Rendiconti di Matematica, Serie VII
Volume 12 , Roma (1992), 35-44

Some properties of certain analytic functions

S. OWA - M. NUNOKAWA - H. SAITOH(*)

RIASSUNTO – *Si introducono due sottoclassi $B_n(\alpha)$ e $C_n(\alpha)$ di funzioni analitiche nel cerchio unitario; se ne studiano le proprietà ed in particolare si considera una loro generalizzazione.*

ABSTRACT – *Two subclasses $B_n(\alpha)$ and $C_n(\alpha)$ of certain analytic functions in the unit disk are introduced. The object of the present paper is to derive some properties of functions belonging to the classes $B_n(\alpha)$ and $C_n(\alpha)$. Further, a generalization of the classes $B_n(\alpha)$ and $C_n(\alpha)$ is considered.*

KEY WORDS – *Analytic function.*

A.M.S. CLASSIFICATION: 30C45

1 – Introduction

Let A_n be the class of functions of the form

$$(1.1) \qquad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in N = \{1, 2, 3, \dots\})$$

which are analytic in the unit disk $U = \{z\colon |z| < 1\}$. A function $f(z)$

(*)This research of the authors was completed at Gunma University, Maebashi, Gunma 371, Japan when the first author was visiting from Kinki University, Higashi-Osaka, Osaka 577, Japan.

belonging to A_n is said to be in the class $B_n(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \alpha \tag{1.2}$$

for some $\alpha(\alpha > 1)$ and for all $z \in U$. Further, a function $f(z) \in A_n$ is said to be a member of the class $C_n(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \alpha \tag{1.3}$$

for some $\alpha(\alpha > 1)$ and for all $z \in U$. Note that $f(z) \in C_n(\alpha)$ if and only if $zf'(z) \in B_n(\alpha)$.

2 – Properties of the class $B_n(\alpha)$

In order to prove our results, we have to recall here the following lemma due to MILLER and MOCANU [2] (also, by MILLER [1]).

LEMMA. *Let $\phi(u, v)$ be a complex valued function,*

$$\phi\colon D \longrightarrow C\,, \quad D \subset C^2 \qquad (C \ \text{ is the complex plane})$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi(u, v)$ satisfies

(i) *$\phi(u, v)$ is continuous in D;*

(ii) *$(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} > 0$;*

(iii) *for all $(iu_2, v_1) \in D$ such that $v_1 \le -n(1 + u_2^2)/2$, $\operatorname{Re}\{\phi(iu_2, v_1)\} \le 0$.*

Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$ be regular in U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$\operatorname{Re}\left\{\phi(p(z), zp'(z))\right\} > 0 \qquad (z \in U)\,,$$

then

$$\operatorname{Re}\{p(z)\} > 0 \qquad (z \in U)\,.$$

Now, we prove

THEOREM 1. *If $f(z) \in B_n(\alpha)$, then*

$$\operatorname{Re}\left(\frac{z}{f(z)}\right)^{1/\gamma} > \frac{\gamma n}{\gamma n + 2\alpha - 2} \qquad (z \in U), \tag{2.1}$$

where $\gamma > 0$ and

$$1 < \alpha \leq 1 + \frac{\gamma n}{2}. \tag{2.2}$$

PROOF. For $f(z) \in B_n(\alpha)$, we define the function $p(z)$ by

$$\left(\frac{z}{f(z)}\right)^{1/\gamma} = \beta + (1-\beta)p(z) \tag{2.3}$$

with

$$\beta = \frac{\gamma n}{\gamma n + 2\alpha - 2}. \tag{2.4}$$

Then, $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ is regular in U. It follows from (2.3) that

$$\frac{zf'(z)}{f(z)} = 1 - \frac{\gamma(1-\beta)zp'(z)}{\beta + (1-\beta)p(z)}, \tag{2.5}$$

or

$$\operatorname{Re}\left\{\alpha - \frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\alpha - 1 + \frac{\gamma(1-\beta)zp'(z)}{\beta + (1-\beta)p(z)}\right\} > 0. \tag{2.6}$$

Letting

$$\phi(u,v) = \alpha - 1 + \frac{\gamma(1-\beta)v}{\beta + (1-\beta)u}, \tag{2.7}$$

$u = u_1 + iu_2$, and $v = v_1 + iv_2$, we see that

(i) $\phi(u,v)$ is continuous in $D = \left(C - \left\{\frac{\beta}{\beta-1}\right\}\right) \times C$;

(ii) $(1,0) \in D$ and $\operatorname{Re}\{\phi(1,0)\} = \alpha - 1 > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -n(1+u_2^2)/2$,

$$\begin{aligned}\operatorname{Re}\left\{\phi(iu_2, v_1)\right\} &= \alpha - 1 + \frac{\gamma\beta(1-\beta)v_1}{\beta^2 + (1-\beta)^2 u_2^2} \\ &\leq \alpha - 1 - \frac{n\gamma\beta(1-\beta)(1+u_2^2)}{2(\beta^2 + (1-\beta)^2 u_2^2)} \\ &= -\frac{(1-\beta)(\gamma n - 2\alpha + 2)u_2^2}{2(\beta^2 + (1-\beta)^2 u_2^2)} \leq 0 .\end{aligned}$$

because $0 < \beta < 1$ and $\gamma n - 2\alpha + 2 \geq 0$.

Thus, the function $\phi(u, v)$ satisfies the condition in Lemma. This shows that $\operatorname{Re}\{p(z)\} > 0 (z \in U)$, that is, that

$$\operatorname{Re}\left(\frac{z}{f(z)}\right)^{1/\gamma} > \beta = \frac{\gamma n}{\gamma n + 2\alpha - 2} \qquad (z \in U) . \tag{2.8}$$

Letting $\gamma = 1$, Theorem 1 leads to

COROLLARY 1. *If $f(z) \in B_n(\alpha)$ with $1 < \alpha < 1 + n/2$, then*

$$\operatorname{Re}\left(\frac{z}{f(z)}\right) > \frac{n}{n + 2\alpha - 2} \qquad (z \in U) . \tag{2.9}$$

Taking $\gamma = 2$ in Theorem 1, we have

COROLLARY 2. *If $f(z) \in B_n(\alpha)$ with $1 < \alpha \leq n + 1$, then*

$$\operatorname{Re}\sqrt{\frac{z}{f(z)}} > \frac{n}{n + \alpha - 1} \qquad (z \in U) . \tag{2.10}$$

3 – Properties of the class $C_n(\alpha)$

For the class $C_n(\alpha)$, we prove

THEOREM 2. *If* $f(z) \in C_n(\alpha)$ *with* $1 < \alpha \leq 1 + n/2$, *then*

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \frac{n+2}{n+2\alpha} \qquad (z \in U). \tag{3.1}$$

Therefore, $f(z)$ *is starlike with respect to the origin in* U.

PROOF. Define the function $p(z)$ by

$$\frac{f(z)}{zf'(z)} = \beta + (1-\beta)p(z). \tag{3.2}$$

Making use of the logarithmic differentiations of both sides in (3.2), we get

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1}{\beta + (1-\beta)p(z)} - \frac{(1-\beta)zp'(z)}{\beta + (1-\beta)p(z)}, \tag{3.3}$$

so

$$\begin{aligned} &\operatorname{Re}\left\{\alpha - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \\ &= \operatorname{Re}\left\{\alpha - \frac{1}{\beta + (1-\beta)p(z)} + \frac{(1-\beta)zp'(z)}{\beta + (1-\beta)p(z)}\right\} > 0. \end{aligned} \tag{3.4}$$

If we define the function $\phi(u, v)$ by

$$\phi(u,v) = \alpha - \frac{1}{\beta + (1-\beta)u} + \frac{(1-\beta)v}{\beta + (1-\beta)u}, \tag{3.5}$$

with $u = u_1 + iu_2$, $v = v_1 + iv_2$, then we obtain that

(i) $\phi(u, v)$ is continuous in $D = \left(C - \left\{\frac{\beta}{\beta - 1}\right\}\right) \times C$;

(ii) $(1, 0) \in D$ and $\operatorname{Re}\{\phi(1, 0)\} = \alpha - 1 > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -n(1+u_2^2)/2$,

$$\begin{aligned}\operatorname{Re}\left\{\phi(iu_2, v_1)\right\} &= \alpha - \frac{\beta}{\beta^2+(1-\beta)^2u_2^2} + \frac{\beta(1-\beta)v_1}{\beta^2+(1-\beta)^2u_2^2} \\ &\leq \alpha - \frac{\beta}{\beta^2+(1-\beta)^2u_2^2} - \frac{n\beta(1-\beta)(1+u_2^2)}{2(\beta^2+(1-\beta)^2u_2^2)} \\ &= -\frac{(1-\beta)(n+2-2\alpha)u_2^2}{2(\beta^2+(1-\beta)^2u_2^2)} \leq 0.\end{aligned}$$

because $0<\beta<1$ and $n+2-2\alpha \geq 0$.

Therefore, applying Lemma, we completes the proof of Theorem 2.

Finally, we derive

THEOREM 3. *If $f(z) \in C_n(\alpha)$, then*

$$\operatorname{Re}\left(\frac{1}{f'(z)}\right)^{1/\gamma} > \frac{\gamma n}{\gamma n+2\alpha-2} \qquad (z \in U), \tag{3.5}$$

where $\gamma > 0$ and

$$1 < \alpha \leq 1 + \frac{\gamma n}{2}. \tag{3.6}$$

PROOF. Defining the function $p(z)$ by

$$\left(\frac{1}{f'(z)}\right)^{1/\gamma} = \beta + (1-\beta)p(z) \tag{3.7}$$

with

$$\beta = \frac{\gamma n}{\gamma n+2\alpha-2}, \tag{3.8}$$

we have

$$\alpha - \left(1+\frac{zf''(z)}{f'(z)}\right) = \alpha - 1 + \frac{\gamma(1-\beta)zp'(z)}{\beta+(1-\beta)p(z)}. \tag{3.9}$$

Therefore, using the same manner of the proof in Theorem 1, we conclude that $\operatorname{Re}\{p(z)\} > 0$ $(z \in U)$, which implies (3.5).

Taking $\gamma = 1$ in Theorem 3, we have

COROLLARY 3. *If* $f(z) \in C_n(\alpha)$ *with* $1 < \alpha < 1 + n/2$, *then*

$$\text{Re}\left(\frac{1}{f'(z)}\right) > \frac{n}{n+2\alpha-2} \qquad (z \in U). \tag{3.10}$$

Therefore, $f(z)$ *is close-to-convex in* U.

Making $\gamma = 2$ in Theorem 3, we have

COROLLARY 4. *If* $f(z) \in C_n(\alpha)$ *with* $1 < \alpha \leq n + 1$, *then*

$$\text{Re}\sqrt{\frac{1}{f'(z)}} > \frac{n}{n+\alpha-1} \qquad (z \in U). \tag{3.11}$$

4 – Generalization of $B_n(\alpha)$ and $C_n(\alpha)$

In this section, we consider a generalization of the classes $B_n(\alpha)$ and $C_n(\alpha)$.

THEOREM 4. *If* $f(z) \in A_n$ *satisfies*

$$\text{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} < \beta \qquad (z \in U) \tag{4.1}$$

for some $\alpha(\alpha > 0)$ *and* $\beta(1 < \beta \leq 1 + \alpha n/2)$, *then*

$$\text{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \frac{\alpha n + 2}{\alpha n + 2\beta} \qquad (z \in U). \tag{4.2}$$

PROOF. Letting

$$\frac{f(z)}{zf'(z)} = \gamma + (1-\gamma)p(z) \tag{4.3}$$

with

$$\gamma = \frac{\alpha n + 2}{\alpha n + 2\beta}, \tag{4.4}$$

we have from (4.1) that

$$\begin{aligned} &\mathrm{Re}\left\{\beta - (1-\alpha)\frac{zf'(z)}{f(z)} - \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} \\ &= \mathrm{Re}\left\{\beta - \frac{1}{\gamma + (1-\gamma)p(z)} + \frac{\alpha(1-\gamma)zp'(z)}{\gamma + (1-\gamma)p(z)}\right\} > 0. \end{aligned} \tag{4.5}$$

Therefore, defining the function $\phi(u, v)$ by

$$\phi(u,v) = \beta - \frac{1}{\gamma + (1-\gamma)u} + \frac{\alpha(1-\gamma)v}{\gamma + (1-\gamma)u}, \tag{4.6}$$

we know that

(i) $\phi(u, v)$ is continuous in $D = \left(C - \left\{\frac{\gamma}{\gamma-1}\right\}\right) \times C$;

(ii) $(1, 0) \in D$ and $\mathrm{Re}\{\phi(1,0)\} = \beta - 1 > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -n(1+u_2^2)/2$,

$$\begin{aligned} \mathrm{Re}\left\{\phi(iu_2, v_1)\right\} &= \beta - \frac{\gamma}{\gamma^2 + (1-\gamma)^2 u_2^2} + \frac{\alpha\gamma(1-\gamma)v_1}{\gamma^2 + (1-\gamma)^2 u_2^2} \\ &\leq \beta - \frac{\gamma}{\gamma^2 + (1-\gamma)^2 u_2^2} - \frac{n\alpha\gamma(1-\gamma)(1+u_2^2)}{2(\gamma^2 + (1-\gamma)^2 u_2^2)} \\ &= -\frac{(1-\gamma)(\alpha n - 2\beta + 2)u_2^2}{2(\gamma^2 + (1-\gamma)^2 u_2^2)} \leq 0. \end{aligned}$$

Thus, with the help of Lemma, we complete the proof of our assertion.

REMARK. If we take $\alpha = 1$ in Theorem 4, then we have Theorem 2. Taking $\alpha = 1/2$ in Theorem 4, we have

COROLLARY 5. *If $f(z) \in A_n$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right\} < 2\beta - 1 \qquad (z \in U) \tag{4.7}$$

for some $\beta(1 < \beta \leq 1 + n/4)$, then

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \frac{n+4}{n+4\beta} \qquad (z \in U). \tag{4.8}$$

Further, letting $\alpha = 1/n$, we have

COROLLARY 6. *If $f(z) \in A_n$ satisfies*

$$\operatorname{Re}\left\{(n-1)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right\} < n\beta - 1 \qquad (z \in U) \tag{4.9}$$

for some $\beta(1 < \beta \leq 3/2)$, then

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{3}{1+2\beta} \qquad (z \in U). \tag{4.10}$$

REFERENCES

[1] S.S. MILLER: *Differential inequalitis and Carathéodory functions*, Bull. Amer. Math. Soc., 81 (1975), 79-81.

[2] S.S. MILLER - P.T. MOCANU: *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65 (1978), 289-305.

Lavoro pervenuto alla redazione il 19 luglio 1990
ed accettato per la pubblicazione il 18 settembre 1991
su parere favorevole di A. Ossicini e di E. Arbarello

INDIRIZZO DEGLI AUTORI:

Shigeyoshi Owa - Department of Mathematics - Kinki University - Higashi-Osaka - Osaka 577 - Japan

Mamoru Nunokawa - Department of Mathematics - Gunma University - Maebashi - Gunma 371 - Japan

Hitoshi Saitoh - Department of Mathematics - Gunma College of Technology - Maebashi - Gunma 371 - Japan