Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 205-216

On the absolute Riesz summability of Fourier series

M.A. SARIGÖL

RIASSUNTO – *Viene dimostrato un teorema sulla sommabilità assoluta di Riesz delle serie di Fourier di indice k; il teorema congloba i risultati di Chow e Pandley.*

ABSTRACT – *In this paper a theorem on the absolute Riesz summability with index k of Fourier series, which includes the results of Chow and Pandey, is proved.*

KEY WORDS – *Convergence factors and summability factors - Absolute convergence - Absolute summability - Summability of Fourier and trigonometric series.*

A.M.S. CLASSIFICATION: 40D15 - 42A28 - 42A24

1 – Introduction

Let $\sum a_n$ be a given infinite series with partial sums s_n. Let u_n^α and r_n^α denote the n-th Cesàro mean of order $\alpha(\alpha > -1)$ of the sequences (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v$$

and

$$r_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v ,$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \text{ for } n \ge 1, \quad A_0^\alpha = 1.$$

The series $\sum a_n$ is then said to be absolutely summable (C,α) with index k, or simply summable $|C,\alpha|_k$, $k \ge 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left| u_n^\alpha - u_{n-1}^\alpha \right|^k < \infty . \tag{1.1}$$

But since $r_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [4]), condition (1.1) can also be written as

$$\sum_{n=1}^{\infty} n^{-1} \left| r_n^\alpha \right|^k < \infty .$$

Let (p_n) be a sequence of positive real numbers such that

$$P_n = p_0 + p_1 + \dots + p_n \to \infty \text{ as } n \to \infty, \quad P_{-1} = p_{-1} = 0.$$

The series $\sum a_n$ is said to be summable $|R,p_n|_k$, $k \ge 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left| t_n - t_{n-1} \right|^k < \infty ,$$

where

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v , \quad (P_n \ne 0).$$

In the special case when $p_n = 1$ for all values of n, $|R,p_n|_k$ summability is the same as $|C,1|_k$ summability.

For any sequence (β_n) we write that

$$\Delta \beta_n = \beta_n - \beta_{n+1} .$$

A sequence (β_n) is said to be convex if $\Delta^2 \beta_n = \Delta(\Delta \beta_n) \ge 0$ for all positive integer n.

Let the formal expansion of a function $f(x)$, periodic with period 2π and integrable in the sense of Lebesgue over $[-\pi, \pi]$, in a Fourier-trigonometric series be given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) .$$

We write

$$\phi(u) = f(x+u) + f(x-u) - 2f(x)$$

$$\varphi(t) = \int_t^\delta \frac{|\phi(u)|}{u} du \,, \Phi(t) = \int_0^t |\phi(u)| du \,, \quad 0 < \delta \leq \pi$$

$$\mu_n = \Big(\prod_{v=1}^{m-1} \log^v n \Big) \big(\log^m n \big)^{1+\epsilon} \,, \quad \varepsilon > 0 \,,$$

where

$$\log^m n = \log\big(\log^{m-1} n\big), \ldots, \log^2 n = \log(\log n) .$$

The following results are known:

THEOREM 1. (CHOW [2], 1941). *If (λ_n) is a convex sequence and the series $\sum n^{-1}\lambda_n$ is convergent, then the series $\sum A_n(x)\lambda_n$ is $|C, 1|$-summable for almost all values of x.*

THEOREM 2. (CHENG [1], 1948). *If*

$$\Phi(t) = 0(t) \,, \; as\, t \to 0+$$

then the series

$$\sum_{n=2}^{\infty} A_n(x)/(\log n)^{1+\epsilon} \,, \quad \varepsilon > 0 \,,$$

is summable $|C, \alpha|$, $\alpha > 1$.

THEOREM 3. (HSIANG [3], 1970). *If*

$$\Phi(t) = 0(t) \,, \; as\, t \to 0+$$

then the series

$$\sum_{n=1}^{\infty} A_n(x)/n^{\alpha}$$

is summable $|C,1|$ *for every* $\alpha > 0$.

THEOREM 4. (HSIANG [3], 1970). *If*

$$\Phi(t) = 0\left\{t/\prod_{v=1}^{m-1} \log^v(1/t)\right\} \quad as \quad t \to 0+,$$

then the series

$$\sum_{n=n_0}^{\infty} A_n(x)/\mu_n$$

is summable $|C,1|$ *for every* $\varepsilon > 0$.

Recently, PANDEY [5] has proved the following theorem, which includes the theorem of Cheng and both the theorems of Hsiang:

THEOREM 5. *If*

$$\varphi(t) = 0\left\{\left(\log^m(1/t)\right)^z\right\} \quad as \quad t \to 0+,$$

then the series

$$\sum_{n=n_0}^{\infty} A_n(x)/\mu_n$$

is summable $|C,1|$ *for* $0 < z < \varepsilon$.

2 – $|R, P_n|_k$ summability factors of Fourier series

In this paper we prove the following theorem, which includes the theorems of Chow and Pandey as special cases, and hence all the previous results.

THEOREM 6. *Let* $(|\lambda_n|)$ *be a non-increasing sequence of real numbers such that* $n|\Delta\lambda_n| = 0(|\lambda_n|)$. *Let* $np_n = 0(P_n)$ *and* $\sum_{v=1}^{n} v^{-1}P_v = 0(P_n)$.

(i) *If*

$$\sum n^{-1}|\lambda_n|^k < \infty,$$

then the series $\sum A_n(x)\lambda_n$ *is summable* $|R, p_n|_k$, $1 \leq k < 2$.

(ii) *If* (w_n) *is a sequence of positive real numbers such that* $n^{-s}w_n \to 0$ *as* $n \to \infty$, *for some* s, $0 < s < 1$, *and if*

$$\varphi(t) = 0\Big(w_{(1/t)}\Big) \quad as \quad t \to 0,$$

$$\sum n^{-1}|\lambda_n|^k w_n^k < \infty,$$

then the series $\sum A_n(x)\lambda_n$ *is summable* $|R, p_n|_k$, $k \geq 1$.

To prove the Theorem, the following results are needed.

THEOREM 7. *Let* t_n *be the* n*-th Cesàro mean of first order of the sequence* (na_n). *Let* (λ_n) *be a sequence of real numbers. If*

$$\sum_{n=1}^{\infty} n^{-1}|\lambda_n|^k|t_n|^k < \infty,$$

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n}|\lambda_n|^k|t_n|^k < \infty,$$

$$\sum_{n=1}^{\infty} n^{k-1}|\Delta\lambda_n|^k|t_n|^k < \infty,$$

$$\sum_{n=1}^{\infty} \left(\frac{np_n}{P_n}\right)^{k-1}\frac{p_n}{P_n}|\lambda_n|^k|t_n|^k < \infty,$$

$$\sum_{n=1}^{\infty} \left(\frac{np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}} = 0\left(\frac{1}{P_v}\right) \quad as \quad v \to \infty,$$

$$\sum_{v=1}^{n} v^{-1}P_v = 0(P_n) \quad as \quad n \to \infty$$

hold, then the series $\sum a_n\lambda_n$ *is summable* $|R, p_n|_k$, $k \geq 1$.

PROOF. Let T_n denote the (R, p_n) mean of the series $\sum a_n\lambda_n$. Then we have

$$\begin{aligned}
T_n - T_{n-1} =& \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} v a_v (v^{-1} P_{v-1}\lambda_v) \\
=& \frac{p_n}{P_n P_{n-1}} \Big\{ \sum_{v=1}^{n-1} v^{-1} P_{v-1}\lambda_v t_v - p_v \lambda_v t_v + P_v \Delta\lambda_v t_v + \\
& + (n+1) n^{-1} P_{n-1} \lambda_n t_n \Big\} \\
=& T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}\,, \text{ say}.
\end{aligned}$$

To prove the theorem, it is enough, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_{n,r}|^k < \infty\,, \quad \text{for} \quad r = 1, 2, 3, 4\,.$$

It follows from Hölder's inequality that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{k-1} |T_{n,1}|^k \leq & \sum_{n=2}^{m+1} \left(\frac{np_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}^k} \Big\{ \sum_{v=1}^{n-1} v^{-1} P_v |\lambda_v|\, |t_v| \Big\}^k \\
\leq & \sum_{n=2}^{m+1} \left(\frac{np_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} v^{-1} P_v |\lambda_v|^k |t_v|^k x \Big\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} v^{-1} P_v \Big\}^{k-1} \\
= & 0(1) \sum_{n=2}^{m+1} \left(\frac{np_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} v^{-1} P_v |\lambda_v|^k |t_v|^k \\
= & 0(1) \sum_{v=1}^{m} v^{-1} P_v |\lambda_v|^k |t_v|^k \sum_{n=v+1}^{\infty} \left(\frac{np_n}{P_n}\right)^{k-1} \frac{p_n}{P_n P_{n-1}} \\
= & 0(1) \sum_{v=1}^{m} v^{-1} |\lambda_v|^k |t_v|^k\,,
\end{aligned}$$

$$\sum_{n=2}^{m+1} n^{k-1}|T_{n,2}|^k \le \sum_{n=2}^{m+1}\left(\frac{np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}p_v|\lambda_v|\,|t_v|\right\}^k$$

$$\le \sum_{n=2}^{m+1}\left(\frac{np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v|\lambda_v|^k|t_v|^k x\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$= 0(1)\sum_{v=1}^{m}p_v|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}}$$

$$= 0(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|\lambda_v|^k|t_v|^k,$$

$$\sum_{n=2}^{m+1} n^{k-1}|T_{n,3}|^k \le \sum_{n=2}^{m+1}\left(\frac{np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}P_v|\Delta\lambda_v|\,|t_v|\right\}^k$$

$$\le \sum_{n=2}^{m+1}\left(\frac{np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}^k}\sum_{v=1}^{n-1}v^{-1}P_v\Big(v|\Delta\lambda_v|\Big)^k x\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}v^{-1}P_v\right\}^{k-1}$$

$$= 0(1)\sum_{v=1}^{m}v^{k-1}|\Delta\lambda_v|^k|t_v|^k, \text{ and}$$

$$\sum_{n=1}^{m} n^{k-1}|T_{n,4}|^k = 0(1)\sum_{n=1}^{m}\left(\frac{np_n}{P_n}\right)^{k-1}\frac{p_n}{P_n}|\lambda_n|^k|t_n|^k.$$

COROLLARY 8. *Let t_n be the n-th Cesàro mean of first order of the sequence (na_n) and write $T_n^{(k)} = \sum_{v=1}^{n}|t_v|^k$. Let $(|\lambda_n|)$ be non-increasing sequence of real numbers such that $n|\Delta\lambda_n| = 0(|\lambda_n|)$. If $np_n = 0(P_n)$, $\sum_{v=1}^{n} v^{-1}P_v = 0(P_n)$ and*

$$\sum_{n=1}^{m} n^{-2}|\lambda_n|^k T_n^{(k)} = 0(1) \text{ as } n \to \infty,$$

then the series $\sum a_n\lambda_n$ is summable $|R,p_n|_k$, $k \ge 1$.

PROOF. Since $np_n = 0(P_n)$ and $|\Delta\lambda_n|^k = 0\left(\frac{|\lambda_n|^k}{n^k}\right)$,

$$\sum_{n=1}^{m}\Big(\frac{np_n}{P_n}\Big)^{k-1}\frac{p_n}{P_n}|\lambda_n|^k|t_n|^k = 0(1)\sum_{n=1}^{m} n^{-1}|\lambda_n|^k|t_n|^k,$$

$$\sum_{n=1}^{m} n^{k-1}|\Delta\lambda_n|^k|t_n|^k = 0(1)\sum_{n=1}^{m} n^{-1}|\lambda_n|^k|t_n|^k, \text{ and}$$

$$\sum_{n=v+1}^{\infty}\Big(\frac{np_n}{P_n}\Big)^{k-1}\frac{p_n}{P_nP_{n-1}} = 0(1)\sum_{n=v+1}^{\infty}\frac{p_n}{P_nP_{n-1}} = 0\Big(\frac{1}{P_v}\Big).$$

Therefore, it is sufficient, by Theorem 7, to show that

$$\sum_{n=1}^{\infty} n^{-1}|\lambda_n|^k|t_n|^k < \infty.$$

Now

$$\sum_{n=1}^{m} n^{-1}|\lambda_n|^k|t_n|^k = \sum_{n=1}^{m-1} T_n^{(k)}\Delta\big(n^{-1}|\lambda_n|^k\big) + T_m^{(k)}m^{-1}|\lambda_m|^k$$

$$= \sum_{n=1}^{m-1} T_n^{(k)}\Big\{|\lambda_n|^k\Delta\Big(\frac{1}{n}\Big) + \frac{1}{n+1}\Delta|\lambda_n|^k\Big\} + T_m^{(k)}\frac{1}{m}|\lambda_m|^k.$$

On the other hand, since

$$\Delta|\lambda_n|^k \le k|\lambda_n|^{k-1}\Delta|\lambda_n| \le k|\lambda_n|^{k-1}|\Delta\lambda_n| = 0\Big(\frac{|\lambda_n|^k}{n}\Big),$$

$$\begin{aligned} T_m^{(k)}\frac{1}{m}|\lambda_m|^k &= T_m^{(k)}\sum_{v=m}^{\infty}\Delta\Big(\frac{1}{v}|\lambda_v|^k\Big) \\ &\le \sum_{v=m}^{\infty} T_v^{(k)}\Big\{|\lambda_v|^k\Delta\Big(\frac{1}{v}\Big) + \frac{1}{v+1}\Delta|\lambda_v|^k\Big\} \\ &= 0(1)\sum_{v=m}^{\infty}\frac{1}{v^2}|\lambda_v|^k T_v^{(k)}, \end{aligned}$$

it follows that

$$\sum_{v=1}^{m}\frac{1}{v}|\lambda_v|^k|t_v|^k = 0(1)\sum_{v=1}^{\infty}\frac{1}{v^2}|\lambda_v|^k T_v^{(k)} < \infty,$$

which completes the proof.

LEMMA 9. *Let $t_n(x)$ be the n-th Cesàro mean of first order of the sequence $(nA_n(x))$ and write $T_n^{(k)}(x) = \sum_{v=1}^{n} |t_v(x)|^k$. Then,*

(i) $$T_n^{(k)}(x) = 0(nw_n^k) \text{ as } n \to \infty, \quad k \geq 1,$$

provided that (w_n) is a sequence of positive real numbers such that $n^{-s}w_n \to 0$ as $n \to \infty$, for some s, $0 < s < 1$, and

$$\varphi(t) = 0(w_{(1/t)}) \text{ as } t \to 0.$$

(ii) $$T_n^{(k)}(x) = 0(n) \text{ as } n \to \infty, \quad 1 \leq k < 2,$$

for almost all values of x.

PROOF. Take $S_n(x) = \sum_{v=0}^{n} A_v(x)$. Then,

$$S_v(x) - f(x) = \frac{2}{\pi} \cdot \int_0^\pi \phi(t) \frac{\sin vt}{t} dt + 0(1) = \frac{2}{\pi} L + 0(1), \text{ say}$$

$$L = \int_0^{1/n} + \int_{1/n}^{\pi} = L_1 + L_2, \text{ say}.$$

Now

$$\int_0^t |\phi(u)| du = \int_0^t u\varphi'(u) du = (u\varphi(u))\Big|_0^t - \int_0^t u^s \varphi(u) u^{-s} du =$$

$$= 0(tw_{(1/t)}) + 0\Big(t^s w_{(1/t)} \int_0^t u^{-s} du\Big) = 0(tw_{(1/t)}).$$

Thus

$$|L_1| = \Big|\int_0^{1/n} \phi(u)\frac{\sin vu}{u}du\Big| \le v\int_0^{1/n} |\phi(u)|du$$

$$\le n\int_0^{1/n} |\phi(u)|du = 0(w_n)\,, \text{ and}$$

$$|L_2| = \Big|\int_{1/n}^{\pi} \phi(u)\frac{\sin vu}{u}du\Big| \le \int_{1/n}^{\pi} \frac{|\phi(u)|}{u}du = 0(w_n)\,.$$

Therefore

$$|L| = 0(w_n)\,,$$

and so

$$\sum_{v=1}^{n} |S_v(x) - f(x)|^k = 0(nw_n^k)\,.$$

Writing

$$u_n(x) = \frac{1}{n+1}\sum_{v=0}^{n} S_v(x)\,,$$

and applying Hölder's inequality, we obtain

$$|u_n(x) - f(x)|^k \le \left\{\frac{1}{n+1}\sum_{v=0}^{n} |S_v(x) - f(x)|\right\}^k$$

$$\le \frac{1}{n^k}\sum_{v=0}^{n} |S_v(x) - f(x)|^k \left\{\sum_{v=0}^{n} 1\right\}^{k-1} = 0(w_n^k)\,,$$

which implies

$$\sum_{v=1}^{n} |u_v(x) - f(x)|^k = 0(nw_n^k)\,.$$

Since $t_n(x) = S_n(x) - u_n(x)$, it follows from Minkowski's inequality that

$$T_n^{(k)}(x) \le \left\{\left(\sum_{v=1}^{n} |S_v(x) - f(x)|^k\right)^{1/k} + \left(\sum_{v=1}^{n} |u_n(x) - f(x)|^k\right)^{1/k}\right\}^k = 0(nw_n^k).$$

PROOF OF LEMMA 9 (ii). Since (see[2])

$$\sum_{v=0}^{n} \{S_v(x) - f(x)\}^2 = 0(n)$$

for almost all values of x, we have

$$\sum_{v=0}^{n} |S_v(x) - f(x)|^k \leq \left\{\sum_{v=0}^{n} |S_v(x) - f(x)|^2\right\}^{k/2} \left\{\sum_{v=0}^{n} 1\right\}^{1-k/2}$$
$$= 0(n)\,, \qquad 1 \leq k < 2\,.$$

The rest can be achieved similarly as in the previous part.

3 – Proof of theorem 6 and special cases

The proof of Theorem 6 follows immediately from Corollary 8 and Lemma 9.

REMARK. If (λ_n) is convex such that $\sum n^{-1}\lambda_n < \infty$, then $\lambda_n \geq 0$ and $n\Delta\lambda_n = 0(\lambda_n)$ (see [6]).

It may be mentioned here that Theorem 1 follows from Theorem 6 (i) by putting $p_n = 1$, $k = 1$ and making use of the remark. While Theorem 5 follows from Theorem 6 (ii) by putting $p_n = 1$, $k = 1$, $\lambda_n = \mu_n^{-1}$ and $w(t) = (\log^m t)^z$.

Acknowledgements

The author is grateful to the referee for his kind remarks and suggestions which improved the presentation of the paper.

REFERENCES

[1] M.T. CHENG: *Summability factors of Fourier series*, Duke Math. J. , 15 (1948), 17-21.

[2] H.C. CHOW: *On the summability factors of Fourier series*, J. London Math. Soc., 16 (1941), 215-226.

[3] F.C. HSIANG: *On* $|C, 1|$ *summability factors of Fourier series at a given point*, Pacific J. Math., 33 (1970), 139-147.

[4] E. KOGBETLIANTZ: *Sur les séries absolument sommables par la méthode des moyennes arithmétiques*, Bull. Sci. Math. 9 (1925), 234-256.

[5] G.S. PANDEY: *Multiplier for* $|C, 1|$ *summability of Fourier series*, Pacific J. Math., 79 (1978), 177-182.

[6] T. SINGH: *A note on* $|\bar{N}, p_n|$ *summability factors for infinite series*, Journal of Mathematical Sciences, 12-13 (1977-1978), 25-28.

Lavoro pervenuto alla redazione il 3 aprile 1991
ed accettato per la pubblicazione il 16 settembre 1991
su parere favorevole di M. Piccardello e di E. Prestini

INDIRIZZO DELL'AUTORE:

M. Ali Sarigöl - Department of Mathematics - Erciyes University - 38039 Kayseri - Turkey