Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 197-203

Fourier Jacobi-Bessel series for Meijer's G-function

S.D. BAJPAI

RIASSUNTO – *Si introduce una nuova classe delle serie di Fourier Jacobi-Bessel per la G-funzione di Meijer.*

ABSTRACT – *In this paper, we present a new class of Fourier Jacobi-Bessel series for Meijer's G-function.*

KEY WORDS – *Fourier Jacobi-Bessel series - Meijer's G-function - Two-dimensional partial differential equation.*

A.M.S. CLASSIFICATION: 33C40 - 35Q99

1 – Introduction

The object of this paper is to introduce a new class of Fourier Jacobi-Bessel series for Meijer's G-function [3, pp. 206-222],and present few of its particular cases. We also show that our Fourier Jacobi-Bessel Series is related to the solution of a two dimensional partial differential equation.

The following formulae are required in the proof:

The integral [1, p. 177, (2.1)]:

$$(1.1)\qquad \int_{-1}^{1}(1-x)^{\rho}(1+x)^{\beta}P_u^{(\alpha,\beta)}(x)G_{p,q}^{m,n}\left[z(1-x)^{\delta}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right]dx$$

$$=\frac{2^{\beta+\rho+1}\Gamma(\beta+u+1)}{\delta^{\beta+1}u!}G_{p+2\delta,q+2\delta}^{m+\delta,n+\delta}\left[z2^{\delta}\,\middle|\,\begin{matrix}\Delta(\delta,-\rho),a_p,\Delta(\delta,\alpha-\rho)\\ \Delta(\delta,\alpha-\rho+u),b_q,\Delta(\delta,-1-\beta-\rho-u)\end{matrix}\right],$$

where δ is a positive integer, $2(m+n) > p+q$, $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $\operatorname{Re}\beta > -1$, $\operatorname{Re}(\rho+\delta b_j) > -1$ $(j = 1,2,\dots,m)$.

The integral [2, p. 37, (2.1)]:

$$(1.2)\qquad \int_0^\infty y^{\sigma-1}J_\nu(y)Y_\mu(y)G_{p,q}^{m,n}\left[zy^{2\lambda}\middle|\begin{matrix}a_p\\ b_q\end{matrix}\right] = \frac{2^{\sigma-1}\lambda^{2(\sigma-1)}}{(2\pi)^{1-\lambda}}\cdot$$

$$\cdot\, G_{p+5\lambda,q+\lambda}^{m,n+2\lambda}\left[2^{2\lambda}\lambda^{4\lambda}z\middle|\begin{matrix}\Delta(\lambda,\frac{2-\sigma-\mu-\nu}{2}),\Delta(\lambda,\frac{2-\sigma+\mu-\nu}{2}),a_p,\\ \Delta(\lambda,\frac{3-\nu+\mu-\sigma}{2}),\Delta(\lambda,\frac{\nu-\mu-\sigma+2}{2}),\Delta(\lambda,\frac{\nu+\mu-\sigma+2}{2}),\\ b_q,\Delta(\lambda,\frac{3-\nu+\mu-\sigma}{2})\,.\end{matrix}\right],$$

where λ is a positive integer, $2(m+n) > p+q$, $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, $\operatorname{Re}(\sigma+\nu\pm\mu+2\lambda b_j) > 0$ $(j = 1,2,\dots,m)$, $\operatorname{Re}(\sigma+2\lambda(a_j-1)) < 1$ $(j = 1,2,\dots,n)$.

The orthogonality property of the Jacobi polynomials [4, p. 285, (5) and (9)]:

$$(1.3)\qquad \int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx$$

$$= \begin{cases} 0 & \text{if } m \neq n,\\ \dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)(\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)} & \text{if } m = n;\end{cases}$$

where $\operatorname{Re}\alpha > -1$, $\operatorname{Re}\beta > -1$.

The orthogonality property of the Bessel functions [5, p. 291, (6)]:

$$(1.4)\qquad \int_0^\infty x^{-1}J_{\nu+2n+1}(x)J_{\nu+2m+1}(x)dx$$

$$= \begin{cases} 0 & \text{if } m \neq n,\\ (4n+2\nu+2)^{-1} & \text{if } m = n, \quad \operatorname{Re}\nu+m+1 > -1.\end{cases}$$

2 – Fourier Jacobi-Bessel series

The Fourier Jacobi-Bessel series to be established is

$$(2.1)\qquad (1-x)^{\rho}y^{\sigma}Y_{\mu}(y)G_{p,q}^{m,n}\left[z(1-x)^{\delta}y^{2\lambda}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right]$$

$$=\frac{2^{\rho+\sigma}\lambda^{2(\sigma-1)}}{\delta^{\beta+1}(2\pi)^{1-\lambda}}\sum_{\substack{r=0\\t=0}}^{\infty}\frac{\alpha+\beta+2r+1)\Gamma(\alpha+\beta+r+1)(\nu+2t+1)}{\Gamma(\alpha+r+1)}\cdot$$

$$\cdot P_r^{(\alpha,\beta)}(x)J_{\nu+2t+1}(y)\cdot$$

$$\cdot G_{p+2\delta+5\lambda,q+2\delta+\lambda}^{m+\delta,n+\delta+2\lambda}\left[\frac{z2^{\delta}}{(2\lambda^2)^{-2\lambda}}\,\middle|\,\begin{matrix}\Delta(\delta,-\alpha-\rho),\Delta(\lambda,\frac{1-\sigma-\mu-\nu-2t}{2}),\\ \Delta(\lambda,\frac{1-\sigma+\mu-\nu-2t}{2}),a_p,\Delta(\delta,-\rho),\\ \Delta(\lambda,\frac{2-\nu-2t+\mu-\sigma}{2}),\\ \Delta(\lambda,\frac{\nu+2t-\mu-\sigma+3}{2}),\Delta(\lambda,\frac{\nu+2t+\mu-\sigma+3}{2})\\ \Delta(\delta,r-\rho),b_q,\Delta(\lambda,\frac{2-\nu-2t+\mu-\sigma}{2}),\Delta(\delta,-1-\alpha-\beta-\rho-r)\end{matrix}\right]$$

valid under the conditions of (1.1), (1.2), (1.3) *and* (1.4).

PROOF. Let

$$(2.2)\qquad f(x,y)=(1-x)^{\rho}y^{\sigma}Y_{\mu}(y)G_{p,q}^{m,n}\left[z(1-x)^{\delta}y^{2\lambda}\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right]$$

$$=\sum_{\substack{r=0\\t=0}}^{\infty}C_{r,t}P_r^{(\alpha,\beta)}(x)J_{\nu+2t+1}(y)\,.$$

Equation (2.2) is valid, since $f(x,y)$ is continuous and of bounded variation in the region $-1<x<1$, $0<y<\infty$. Multiplying both sides of (2.2) by $y^{-1}J_{\nu+2v+1}(y)$, integrating with respect to y from 0 to ∞, and using (1.2) and (1.4), then multiplying both sides of the resulting expression by $(1-x)^{\alpha}(1+x)^{\beta}P_u^{(\alpha,\beta)}(x)$, integrating with respect to x from -1 to 1 and using (1.1) and (1.3), the value of $C_{r,t}$ is obtained. Substituting this value of $C_{r,t}$ in (2.2), the Fourier Jacobi-Bessel series

(2.1) is established.

NOTE: On applying the same procedure as above, we can establish three other forms of two-dimensional expansions of this class with the help of alternative forms of (1.1) and (1.2).

3 – Particular cases

Since on specializing the parameters Meijer's G-function yields almost all special functions appearing in applied mathematics and physical sciences [3]. Therefore, the result presented in this paper is of a general character and hence may encompass several cases of interest. However, we present below only few particular cases of our Fourier Jacobi-Bessel series.

In (2.1), putting $\delta = \lambda = 1$ and using the identity [1, p. 180, (2.6)], it reduces to the form

$$(3.1) \qquad (1-x)^{\rho} y^{\sigma} Y_{\mu}(y) G^{m,n}_{p,q}\left[z(1-x)y^2 \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right]$$

$$= 2^{\rho+\sigma} \sum_{\substack{r=0 \\ t=0}}^{\infty} \frac{(-1)^r(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+r+1)(\nu+2t+1)}{\Gamma(\alpha+r+1)} \cdot$$

$$\cdot P_r^{(\alpha+\beta)}(x) J_{\nu+2t+1}(y) \cdot$$

$$\cdot G^{m,n+4}_{p+7,q+3}\left[8z \,\middle|\, \begin{matrix} -\rho, -\alpha-\rho, \frac{1-\alpha-\mu-\nu-2t}{2}, \frac{1-\sigma+\mu-\nu-2t}{2}, a_p, \\ \frac{2-\nu-2t+\mu-\sigma}{2}, \frac{\nu+2t-\mu-\sigma+3}{2}, \frac{\nu+2t+\mu-\sigma+3}{2}, \\ b_q, r-\rho, -1-\alpha-\beta-\rho, 2-\nu-2t+\mu-\sigma \end{matrix}\right],$$

valid under the conditions analogous to (2.1).

In (3.1), setting $m = 1$, $n = p$, $b_1 = 0$ and using [4, p. 439, (4)], we get

(3.2) $$(1-x)^\rho y^\sigma Y_\mu(y)_pF_{q-1}\left[\begin{matrix}1-a_1,\dots,1-a_p;-z(1-x)=y^2\\ 1-b_2,\dots,1-b_q;\end{matrix}\right]$$

$$=2^{\rho+\sigma}\sum_{\substack{r=0\\t=0}}^{\infty}\frac{(-1)^r(\alpha+\beta+2r+1)\Gamma(\alpha+\beta+r+1)(\nu+2t+1)}{\Gamma(\alpha+r+1)}\cdot$$

$$\cdot P_r^{(\alpha+\beta)}(x)J_{\nu+2t+1}(y)\cdot$$

$$\cdot\frac{\Gamma(1+\rho)\Gamma(1+\alpha+\rho)\Gamma(\frac{1+\sigma+\mu+\nu+2t}{2})\Gamma(\frac{1+\sigma-\mu+\nu+2t}{2})}{\Gamma(1-r+\rho)\Gamma(2+\alpha+\beta+\rho+r)\Gamma(\nu+2t+\sigma-\mu-1)\Gamma(\frac{2-\nu-2t+\mu-\sigma}{2})}\cdot$$

$$\cdot\frac{1}{\Gamma(\frac{\nu+2t-\mu-\sigma+3}{2})\Gamma(\frac{\nu+2t+\mu-\sigma+3}{2})}\cdot$$

$$\cdot\, {}_{p+7}F_{q+2}\left[\begin{matrix}1-a_1,\dots,1-a_p,1+\rho,1+\alpha+\rho,\dfrac{1+\sigma+\mu+\nu+2t}{2},\\ \dfrac{1+\sigma-\mu+\nu+2t}{2},\dfrac{1+\nu+2t-\mu+\sigma}{2},\dfrac{\mu+\sigma-\nu-2t-1}{2},\\ \dfrac{\sigma-u-\nu-2t-1}{2};\\ 1-b_2,\dots,1-b_q,1-r+\rho,2+\alpha+\beta+\rho+r,\nu+2t+\sigma-r-1;-8z\end{matrix}\right],$$

valid under the conditions analogous to (2.1).

In (3.2), putting $z=0$, we obtain

(3.3) $$(1-x)^\rho y^\sigma Y_\mu(y)=2^{\rho+\sigma}\sum_{\substack{r=0\\t=0}}^{\infty}\frac{(-1)^r(\alpha+\beta+2r+1)}{\Gamma(\alpha+r+1)}\cdot$$

$$\cdot\frac{\Gamma(\alpha+\beta+r+1)(\nu+2t+1)\Gamma(1+\rho)\Gamma(1+\alpha+\rho)\Gamma(\frac{1+\sigma+\mu+\nu+2t}{2})}{\Gamma(1-r+\rho)\Gamma(2+\alpha+\beta+\rho+r)\Gamma(\nu+2t+\sigma-\mu-1)}\cdot$$

$$\cdot\frac{\Gamma(\frac{1+\sigma-\mu+\nu+2t}{2})}{\Gamma(\frac{2-\nu-2t+\mu-\sigma}{2})}P_r^{(\alpha,\beta)}(x)J_{\nu+2t+1}(y),$$

valid under the conditions analogous to (2.1).

4 – Two-dimensional partial differential equation

Let us consider the following two-dimensional partial differential equation

$$(4.1)\quad \frac{\partial u}{\partial t} = (1-x^2)\frac{\partial^2 u}{\partial x^2} + [\beta-\alpha-(\alpha+\beta+2)x]\frac{\partial u}{\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} + y^2 u\,,$$

where $u \equiv u(x,y,t)$, and $u(x,y,0) = f(x)$.

To solve (4.1), let us assume that (4.1) has a solution of the form:

$$(4.2)\qquad u(x,y,t) = e^{-[n(n+\alpha+\beta+1)-m^2]t} X(x)Y(y)\,.$$

The substitution of (4.2) into (4.1) yields the differential equation:

(4.3)

$$Y(y)[(1-x^2)X'' + \{\beta-\alpha-(\alpha+\beta+2)x\}X' + n(n+\alpha+\beta+1)X] +$$
$$+ X(x)[y^2Y'' + yY' + (y^2-m^2)Y] = 0\,.$$

We have, Jacobi equation

$$(4.4)\quad (1-x^2)X'' + \{\beta-\alpha-(\alpha+\beta+2)x\}X' + n(n+\alpha+\beta+1)X = 0\,,$$

with its solution $X = P_n^{(\alpha,\beta)}(x)$.

And, we have, Bessel equation

$$(4.5)\qquad y^2Y'' + yY' + (y^2-m^2)Y = 0\,,$$

with its solution $Y = J_m(y)$.

In view of (4.4) and (4.5), we conclude that to each eigenvalue given by (4.3), there corresponds the solution of (4.1), called an eigenfunction or eigenstate given by

$$(4.6)\qquad u(x,y,t) = e^{-[n(n+\alpha+\beta+1)-m^2]t} P_n^{(\alpha,\beta)}(x) J_m(y)\,.$$

In view of the principle of superposition, the solution of (4.1) is given by

$$(4.7)\qquad u(x,y,t) = \sum_{\substack{n=0\\m=0}}^{\infty} A_{n,m} e^{-n[(n+\alpha+\beta+1)-m^2]t} P_n^{(\alpha,\beta)}(x) J_m(y)\,.$$

In (4.7), putting $t = 0$, we have

$$f(x,y) = u(x,y,0) = \sum_{\substack{n=0 \\ m=0}}^{\infty} A_{n,m} P_n^{(\alpha,\beta)}(x) J_m(y)\,. \tag{4.8}$$

It is interesting to note that (2.2) is of the same form as (4.8).

Acknowledgement

The author wishes to express his sincere thanks to the learned referee for some useful suggestions for the improvement in this paper.

REFERENCES

[1] S.D. BAJPAI: *Some expansion formulae for Meijer's G-function*, Vijnana Parishad Anusandhan Patrika, 11-3 (1968), 177-191.

[2] S.D. BAJPAI: *Expansion formulae for the products of Meijer's G-function and Bessel functions*, Portugal. Math. **33** (1974), 35-41.

[3] A. ERDÉLYI: *Higher trascendental functions*, Vol. 1. McGraw-Hill, New York, (1953).

[4] A. ERDÉLYI: *Tables of integral transforms*, Vol. 2. McGraw-Hill, New York, (1954).

[5] Y.L. LUKE: *Integrals of Bessel functions*, McGraw-Hill, New York, (1962).

Lavoro pervenuto alla redazione il 26 marzo 1991
ed accettato per la pubblicazione il 17 luglio 1991
su parere favorevole di L. Gatteschi e di P. Benvenuti

INDIRIZZO DELL'AUTORE:

S.D. Bajpai - Department of Mathematics - University of Bahrain - P.O. Box 32038, Isa Town - Bahrain