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A theorem on nonnegatively curved locally conformal Kaehler manifolds

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RIASSUNTO – *Sia M^n una varietà localmente conformemente Kaehleriana completa, con curvatura sezionale nonnegativa, curvatura nulla potendo avere soltanto i due-piani che contengano il campo vettoriale di Lee B. Siano V^r e W^s due sottovarietà compatte complesse, tangenti a B. Si dimostra che se $r+s \geq n$ allora V^r e W^s si intersecano. Si osserva anche che, mutatis mutandis, in simili condizioni, un risultato analogo vale per un ambiente Sasakiano.*

ABSTRACT – *Let M^n be a complete locally conformal Kaehler manifold with nonnegative sectional curvature, zero curvature having, eventually, the two-planes containing the Lee vector field B. Let V^r and W^s be two compact complex submanifolds tangent to B. It is proved that if $r+s \geq n$ then V^r and W^s must intersect. One also observes by passing that, mutatis mutandis, in similar conditions, an analogous result is valid for a Sasakian ambient too.*

KEY WORDS – *Locally conformal Kaehler manifold - Second variation formula.*

A.M.S. CLASSIFICATION: 53C55 - 53C21

1 – Introduction

In [2] T. FRANKEL proved the following:

THEOREM A. *Let M^n be a complete connected manifold with positive Riemannian sectional curvature and let V^r and W^s be compact totally geodesic submanifolds. If $r+s \geq n$ then V^r and W^s have a nonempty intersection.*

THEOREM B. *Let M^n be a complete connected Kaehler manifold with positive sectional curvature and let V^r and W^s be compact complex analytic submanifolds. If $r + s \geq n$ then V^r and W^s have a nonempty intersection.*

As far as we know, these results have been extended by A. GRAY [3] for the case of nearly Kaehler ambient space and by S. MARCHIAFAVA [4] for a quaternionic Kaehler one. In this note we show that a result analoguos with Theorem B is true for locally conformal Kaehler (l.c.K.) manifolds and also for normal contact metric (Sasakian) manifolds. The proof will follow essentially Frankel's original.

All manifolds are supposed to be connected and all geometric objects we use are differentiable of class C^∞.

We shall briefly recall some basic facts about l.c.K. manifolds. We refer to [5], [6], [7], [8] for details and examples.

A Hermitian manifold M^n with complex structure $\mathcal{J}$ and Hermitian metric g, of complex dimension n is l.c.K. if there exists a globally defined closed one form ω (the Lee form) which satisfies $d\Omega = \omega \wedge \Omega$ where $\Omega(X,Y) = g(X, \mathcal{J}Y)$ is the fundamental two-form of M^n. Let $B = \omega^\#$ and $\theta = \omega \circ \mathcal{J}$. The following formula characterizes the l.c.K. class:

$$(1.1)\quad (\nabla_X \mathcal{J})Y = (1/2)\Big\{g(X,Y)\mathcal{J}B - g(X,\mathcal{J}Y)B + \theta(Y)X - \omega(Y)\mathcal{J}X\Big\}.$$

If ω is parallel with respect to the Levi-Civita connection ∇ then M^n is called a generalized Hopf manifold (g.H.m.). A typical example is the Hopf manifold $H^n = (C^n - 0)/\Delta_\lambda$ (where Δ_λ is the cyclic group generated by the transformation $z \mapsto \lambda z$, $\lambda \in C - \{0\}$, $|\lambda| \neq 1$) with the metric $(\sum dz^k dz^{-k})/(\sum z^k z^{-k})$. We notice that on a g.H.m. the two planes containing B have zero sectional curvature.

The proper analogue in l.c.k. geometry of the complex submanifolds of Kaehler manifolds are the complex submanifolds tangent to the Lee vector field. These inherit the l.c.K. structure.

Now we can state the announced results:

THEOREM 1. *Let M^n be a complete l.c.K. manifold and V^r, W^s two compact complex submanifolds tangent to the Lee vector field B of M^n. If M^n has nonnegative sectional curvature, zero curvature having* *eventually, only the two-planes containing B and if $r+s \geq n$ then V^r and W^s have a nonempty intersection.*

COROLLARY. *Let M^n be a complete g.H.m. with nonsingular Lee vector field and which satisfies the same curvature restrictions as above. Then any two compact complex submanifolds tangent to the Lee vector field must intersect (no restriction on the dimensions).*

REMARK. The Hopf manifold has a nonsingular Lee vector field.

Now we refer to [9] for the definition and properties of Sasakian manifolds $(M^{2n+1}, \varphi, \xi, \eta, g)$ and their invariant submanifolds.

In this context the following result holds:

THEOREM 2. *Let M^{2n+1} be a complete Sasakian manifold and V^r, W^s two compact invariant submanifolds tangent to the structure vector field ξ. If M has nonnegative sectional curvature, zero curvature having, eventually, only the two-planes containing ξ and if $r+s \geq 2n+1$ then V^r and W^s must intersect.*

2 – Proof

Suppose, ad absurdum, that V^r and W^s do not intersect. There is a unique minimal geodesic $c\colon [0,L] \to M^n$ parametrized by arclength which realizes the distance between V^r and W^s. Obviously c intersects $V^r(W^s)$ orthogonally in $p = c(0)(q = c(L))$. Let V_q be the linear subspace of T_qM^n obtained by parallel translation of T_pV^r along c. Applying Grassman's theorem to the linear subspaces V_q and T_qW^s (both orthogonal to T_qc) we find:

$$\dim(V_q \cap T_qW^s) \geq r+s-(n-1) \geq 1\,.$$

So there is at least one unitary vector X_q in $V_q \cap T_qW^s$; this one must correspond by parallel translation long c to a unitary vector X_p in T_pV^r. Let X be the parallel (unitary) vector field on c obtained by parallel translation of X_p (so $X_{c(L)} = X_q$). The idea is to make variations of c in the direction of X and $\mathcal{J}X$ and to show that at least one second variation $L''_X(0)$ or $L''_{\mathcal{J}X}(0)$ is strictly negative, thus contradicting the minimality of c. The following formula for the second variation is well-known (see e.g. [1]):

$$L''_X(0) = g(\nabla_X X, \dot{c})(q) - g(\nabla_X X, \dot{c})(p) - \int_0^L K(\dot{c}, X)dt$$

where $\dot{c}$ is the velocity vector field of c and K the sectional curvature of M^n. Now $\mathcal{J}X$ will not be in general parallel on c. Still remains true $g(\dot{c}, X) = 0$ in p and q. To compute $L''_{\mathcal{J}X}(0)$ we first evaluate, using (1.1):

$$\begin{aligned}
\nabla_{\mathcal{J}X}\mathcal{J}X &= (\nabla_{\mathcal{J}X}\mathcal{J}) + \mathcal{J}(\nabla_{\mathcal{J}X}X) = \\
&= \frac{1}{2}\Big\{g(\mathcal{J}X, X)\mathcal{J}B - g(\mathcal{J}X, \mathcal{J}X)B + \theta(X)\mathcal{J}X + \omega(X)X\Big\} + \\
&+ \mathcal{J}\nabla_X \mathcal{J}X + \mathcal{J}[\mathcal{J}X, X] = \\
&= \frac{1}{2}\Big\{-B + \theta(X)\mathcal{J}X + \omega(X)X\Big\} + \mathcal{J}(\nabla_X\mathcal{J})X - \nabla_X X + \\
&+ \mathcal{J}[\mathcal{J}X, X] = \\
&= -B + \omega(X)X + \theta(X)\mathcal{J}X - \nabla_X X + \mathcal{J}[\mathcal{J}X, X]\,.
\end{aligned}$$

As B, X, $\mathcal{J}X$ (hence $\mathcal{J}[\mathcal{J}X, X]$) are tangent in $p(q)$ to $V^r(W^s)$ we have

$$L''_{\mathcal{J}X}(0) = -g(\nabla_X X, \dot{c})(q) + g(\nabla_X X, \dot{c})(p) - \int_0^L K(\dot{c}, X)dt\,.$$

Thus $L''_X(0) + L''_{\mathcal{J}X}(0) = -\int_0^L \{K(\dot{c}, X) + K(\dot{c}, \mathcal{J}X)\}dt < 0$ because $\dot{c}$ beeing orthogonal in p, q to the submanifolds cannot be the restriction of B on c, so at least one of $K(\dot{c}, X)$ and $K(\dot{c}, \mathcal{J}X)$ is strictly positive. Thus one of $L''_X(0)$ and $L''_{\mathcal{J}X}(0)$ must be strictly negative and the proof is complete.

If now M^n is a g.H.m. like in the Corollary, we can renounce to the restriction on the dimensions of V^r and W^s. Indeed, beeing parallel on M^n, B is in particular parallel on c; thus $B_q \in V_q \cap T_q W^s$ plays the part of X_q in the above proof. Then $L''_B(0) = 0$ but $L''_{\mathcal{J}B}(0) = -\int_0^L K(\dot{c}, \mathcal{J}B) < 0$ and the conclusion follows.

Finally, to obtain Theorem 2 we rewrite, mutatis mutandis (φ for $\mathcal{J}$, ξ for B etc....) the above proof.

REMARK 1. Theorem 1 does not apply to compact l.c.K. manifolds with constant sectional curvature because these are in fact flat Kaehler manifolds [7].

REMARK 2. The Corollary does not apply to conformally flat g.H.m. (e.g. the Hopf manifold). Indeed, if we denote by $k = |\omega|/2$ and let $u = \omega/|\omega|$, then from the formula (3.22) in [8] we deduce that the sectional curvature is given by:

$$K(X,Y) = k^2\left\{1 - [u(X)]^2 - [u(Y)]^2\right\}.$$

Hence it cannot be everywhere nonnegative (nor nonpositive).

REMARK 3. Compactness is essential in Theorem 1 and in the Corollary, that is it isn't imposed only by the method of proof. In fact, a g.H.m. with nonsingular Lee vector field is foliated by codimension one totally geodesic submanifolds (which obviously do not intersect). The foliation is given by the kernel of ω [8].

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