWELL-RATIU [10], MARTINEZ-ALONSO [11], ANTONOWICZ and FORDY [12]. A theoretical study of these structures will be considered in more detail elewhere). Then if one chooses D and D' as:

$$D\xi = \sum_{i=0}^{n} (D\xi_{n-i})\, z^i \qquad D'\xi = 0 \tag{2.28}$$

one can easily verify that the actions defined by

$$\varphi_\xi(u) = D\xi - ad_\xi u \qquad \psi_\xi(u) = -ad'_\xi u \tag{2.29}$$

are compatible actions, so that $M = U_n$ is a Nijenhuis manifold. Moreover, by using the pairing (2.16) one can associate with the vector fields (2.29) two Poisson tensors, which are defined as $P_u\xi = \varphi_\xi(u), Q_u\xi = \psi_\xi(u)$. So, the vector space U_n can be endowed with a bi-Hamiltonian structure (indeed, as it has been previously pointed out, one can define in U_n a multi-Hamiltonian structure, given by $(n+1)$ compatible Poisson tensors).

Of course, it is also possible to have different Lie algebraic structures acting on a manifold M whose actions are not compatible. A well-known example is obtained by considering the solutions R of the modified Yang-Baxter equation on any Lie algebra:

$$ad_{R\xi}R\eta - Rad_{R\xi}\eta - Rad_\xi R\eta = \alpha ad_\xi\eta \qquad (\alpha \in \mathbb{C}) \tag{2.30}$$

For any such R, the algebra $\mathcal{G}$ can be endowed with a second Lie algebraic structure ad' which is defined as [13]:

$$ad'_\xi\eta = ad_{R\xi}\eta + ad_\xi R\eta \tag{2.31}$$

The two actions given by $\varphi_\xi(u) = ad_\xi u$ and $\psi_\xi(u) = ad'_\xi u$ are not compatible, so that the algebra has not a Nijenhuis structure which is directly constructed by means of these actions. □

3 – The $GN\tau$ scheme for KdV and KP hierarchies

Let us discuss in more detail, for the chiral model of Example II, Sect. 2, the conditions enabling one to introduce in a rigorous way a $\mathcal{G}N\tau$ structure, i.e. a Nijenhuis tensor field with a symmetry algebra and a master-symmetry.

Let A be an associative algebra with unit and with a derivation D, $V \subset A$ a subalgebra where D is invertible, $K = \operatorname{Ker} D$ and $K_v \subset K$ an Abelian subalgebra such that $[K_v, V] \subset V$.

The manifold M is the affine hyperplane modelled on V, $M = V + \{c\}$, $c \in K_v$. By taking $\mathcal{G} = V$ in the general scheme, we consider the actions of V on M with the infinitesimal generators (2.17); they allow us to define in M the Nijenhuis tensor N given by:

$$N_u \varphi = [u, D^{-1}\varphi] \qquad (u \in M, \varphi \in V) \tag{3.1}$$

The symmetry algebra of N is given by the action of K_v on M by the vector fields $\varphi_b(u) = [u, b], b \in K_v$, fulfilling the condition $L_{\varphi_b}(N) = 0$. The recursion scheme for the KP hierarchy is obtained by the following choices:

i) A is the algebra of differential operators in $X \equiv \partial/\partial x$, $Y \equiv \partial/\partial y$, with coefficients in $C^\infty(\mathbb{R}^2)$

$$a \in A: \qquad a = \sum_{j,k} a_{jk}(x, y) X^j Y^k \tag{3.2}$$

ii) D is given by: $Da = \sum_{j,k} a_{jk,x} X^j Y^k \qquad (a_{jk,x} = \partial a_{jk}/\partial x)$

iii) V, K, K_v are the subalgebras whose coefficients are respectively rapidly vanishing for $|x| \to \infty$, y-independent and costant.

iv) $c = X^2 + Y$.

The equations of the KP hierarchy can be obtained by taking $u = q + X^2 + Y$, where q is a function (as for the construction of the equations following this scheme, see [5]). As a particular case, the recursion scheme for the KdV hierarchy is obtained from the previous one by replacing Y with $z \in \mathbb{C}$, i.e. by choosing the algebra of the polynomials in $z \in \mathbb{C}$ whose coefficients are polynomials in $X \equiv \partial/\partial x$.

The previous choices for M do not allow one to introduce the scaling τ_o (and consequently the master-symmetry $\tau = N\tau_o$); indeed, τ_o cannot be interpreted as a vector field in M, being $\tau_o(u) = u$. However, a $\mathcal{G}N\tau$ structure can be rigorously constructed by a different choice of the manifold M. To this end, let us assume that V be an ideal of A and that there is an element $x \in A$ such that $Dx = 1$, $[x, K_v] \subset K_v$ (clearly, if x exists it is not unique). Under these assumptions, let us consider the subalgebra $W = V \oplus K_v$ and let us define the mapping $\mathcal{J}: W \to A$ depending on x and given by

$$\mathcal{J}(\varphi_v + \varphi_k) = D^{-1}\varphi_v + x\varphi_k \qquad (\varphi_v \in V, \varphi_k \in K_v) \tag{3.3}$$

Clearly, $\mathcal{J}$ is a right-inverse of D: $D\mathcal{J} = 1$, and moreover it is $\mathcal{J}D_{|v} = 1$. Then if $M' = W$, one can easily verify that M' is a Nijenhuis manifold with a Nijenhuis tensor $N: W \times W \to W$ defined us:

$$N_u\varphi = [u, \mathcal{J}\varphi] \tag{3.4}$$

Moreover, τ_o: $\tau_o(u) = u$ and $\tau = N\tau_o$ are well-defined as vector fields in M', and they can be proved to be a scaling and a master-symmetry for N. The symmetry algebra is still given by the vector fields φ_b as in the previous case; so, one has a complete $\mathcal{G}N\tau$ structure defined in $M = W$. The proof that the tensor field (3.4) is a Nijenhuis tensor is easily obtained by directly verifying that $T(N) = 0$, and it is essentially based on the property that

$$\mathcal{J}D[\mathcal{J}\varphi, \mathcal{J}\psi] = [\mathcal{J}\varphi, \mathcal{J}\psi] \qquad (\forall\varphi, \psi \in W) \tag{3.5}$$

following from the assumptions on K_v and x. At last, one can remark that the Nijenhuis tensor (3.4) is not obtained by the actions φ_ξ, ψ_ξ, since φ_ξ is not an invertible mapping in W.

If the $\mathcal{G}N\tau$ scheme is realized in order to obtain the KP and KdV hierarchies, the abstract element $x \in A$ becomes simply the function x; so, the master symmetry $\tau(u) = [u, \mathcal{J}u]$ becomes:

$$\tau(u) = (D^{-1}q_y + q_x - x(q_{xx} + q_y)) + 2(q - xq_x)X + 2XY + 2X^3 \tag{3.6}$$

at any point $u = q + X^2 + Y$ for the KP case and

$$\tau(q) = (q_x - xq_{xx}) + 2(q - xq_x)X + 2zX + 2X^3 \tag{3.7}$$

at any point $u = q + z + X^2$ for the KdV case (actually, it can be proved that the entire hierarchies of KP and KdV equations can be obtained by τ although in a different way w.r.t. the usual master-symmetry approach [14], [15]; however, this problem will be considered in more detail elsewhere [16]).

REMARK. The choice of the algebra of differential operators allows one to construct the recursion scheme for KP and KdV, but not the related bihamiltonian structure corresponding to the actions (2.17). Indeed, since in our scheme the derivation D is interpreted as a tensor field of type (2,0), we need a non-degenerate bilinear form enabling us to identify T^*M with TM itself. Unlikely, as far as we know the algebra of differential operators can be put in duality only with the algebra of pseudo-differential operators, by means of the well-known trace-form introduced by ADLER [2]. In our opinion, the problem of the construction of a bi-Hamiltonian structure related with the chiral model is not completely solved neither in the framework of the so-called bilocal formalism [7], [14], based on the use of the algebra of integral operators with distributional kernels and of a suitable bilinear form between two spaces of *admissible functions*. Indeed, these spaces seem to us to be not clearly characterized.

4 – The WKI hierarchy

In this section we discuss the integrability structure which is associated with the so-called WKI hierarchy [6]. To this end, let us consider the algebra $A = gl(2, C^\infty(\mathbb{R}))$, the subalgebra $V = gl(2, S(R))$, the subalgebra K of the constant 2×2 matrices and the derivation D w.r.t. the independent variable x (the spatial dimension of the equations of the hierarchy). The manifold M is the affine hyperplane modelled on V: $M = V + \{\sigma_3\}, \sigma_3 = \mathrm{diag}(1, -1)$ being the Pauli matrix. In this case, at any point $u \in M$ it is possible to identify $T_uM \simeq T_u^*M = V$ by the pairing:

$$(4.1) \qquad < \alpha, \varphi >= Tr \int_{-\infty}^{+\infty} \alpha_o \varphi_o dx + Tr \int_{-\infty}^{+\infty} \alpha_d \varphi_d dx$$

where the indices o and d denote the off-diagonal and the diagonal parts respectively and Tr is the trace of matrices. Clearly, $D_{|V}$ is invertible and $D: T^*M \to TM$ is skew-symmetric w.r.t. (4.1). In this case we can consider the chiral model of Example II, Sect. 2, giving rise to the bi-Hamiltonian integrability structure which is defined by the Poisson tensors P, Q:

$$P_u \alpha = D\alpha \qquad Q_u \alpha = [u, \alpha] \tag{4.2}$$

the Nijenhuis tensor N:

$$N_u \varphi = Q_u P_u^{-1} \varphi = [u, D^{-1}\varphi] \tag{4.3}$$

and the symmetry algebra:

$$\varphi_b = [u, b] \qquad (b \in K_v, [K_v, V] \subset V) \tag{4.4}$$

The transversal submanifold S (see Lemmas 1.1, 1.2) for the WKI hierarchy is given by

$$S = \{u \colon u_d = \sigma_3\} \tag{4.5}$$

Indeed, at any point of S the Nijenhuis tensor N has a constant Riesz index, $\mathrm{ind}(N) = 1$, and the splitting condition is verified, since $TS = \{\varphi \colon \varphi_d = 0\}$ and:

$$\mathrm{Im}\, N = \{\varphi \colon Tr \quad \varphi = Tr \ \ u\varphi = 0\} \tag{4.6}$$

$$\mathrm{Ker}\, N = \{\varphi \colon \varphi = \lambda_x \mathbf{1} + (\mu u)_x, \lambda, \mu \in \mathcal{S}(\mathbb{R})\} \tag{4.7}$$

So, one can construct the mapping $\Pi \colon T(S, M) \to TS$ and its dual $\Pi^* \colon T^*(S, M) \to \mathrm{Im}\, N^*$ w.r.t the pairing (4.1):

$$\Pi \colon \varphi \mapsto \varphi_o - \frac{1}{2} D(u_o D^{-1} Tr(\varphi\sigma_3)) \tag{4.8}$$

$$\Pi^* \colon \alpha \mapsto \alpha_o - \frac{1}{2}\sigma_3 D^{-1} Tr(u_o D\alpha_o) \tag{4.9}$$

Now, the integrability structure (4.2)-(4.4) can be reduced by the two Lemmas cited in the Introduction. One easily obtains:

$$\overline{P} \colon \varphi_o = D\alpha_o + \frac{1}{2} D(u_o D^{-1} Tr(u_o D\alpha_o)) \tag{4.10}$$

$$\overline{Q}\colon \varphi_o = [\sigma_3, \alpha_o] - \frac{1}{2}[u_o, \sigma_3]D^{-1}Tr(u_o D\alpha_o) - \\ - \frac{1}{2}D(u_o D^{-1}Tr(\sigma_3[u_o, \alpha_o])) \tag{4.11}$$

Moreover, the Nijenhuis tensor is:

$$\overline{N} = \overline{Q}\,\overline{P}^{-1}: \\ \overline{N}\varphi_o = [\sigma_3, D^{-1}\varphi_o] - \frac{1}{2}D\left(u_o D^{-1}Tr([\sigma_3, u_o]D^{-1}\varphi_o)\right) \tag{4.12}$$

and the symmetries φ_b become:

$$\overline{\varphi}_b\colon \overline{\varphi}_b = [u_o, b_d] + [\sigma_3, b_o] - \frac{1}{2}D\left(u_o D^{-1}Tr(\sigma_3[u_o, b_o])\right) \tag{4.13}$$

REMARK I. Actually, the recursion operator of the WKI hierarchy is $\overline{N}^{-1}$; for its construction, see [17], where $\overline{N}$ has been obtained just by the present approach. As for the bi-Hamiltonian structure $\overline{P}$, $\overline{Q}$, the reduction technique of this section has not been previously applied to the WKI structure at the best of our knowledge. □

REMARK II. With reference to the Restriction Technique for Poisson manifolds discussed in [1], Sect. 3, we remark that P can be restricted to S, in contrast with Q. Indeed, at any point of S it is:

$$TS^\circ = \{\alpha = \alpha_d\} \quad T_P^*(S) = \{\alpha = \alpha_o\} \quad T_Q^*(S) = \{\alpha\colon [u_o, \alpha_o] = 0\} \tag{4.14}$$

so that P clearly fulfils the conditions for the restriction of Poisson tensors: $T^*(S, M) = TS^\circ + T_P^*(S)$. On the contrary, Q does not fulfil this condition and consequently it cannot be restricted to S. Moreover, it is not possible to reduce the structure (4.2) as a $P\Omega$ structure (on account of the invertibility of $P = D$) since the submanifold S does not fulfil the condition of the *Restriction Lemma for $P\Omega$ manifolds* stated in [1], Sect. 4. So, the reduction technique on a transversal submanifold is the unique method to obtain the bi-Hamiltonian structure of the WKI hierarchy. □

Again, in order to introduce the master-symmetry structure, one has to modify the manifold M. Following the same approach as in the previous section, we consider the manifold $M' = V \oplus K$ and the Nijenhuis tensor N: $N_u\varphi = [u, \mathcal{J}\varphi]$, so that τ_o: $\tau_o(u) = u$ and $\tau = N\tau_o$ are well-defined and can be correctly interpreted as vector fields in M'. The reduction technique on a transversal submanifold can be applied to N and τ, since at any point of the submanifold S given by (4.5) the conditions required by Lemma 1.1 are fulfilled. Indeed, Im N is still given by (4.6), whereas Ker N is defined as

$$\text{Ker } N = \{\varphi\colon \mathcal{J}\varphi = \lambda \mathbf{1} + \mu u,\ \lambda, \mu \in C^\infty(\mathbb{R})\} \tag{4.15}$$

so that the mapping Π becomes:

$$\Pi\colon \varphi \mapsto \varphi_o - \frac{1}{2} D(u_o \mathcal{J} Tr(\varphi\sigma_3)) \tag{4.16}$$

In particular, the scaling τ_o gives rise to:

$$\overline{\tau}_o = \Pi\tau_o\colon \overline{\tau}_o(u) = -x u_{0x} \tag{4.17}$$

We are not going into further details on the explicit construction of the equations of the WKI hierarchy, which can be obtained either by N or by τ following well-known techniques holding for integrable equations in one spatial dimension (e.g. see [15] and references therein). Instead, we end this section by observing that one can introduce in $M' = V \oplus K$ the pairing

$$< \alpha, \varphi > = Tr \int_{-\infty}^{+\infty} \alpha_v \varphi_v dx + Tr\ \alpha_k \varphi_k \tag{4.18}$$

where $\alpha = \alpha_v + \alpha_k$, $\varphi = \varphi_v + \varphi_k$. It is straightforward to verify that D is skew-symmetric w.r.t. (4.18) and that P and Q are still Poisson tensors; since P is not invertible in M', one has a bi-Hamiltonian manifold but not a Poisson-Nijenhuis manifold. Otherwise stated, it is not $Q = NP$, as it follows from $NP\alpha = [u, \mathcal{J}D\alpha]$ and from the fact that $\mathcal{J}$ is not a left-inverse of D.

5 – The $AKNS$ and DS hierarchies

In this section we apply the reduction technique to the $\mathcal{G}N_T$ structure of Example I, Sect. 2, in order to obtain the recursion scheme for the $AKNS$ hierarchy in two spatial dimensions; as is known, by further reduction of this structure one recovers the recursion scheme of other interesting hierarchies, and in particular that of the Davey-Stewartson (DS) equation [7].

So, under the assumptions made in Example I, Sect. 2, the Nijenhuis tensor N: $N\varphi_\xi = \psi_\xi$ is defined by the two infinitesimal generators:

$$\varphi_\xi = D\xi + [u, \xi] \qquad \psi_\xi = [a, \xi] \tag{5.1}$$

We make the following choices (to be verified in any specific realization):
i) $A = gl(2, B)$, where B is an associative algebra with unit and with a derivation D
ii) $a = \sigma_3 = \mathrm{diag}(1, -1)$
iii) $M = \mathcal{G} + \{\alpha\sigma_3\}$, where $\alpha \in K = \mathrm{Ker}\, D$ is assumed to be such that the restriction to $\mathcal{G}$ of the mapping D_α: $D_\alpha = D + ad_{\alpha\sigma_3}$ be invertible, where we denote still by D the component-wise derivation in $gl(2, B)$.
iv) the reduction submanifold $S \subset M$ is defined by $S = \{u\colon u_d = \alpha\sigma_3\}$.

Since at any point of S it is

$$\mathrm{Im}\, N^2 = \mathrm{Im}\, N = \{\varphi\colon \varphi_d = 0\} \tag{5.2}$$

$$\mathrm{Ker}\, N^2 = \mathrm{Ker}\, N = \{\varphi\colon \varphi = D_\alpha\xi_d + [u_o, \xi_d], \forall \xi_d \in \mathcal{G}\} \tag{5.3}$$

it follows that $\mathrm{ind}(N) = 1$ and that the splitting condition (1.1) is verified. So, the mapping Π: $T(S, M) \to TS$ is given by

$$\Pi\colon \varphi \mapsto \varphi_o - [u_o, D_\alpha^{-1}\varphi_d] \tag{5.4}$$

and the Reduction Lemma 1.1 can be applied. As for the Nijenhuis tensor

N, on account of (5.1) it is simpler to explicitly compute $\overline{N}^{-1}$, which takes the form:

$$\overline{N}^{-1}\varphi_o = \frac{1}{2}\sigma_3 D\varphi_o - \frac{1}{2}\Big[u_o, D_\alpha^{-1}[u_o, \sigma_3\varphi_o]\Big] \tag{5.5}$$

(one has to recall that for 2×2 matrices the equation $\varphi_o = [\sigma_3, \xi_o]$ has the unique solution $\xi_o = \frac{1}{2}\sigma_3\varphi_o$).

As for the symmetries of N, they are given by the vector fields $\varphi_c = [u, c_d]$, $c \in gl(2, K)$. So, it follows from Lemma 1.1 and from (5.4) that

$$\overline{\varphi}_c = [u_o, c_d] \tag{5.6}$$

Let us consider two realizations of the previous abstract scheme. The $AKNS$ hierarchy in one spatial dimension is obtained by the following choice:

$$B = C^\infty(\mathbb{R}) \qquad \mathcal{G} = \mathcal{S}(\mathbb{R}) \qquad \alpha = 0 \tag{5.7}$$

the usual parametrization of S being

$$j : (q; r) \longmapsto u = \left(\begin{array}{c|c} 0 & q \\ \hline r & 0 \end{array}\right) \qquad (q, r \in \mathcal{G}) \tag{5.8}$$

A recursion scheme in two spatial dimensions is obtained by choosing the algebra B of differential operators in Y and by taking $\alpha = Y$. In this case the operator D_α acts on any $\xi \in TS$ as:

$$D_\alpha \left(\begin{array}{c|c} 0 & \dot{q} \\ \hline \dot{r} & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & D_+\dot{q} \\ \hline D_-\dot{r} & 0 \end{array}\right) \qquad (D_\pm = \partial_x \pm \partial_y) \tag{5.9}$$

and the Lenard bicomplex is reduced on the manifold of zero-order differential operators $q, r \in \mathcal{S}(\mathbb{R})$. The equations of the DS hierarchy can be obtained by taking the coefficients of the differential operators in $C^\infty(\mathbb{C})$ and by restricting the Lenard bicomplex to the submanifold $r = \overline{q}$.

So, the manifold of the $AKNS$ hierarchy in one spatial dimension is given by the vector space V of 2×2 off-diagonal matrices with entries in $\mathcal{S}(\mathbb{R})$, whereas for two spatial dimensions M is the affine hyperplane modelled on V. A more remarkable difference between the two cases is the following one: for $B = C^\infty(\mathbb{R})$ there is a bilinear form, such as the previous mapping (4.1) written for the WKI hierarchy, enabling one to define two Poisson structures related with φ_ξ and ψ_ξ, i.e. the well-known bi-Hamiltonian structure for $AKNS$. If B is the algebra of differential operators, it is not possible to identify $\mathcal{G}^* \simeq \mathcal{G}$ for the same reasons as in the Remark of Sect. 3.

As a final remark, the problem of a rigorous definition and reduction of the master-symmetry for the abstract structure can be handled as well as for the previous KP and WKI structures. Let us consider the case of one spatial dimension ($\alpha = 0, D_\alpha = D$); then $\tau\colon \tau(u) = a$ is not a vector field in $M = \mathcal{G}$, but $\tau \in \mathcal{X}(M')$ if the enlarged manifold $M' = \mathcal{G} \oplus K$ is considered. One can directly verify that M' is a Nijenhuis manifold, whose tensor is given by eq. (5.5) with D^{-1} replaced by the right-inverse $\mathcal{J}$. Now, $\tau = a$ is a vector field in M' and its reduction give rise to

$$\bar{\tau} = \Pi\tau = -[u_o, x\sigma_3] \tag{5.10}$$

corresponding to

$$\bar{\tau} = -2\begin{pmatrix} xq \\ -xr \end{pmatrix} \tag{5.11}$$

in the parametrization (5.8) of the submanifold S.

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