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# On the distance between connections

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RIASSUNTO – *Si fa vedere che esiste una metrica naturale sulle fibre dello spazio delle connessioni principali sul fibrato dei riferimenti. Tale metrica è data da un'estensione dell'idea di metrica di fibrato (o b-metrica) indotta da una data connessione, e fornisce una maniera naturale di definire la distanza di due connessioni assegnate. Si studia il problema di ridurre tale distanza a una distanza sopra la varietà di base e, in particolare, si trova una relazione tra la distanza di due connessioni lungo una data curva e la b-incompletezza della medesima curva rispetto alle due connessioni. Questo risultato è legato al problema della stabilità delle singolarità spazio-temporali.*

ABSTRACT – *It is shown that there is a natural metric in the fibres of the space of principal connections over the frame bundle. This metric is given by an extension of the idea of bundle-metric induced by a given connection, and provides a natural way of defining the distance between two given connections. The problem of reducing this distance to a distance over the base manifold is examined and, in particular, a relation between the distance of two given connections along a curve and the bundle-incompleteness of that curve with respect to the given connections is found. This result is related to the problem of the stability of space-time singularities.*

KEY WORDS – *Connections - Bundle metric.*

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## 1 – Notation and preliminaries [5,6]

Let $M$ be a manifold and $p\colon LM \to M$ its frame bundle. The action of the structure group $G \equiv Gl(n, \mathbb{R})$ on $LM$ can be naturally extended to an action on the first-order jet space $JLM$. Namely, for each $g \in G$, we take the jet prolongation of $R_g\colon LM \to LM$, i.e.

$$JR_g\colon JLM \to JLM.$$

A principal connection on $p$ is a section $\mathbf{c}\colon LM \to JLM$ which is invariant with respect to this extended action. Thus, the assignment of a principal connection is equivalent to that of a section

$$c\colon M \to C \equiv JLM/G.$$

More precisely, we have the canonical isomorphism over $LM$

$$\xi\colon C \underset{M}{\times} LM \to JLM$$

and, for any section $c\colon M \to C$, we have the connection

$$\mathbf{c} \equiv \xi \circ (p^{*}c)\colon LM \to JLM.$$

The map $\xi$ is called the *system of principal connections* on $M$. The bundle $p^C\colon C \to M$ is called the *bundle of principal connections* on $M$[(1)].

A local chart on $M$ induces naturally a local fibred chart $(x^\lambda, y^\lambda_\mu)\colon LM \to \mathbb{R}^{m(1+m)}$, where $b^\lambda_\mu \equiv y^\lambda_\mu(b)$ is defined as the $\lambda$-component, in the basis $\{\partial x_\lambda\}_{\lambda=1,\dots,m}$, of $b_\mu \in TM$, which is the $\mu$-th element of $b \equiv (b_\mu)_{\mu=1,\dots,m} \in LM$. In other terms $y^\lambda_\mu(b) \equiv \langle dx^\lambda, b_\mu\rangle$.

The induced fibred chart on $JLM$ will be denoted as $(x^\lambda, y^\lambda_\mu, y^\lambda_{\nu\mu})\colon JLM \to \mathbb{R}^{m(1+m+m^2)}$. Moreover we have the naturally induced fibred chart $(x^\lambda, v^\lambda_{\nu\mu})$ on $JLM/G$ defined by

$$v^\lambda_{\nu\mu}([\bar{b}]_G) \equiv y^\lambda_{\nu\mu}(\bar{b}_\theta) = ((y^{-1})^\lambda_\rho y^\rho_{\nu\mu})(\bar{b})$$

where $\bar{b} \in JLM$ and $\bar{b}_\theta$ is the unique element of $[\bar{b}]_G \subset JLM$ such that $p_{LM}(\bar{b}_\theta) = \{\partial_i\}$ ($p_{LM}\colon JLM \to LM$ is the natural projection). The coordinate expression of the map $\xi$ can be written as

$$(x^\lambda, y^\lambda_\mu, y^\lambda_{\nu\mu}) \circ \xi = (x^\lambda, y^\lambda_\mu, \xi^\lambda_{\nu\mu})$$

(1) This is a particular case of a quite general concept of system[8], describing a situation in which we have a two-fibred manifold $F \to E \to B$ and there is a distinguished subspace of the space of all sections $E \to F$, where the sections of this subspace can be characterized as sections of some bundle with base $B$; in our case, invariant sections $LM \to JLM$ can be characterized as sections $M \to C$.

where $\xi^{\lambda}_{\nu\mu}\colon C \underset{M}{\times} LM \to \mathbb{R}$.

The assignment of a connection $\mathbf{c}\colon LM \to JLM$ is equivalent to that of a vertical-valued form

$$\omega_{\mathbf{c}}\colon LM \to T^{*}LM \underset{LM}{\otimes} VLM\,.$$

The connection $\mathbf{c}$ is invariant if and only if $\omega_{\mathbf{c}}$ is invariant. Moreover, $\omega_{\mathbf{c}}$ is characterized by the form

$$\omega_{c}\colon M \to T^{*}M \underset{LM}{\otimes} A$$

where $A \equiv VLM/G$ is the adjoint bundle (thus we actually have a system of vertical-valued forms). Because of the canonical isomorphism $\sigma\colon VLM \to LM \times \mathbb{R}^{m^2}$ ($\mathbb{R}^{m^2}$ is the Lie algebra of $G$), the assignment of a principal connection $\mathbf{c}$ is also equivalent to that of the connection form $\Omega_{\mathbf{c}} \equiv \sigma \circ \omega_{\mathbf{c}}\colon LM \to T^{*}LM \otimes \mathbb{R}^{m^2}$, or equivalently $\Omega_{\mathbf{c}}\colon TLM \to \mathbb{R}^{m^2}$, which has the coordinate expression

$$(\Omega_{\mathbf{c}})^{\alpha}_{\beta} = (y^{-1})^{\alpha}_{\rho}(\dot{y}^{\rho}_{\beta} - C^{\rho}_{\lambda\sigma} y^{\sigma}_{\beta} \dot{x}^{\lambda})$$

where $C^{\rho}_{\lambda\sigma}$ are the structure constants. Hence we have the system of vector-valued forms

$$\Omega_{\xi}\colon C \underset{M}{\times} LM \to T^{*}LM \otimes \mathbb{R}^{m^2}$$

characterized by $\Omega_{\mathbf{c}} = \Omega_{\xi} \circ (c, \mathrm{id}_{LM})$ for all $c\colon M \to C$.

## 2 – Bundle metric and extensions to $C \underset{M}{\times} LM$

Given a principal connection on $p\colon LM \to M$, consider the symmetric bilinear form

$$f_{\mathbf{c}}\colon TLM \underset{LM}{\times} TLM \to \mathbb{R}\colon (w, z) \longmapsto \Omega_{\mathbf{c}}(w) \cdot \Omega_{\mathbf{c}}(z)$$

where the dot stands for the standard scalar product in $\mathbb{R}^{m^2}$. This bilinear form is degenerate, since it vanishes on c-horizontal vectors. However, its restriction to the fibres of $p\colon LM \to M$ is a true Riemannian metric. As we have one such form for each connection of the system, we actually have a system of symmetric bilinear forms

$$f_\xi\colon C \underset{M}{\times} LM \to T^*LM \underset{LM}{\otimes} T^*LM\,.$$

Now, we set

$$\tilde{C} \equiv C \underset{M}{\times} LM\,;$$

by using the canonical projection $(\pi_C \circ \pi^1, T\pi^2)\colon T\tilde{C} \to C \underset{M}{\times} TLM$, $f_\xi$ can also be viewed as a symmetric bilinear form on $\tilde{C} \equiv C \underset{M}{\times} LM$. Namely, we have the map

$$\tilde{f}_\xi\colon T\tilde{C} \underset{C}{\times} T\tilde{C} \to \mathbb{R}\colon ((w_C, w_L), (z_C, z_L)) \longmapsto f_\xi(w_L, z_L)\,.$$

Furthermore, we have the canonical 1-form

$$\Theta\colon TLM \to \mathbb{R}^m\colon w \longmapsto (w^\lambda)$$

where $(w^\lambda)$ are the components of $Tp(w) \in TM$ in the basis $\pi_{LM}(w) \in LM$. In coordinates

$$\Theta^\lambda = (y^{-1})^\lambda_\mu \dot{x}^\mu\,.$$

This enables us to define, for each principal connection, a true Riemannian metric:

$$g_c \equiv f_c + \Theta \cdot \Theta\colon TLM \underset{LM}{\times} TLM \to \mathbb{R}\colon (w, z) \longmapsto \Theta(w) \cdot \Theta(z) + f_c(w, z)$$

whose restriction to the fibres of $p\colon LM \to M$ coincides with $f_c$.

This is exactly the *bundle-metric* that was introduced by SCHMIDT [9] for the study of space-time singularities and independently by MARATHE [7] for different purposes. The manifold $M$ is called *bundle-complete* with respect to c if $LM$ is $g_c$-complete [4, 9].

As such metric is given for each principal connection c on $LM \to M$, we have a system of Riemannian metrics

$$g_\xi\colon C \underset{M}{\times} LM \to T^*LM \underset{LM}{\otimes} T^*LM\,.$$

We have also the symmetric bilinear form on $\widetilde{C}$

$$\tilde{g}_\xi \equiv \tilde{f}_\xi + (\Theta\cdot\Theta)\circ T\tilde{p} : T\widetilde{C} \underset{C}{\times} T\widetilde{C} \to \mathbb{R} : ((W_C, W_L), (Z_C, Z_L)) \longmapsto g_\xi(W_L, Z_L) .$$

One sees immediately that $\tilde{g}_\xi$ is degenerate, since it vanishes on $\pi^2$-vertical vectors. It is natural to ask whether it can be extended to a Riemannian metric. The answer is that there is no distinguished way of choosing such an extension, but we shall see that the assignment of a section $k: C \to JC$ (hence a connection on $p^C$) determines one such choice.

## 3 – The canonical metric in the fibres of $\pi^2 : \widetilde{C} \to LM$

Next, we show that there is a canonical metric defined on each fibre of $\pi^2 : \widetilde{C} \to LM$. The construction is similar to that of the Schmidt-Marathe bundle metric on $LM$, but slightly more complicated.

First, we note that there is a natural linear map over $LM$

$$\Theta^* : T^*M \underset{M}{\times} LM \to \mathbb{R}^m$$

which associates with any $(\alpha, b)$ the components of the covector $\alpha$ in the dual basis of $b$. In coordinates:

$$(\Theta^*)_\lambda = y^\mu_\lambda \dot{x}_\mu .$$

Second, observe that $\sigma: VLM \to LM \times \mathbb{R}^{m^2}$ gives rise to the isomorphism (indicated by the same symbol) $\sigma: LM \underset{M}{\times} VLM/G \to M \times \mathbb{R}^{m^2}$ over $p: LM \to M$. Since $VC \equiv T^*M \underset{M}{\otimes} (VLM/G)$, we have the isomorphism over $p$

$$(\mathrm{id}_{T^*M} \otimes \sigma) : LM \underset{M}{\times} T^*M \underset{M}{\otimes} (VLM/G) \equiv LM \underset{M}{\times} VC \longrightarrow T^*M \otimes \mathbb{R}^{m^2} .$$

Then, $(\mathrm{id}_{T^*M} \otimes \sigma)_b : T^*M \underset{M}{\otimes} (VLM/G) \to T^*M \otimes \mathbb{R}^{m^2}$ is a linear isomorphism over $M$ for each section $b: M \to LM$. By composition with $\Theta^*$ we obtain a map

$$\tau \equiv (\Theta^* \otimes \mathrm{id}_{\mathbb{R}^{m^2}}) \circ (\mathrm{id}_{LM}, \mathrm{id}_{T^*M} \otimes \sigma) : LM \underset{M}{\times} VC \longrightarrow \mathbb{R}^m \otimes \mathbb{R}^{m^2} \equiv \mathbb{R}^{m^3} .$$

Let $(x^\lambda, v^\lambda_\mu, \dot{v}^\lambda_{\nu\mu})$ be the induced fibred coordinate chart on $VC$. Then the coordinate expression of $\tau$ is

$$\tau^\lambda_{\nu\mu} = y^\alpha_\nu \dot{v}^\lambda_{\alpha\mu} \,.$$

Then we define a map

$$h\colon LM \underset{M}{\times} VC \underset{C}{\times} VC \longrightarrow \mathbb{R}\colon (b, z, z') \longmapsto (\tau_b(z)) \cdot (\tau_b(z'))$$

where the dot stands for the standard scalar product in $\mathbb{R}^{m^3}$. For any fixed $b \in L_x M, h_b\colon (VC)_x \times (VC_x) \to \mathbb{R}$ is a Riemannian metric on the manifold $C_x$. Now, since $V_{LM}\widetilde{C} \equiv LM \underset{M}{\times} VC$, the map $h$ is a metric on the fibres of $\pi^2\colon \widetilde{C} \to LM$, called the *$\pi^2$-vertical metric*.

This metric induces a distance $d_h$ on each fibre of $\pi^2$, which can be used for defining in a natural way a distance between two principal connections on $LM \to M$. Namely, take any two sections $c, c'\colon M \to C$. Then we have the map

$$d(c, c')\colon LM \to \mathbb{R}^+\colon b \longmapsto d_h(p^* c(b), p^* c'(b))\,.$$

Note that $d(c, c')$ is continuous but, in general, may be not differentiable.

## 4 – Reductions of the $\pi^2$-vertical metric

A *reduction* of $h$ to $M$ is a map

$$h_M\colon VC \underset{C}{\times} VC \to \mathbb{R}$$

such that $h_M \circ p = h$. Then, $h_M$ is a metric in the fibres of $p^C\colon C \to M$. Thus a reduction of $h$ induces a distance over $M$ between sections of $p^C$, given by

$$d(c, c')\colon M \to \mathbb{R}^+\colon x \longmapsto d_{h_M}(c(x), c'(x))\,.$$

It is particularly interesting to examine cases in which $h_M$ arises in a natural way. A first obvious observation is that, if $M$ is parallelizable, then there is a reduction of $h$ for any choice of a global section $b\colon M \to LM$, given by

$$h_M(w,z) \equiv h(b(x), w, z)$$

where $x \equiv p^C \circ \pi_C(w) \in M$. We have seen that this kind of reduction can be performed at least locally, even if there is no global parallelization.

Let now $c\colon M \to C$ be a section such that the curvature tensor of the induced connection $\mathbf{c}$ vanishes. Then we have, at least locally, the $\mathbf{c}$-horizontal sections $M \to LM$. The choice of just one element $\bar{b} \in LM$ determines the horizontal section through $\bar{b}$ and then a local reduction of $h$ and the distance, over $M$, between $c$ and any other section $c'\colon M \to C$.

## 5 – Distance along a curve and its relation with $b$-completeness

Let $\gamma\colon I \subset \mathbb{R} \to M$ be a curve and $\gamma^\dagger\colon I \to LM$ a *lift* of $\gamma$, that is a curve such that $p \circ \gamma^\dagger = \gamma$. Then the distance between two connections $\mathbf{c}$ and $\mathbf{c}'$ along $\gamma^\dagger$ is naturally defined as the function

$$d(c,c') \circ \gamma^\dagger\colon I \to \mathbb{R}^+ .$$

It is interesting to compare the lengths of $\gamma^\dagger$ in the two induced metrics $g_c$ and $g_{c'}$, in relation to the study of bundle-completeness and incompleteness stability. Let us show that there is a function $r\colon I \to \mathbb{R}^+$ with the property that if $d(c,c') \circ \gamma^\dagger < r$ then the $g_c$-length of $\gamma^\dagger$ is finite iff its $g_{c'}$-length is finite. Actually, the difference $(c^\lambda_{\nu\mu} - c'^\lambda_{\nu\mu}) \circ \gamma^\dagger$ between the components of the two connections along the curve is bounded if $d(c,c') \circ \gamma^\dagger$ tends to zero with sufficient rapidity when the parametre approaches an extremum of the interval $I$. Thus, also the difference between the two norms of the tangent vector field $d\gamma^\dagger$ to the curve $\gamma^\dagger$ is bounded, since

$$(\Omega_{\mathbf{c}} - \Omega_{\mathbf{c}'})^\lambda_\mu \circ \gamma^\dagger = (\gamma^{-1})^\lambda_\rho \gamma^\beta_\mu \gamma^\alpha ( - c^\rho_{\alpha\beta} + c'^\rho_{\alpha\beta}) \circ \gamma .$$

A consequence of this fact is the following: if $\gamma$ is a bundle-incomplete curve with respect to $\mathbf{c}$, i.e. $\gamma^\dagger$ is an inextendible curve of finite $g_c$-length, then $\gamma$ is bundle-incomplete with respect to $\mathbf{c}'$ if the distance of the two connections along $\gamma^\dagger$ is sufficiently small. This result thus establishes a kind of stability of the bundle-incompleteness of a manifold, and is consistent with similar results obtained in [2, 3].

We observe that this condition is not at all necessary. Actually, it is easy to see by examples that in some cases bundle-incompleteness may be preserved also if the distance between the two connection diverges. It would be quite interesting, from the viewpoint of the stability of singularities, to find out the superior limit of the distance of the two connections, along $\gamma$, such that incompleteness is preserved.

## 6 – Riemannian metrics on $\widetilde{C}$

We observe that $\tilde{g}_\xi : T\widetilde{C} \underset{C}{\times} T\widetilde{C} \to \mathbb{R}$ is a degenerate metric on $\widetilde{C}$, the degeneracy arising from the fact that $\tilde{g}_\xi$ vanishes on $\pi^2$-vertical vectors, i.e. elements of $VC \underset{M}{\times} LM$. Now, we saw that $h$ acts exactly on such vectors. Thus, one is tempted to "assemble" the two metrics and construct a Riemannian metric on $\widetilde{C}$. However, this is not possible in general for the following reason.

Take $(w_C, w_L) \in T\widetilde{C} \equiv TC \underset{TM}{\times} TLM$. Then $h$ cannot be applied to $(w_C, w_L)$ because there is no distinguished way of saying which is the vertical part of $w_C$. Actually, we know that what is needed to project $w_C$ on $VC$ is exactly a connection $k: C \to JC$, via its associated vertical-valued form $\omega_k : TC \to VC$. Given $k$, we have the Riemannian metric

$$g_k \equiv \tilde{g}_\xi + h \circ (\omega_k \times \omega_k) : T\widetilde{C} \underset{C}{\times} T\widetilde{C} \to \mathbb{R}$$

or

$$g_k((w_C, w_L), (z_C, z_L)) \equiv g_\xi(w_L, z_L) + h(\omega_k(w_C), \omega_k(z_C)) .$$

In [1], the question of Riemannian metrics on $\widetilde{C}$ has beeen addressed using a quite general approach to systems of connections and over connections, as developed in [8]. One sees that there exists a distinguished system of metrics on $\widetilde{C}$. The structure of this system is rather complicate; consistently with our previous result, one sees that it has a natural subsystem where the bundle of metrics can be identified with the bundle of affine connections on $C \to M$.

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